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HENCKY–PRANDTL NETS AND CONSTANT PRINCIPAL STRAIN MAPPINGS WITH ISOLATED SINGULARITIES

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Abstract. The work presented in this paper is motivated in large measure by the appearance of Hencky–Prandtl nets (HP-nets) in the context of planar quasi-isometries with constant principal stretching factors (cps-mappings) and by compelling analogies between such mappings and those given by analytic functions of one complex variable. We study the behavior of HP-nets in the vicinity of isolated singularities and use the results of this analysis to show that if an HP-net is regular in the entire plane except for isolated singularities, then it can have at most two of them, and that all possibe nets of this kind fall into five classes each of which depends on a small number of parameters. In light of the relationship between HP-nets and cps-mappings it follows that an analogous statement holds for the latter as well, and this connection is further exploited to prove that HP-nets regular except for isolated singularities in smoothly bounded Jordan domains have nontangential limits in the appropriate sense at almost all boundary points. The treatment includes, in addition, an interpretation of cps-mappings with isolated singularities as deformations produced by the cryptocrystalline solidification with microscopic flaws of a planar film and a discussion of the problem of just how the singularities of such mappings can actually be distributed in a given domain.

Introduction

Disregarding considerations of regularity and connectivity, two mutually orthogonal one-parameter families of curves (called *characteristics*), covering a given plane domain D, form a *Hencky–Prandtl net* (abbreviated as *HP-net*) if for any two fixed curves C_1, C_2 belonging to one of the families, the change in the inclination of the tangent is the same along all subarcs of curves of the *other* family which join a point of C_1 to a point of C_2. Such nets are of importance in the theories of plasticity (see [Hi]) and optimal design (see [Hem]), and there is an extensive literature dealing with the analytic and numerical construction of HP-nets that satisfy various boundary conditions arising in connection with these theories as well as other applied problems. The local theory of such nets seems to have been worked out fully by Prandtl [Pra], Hencky [Hen] and Carathéodory–Schmidt [CS]. A paper of Collins [C] contains an excellent discussion of numerous aspects of the theory of Hencky–Prandtl nets and an encyclopedic bibliography. Moreover, G. Strang and

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R.V. Kohn [SK] have described a problem which involves construction of HP-nets in both the plasticity and optimal design contexts in different parts of a given domain.

The present discussion is motivated by yet another connection in which HP-nets arise, namely that of planar deformations for which the principal strains are distinct constants, henceforth called *cps-mappings*. Indeed, it is not difficult to show (see Proposition 1.1) that two mutually orthogonal families of curves covering a simply connected domain are the families of the lines of principal strain of a cps-mapping if and only if they form an HP-net. This class of mappings presents itself as a natural object of study in various ways.

First of all, they constitute a simple class of quasi-isometries (essentially deformations with bounded principal stretching factors) introduced and studied by F. John ([J1], [J2], [J3]). It is quite likely that for many of the as yet unresolved distortion questions for quasi-isometries raised by John, extremal behavior is displayed by cps-mappings and their higher dimensional analogues. Regardless of whether this proves to be the case, cps-mappings form a nontrivial but nonetheless tractable class of quasi-isometries whose study yields valuable insights into the extent of global distortion consistent with given bounds on local stretching.

Secondly, although governed by a nonlinear hyperbolic system (equations (1.1) in Section 1.2) cps-mappings bear, in many aspects of their behavior, notable similarities to conformal mappings, more precisely to conformal mappings f for which $\mathrm{Re}\{\log f'(z)\}$ is bounded. There are several ways in which the analogy can be drawn, but for the purposes at hand it is enough to say that the function which gives the inclination of the tangent line to the curves of either of the families of the associated HP-net (to be referred to as an HP-function in the sequel) takes the role of the harmonic function $\arg f'(z)$. A simple, but striking instance of this similarity is the HP-version of Liouville's theorem on bounded harmonic functions: an HP-function regular in the entire plane is necessarily a constant. Moreover, possibilities for developing a distortion theory for cps-mappings paralleling parts of the classical geometric theory of functions of one complex variable are described at some length in [G1, Section 4] where a numerically sharp result in this vein is established. In this paper we follow the function theory model in an investigation of isolated singularities of HP-nets and cps-mappings. In Section 1 we set down formal definitions of HP-nets and cps-mappings with minimal regularity requirements, discuss their basic properties (with proofs included for the sake of completeness), describe several procedures for the construction of HP-nets and define several specific nets which play a fundamental role in the succeeding development. The following two sections are devoted to an analysis of the behavior of HP-nets in the vicinity of an isolated singularity. Isolated singularities fall into two distinct categories depending on whether the associated HP-function is bounded on some characteristic terminating at the singular point or not; these two cases are fully analyzed in Sections 2 and 3, respectively (see Theorems 2.1 and 3.1). The results of this analysis yield information about the global consequences of the presence of a given singularity which stem from the hyperbolic nature of the underlying equations. It turns out that there are severe restrictions on the distribution of isolated singularities of an HP-net in a given domain, the exact form of which depends to a large extent on its shape—the less contorted the boundary the more difficult it is for there to be a lot of singularities. The ultimate manifestation of this phenomenon (that is, in the case in which there is no boundary) is contained in one of the main results: in Section 4 we show that an HP-net regular in the entire plane except for isolated singularities can have at most two singularities and that every such net belongs to one of five families, each of which is described by a few parameters (see Theorem 4.1). The small number of possibilities for such HP-nets is most consistent with the analogy we have been pursuing—a harmonic function whose conjugate is bounded and regular in the whole plane except for isolated singularities must be a constant. In a somewhat different direction, we use the relationship between HP-nets and cps-mappings to show in Section 5 that an HP-function regular except for isolated singularities in a smoothly bounded Jordan domain D possesses nontangential limits at almost all points of ∂D. This result closely parallels the classical Fatou theorem (see [Pri]) on the boundary behavior of bounded harmonic functions (and their conjugates). This section also contains a construction which shows that, in spite of the aforementioned limitations on the distribution of isolated singularities, on any such domain there are cps-mappings having infinitely many of them (Theorem 5.2).

Finally, deformations with constant principal strains are of concrete interest in connection with models of real situations, several of which are briefly discussed in [Y]. Consider, for example, a thin liquid film on a plane surface which upon solidification takes on a cryptocrystalline structure, that is, at each point a suitably oriented infinitesimal square of the original liquid becomes an (again, suitably oriented infinitesimal) rectangular crystal whose side lengths are constant multiples of the side length of the square. In this light global geometric results for cps-mappings acquire new significance in as much as they tell one about the extent to which the shape of the original film can change as a result of such a solidification process. Furthermore, one can interpret isolated singularities in this context as microscopic flaws in the crystallized lamina, as is explained in some detail in the final paragraph of Section 5.

1. Preliminaries

1.1. Notation and terminology. For convenience we treat the plane as $\mathbf{C}$, rather than as $\mathbf{R}^2$, and denote planar vectors as complex numbers. Let $D \subset \mathbf{C}$ be a domain and let θ be a locally Lipschitz continuous real-valued function on D. The complete integral curves of the fields $e^{i\theta}$ and $ie^{i\theta}$ will be called 1- and 2-characteristics of θ, respectively. The convention that $\{i,j\} = \{1,2\}$ will hold throughout. Arcs of i-characteristics will be called i-arcs, or less specifically, characteristic arcs. With reference to a given θ a characteristic arc joining points

$a, b \in D$ will be denoted by ab and we shall use the abbreviation

$$\Delta\theta(ab) = \theta(b) - \theta(a).$$

A domain $Q \subset D$ will be said to be a characteristic quadrilateral of θ if ∂D is a Jordan curve lying in D containing four points a, b, c, d occurring in that order when ∂D is traversed (in either the positive or negative sense) and such that ab and cd are i-arcs and bc and da are j-arcs. We will refer to such a Q as $abcd$ and use the abbreviation

$$\Delta^2(abcd) = \Delta\theta(bc) - \Delta\theta(ad) = \Delta\theta(dc) - \Delta\theta(ab).$$

Furthermore, D_1 and D_2 will denote differentiation with respect to arc length in the directions $e^{i\theta}$ and $ie^{i\theta}$, respectively; that is, for differentiable u

$$\begin{aligned} D_1 u(z) &= \cos\bigl(\theta(z)\bigr) u_x(z) + \sin\bigl(\theta(z)\bigr) u_y(z), \\ D_2 u(z) &= -\sin\bigl(\theta(z)\bigr) u_x(z) + \cos\bigl(\theta(z)\bigr) u_y(z). \end{aligned}$$

We use the symbol $\lambda(E)$ to denote the 1-dimensional measure of the set E, so that in particular $\lambda(C)$ is the length of the simple arc C.

1.2. Definition and basic properties of HP-nets and cps-mappings. In dealing with HP-nets it is frequently more convenient to work with the function θ which gives the inclination of the tangent to the curves belonging to one or the other of the two families that make up the net, rather than with the net itself. Since, however, we shall be working with domains that are not simply connected, an inevitable contingency in any discussion of singularities, a minor complication arises; namely, that upon going around a hole θ may change its value by a multiple of π, (*not* 2π). Thus, in the following definition we consider functions which, although not necessarily single-valued in the entire domain D under consideration, do have a single-valued branch on any simply-connected subdomain of D.

Definition 1.1. Let $D \subset \mathbf{C}$ be a domain. A (possibly multivalued) function θ on D is called an *HP-function* if it satisfies the following conditions:

(i) Every point p in D has a neighborhood E on which θ has a Lipschitz continuous branch and satisfies $\Delta^2(abcd) = 0$ for all characteristic quadrilaterals $abcd$ of θ contained in E.

(ii) $e^{2i\theta}$ is single-valued in D.

The set of all HP-functions on D will be denoted by $\mathrm{HP}(D)$.

It is necessary to consider $e^{2i\theta}$ in (ii), rather than $e^{i\theta}$, since, as noted above, in going around a hole in D, θ might change by a multiple of π. We will use the term *HP-net* to refer to the families of integral curves of the fields $e^{i\theta}$ and $ie^{i\theta}$; through each point of D there passes exactly one curve of each family. It is evident that the corresponding net remains unchanged if we add $\frac{1}{2}\pi$ to an HP-function;

this causes, of course, an interchange of the two families of characteristics. We use the notation $\mathrm{HP}(D)$ to denote the set of all HP-nets, as well as the set of all HP-functions, on D; this minor ambiguity will cause no confusion.

To discuss cps-mappings we need the following notation:

$$T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad S(m_1, m_2) = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

where here and in what follows m_1 and m_2 are distinct positive numbers.

Definition 1.2. A mapping of a domain $D \subset \mathbf{C}$ into $\mathbf{C}$ is called an (m_1, m_2)-*mapping* if each point of D has a neighborhood N in which there are two Lipschitz continuous functions $\theta = \theta_f$ and $\phi = \phi_f$ such that the Jacobian matrix J_f of f is given by $T(-\phi)S(m_1, m_2)T(\theta)$.

If θ and ϕ are Lipschitz continuous in a simply-connected domain D, then $T(-\phi)S(m_1, m_2)T(\theta)$ is the Jacobian matrix of a mapping if and only if

$$D_1(m_1\theta - m_2\phi) = 0 \quad \text{and} \quad D_2(m_1\phi - m_2\theta) = 0 \quad \text{a.e. in } D, \tag{1.1}$$

or in other words

$$m_i\theta - m_j\phi \text{ is constant along } i\text{-arcs of } \theta. \tag{1.2}$$

To see this, it is enough to show, in light of the Lipschitz continuity of θ and ϕ, that these conditions simply amount to the formal necessary and sufficient compatibility conditions on the entries of a matrix in order for it to be the Jacobian of a mapping. For a given fixed $p \in D$, let $\theta_0 = \theta(p)$ and $\phi_0 = \phi(p)$, and let $\bar{D}_1 u$, $\bar{D}_2 u$ denote the directional derivatives of u at p in the directions $e^{i\theta_0}$ and $ie^{i\theta_0}$, respectively. Then functions A and B give $\bar{D}_1 u$ and $\bar{D}_2 u$ if and only if $\bar{D}_2 A = \bar{D}_1 B$. (Note that at p $\bar{D}_1\bar{D}_2 u$ is not the same as $D_1 D_2 u$ since the latter involves derivatives of θ and the former does not. The compatibility conditions can, of course, be formulated in terms of $D_1 D_2 u$ and $D_2 D_1 u$—see (1.4) below—and we shall make subsequent use of that formulation also.) Let $f = u + iv$. Then $J_f = T(-\phi)S(m_1, m_2)T(\theta)$ is equivalent to

$$\begin{bmatrix} \bar{D}_1 u & \bar{D}_2 u \\ \bar{D}_1 v & \bar{D}_2 v \end{bmatrix} = T(-\phi)S(m_1, m_2)T(\theta)T(-\theta_0) = T(-\phi)S(m_1, m_2)T(\theta - \theta_0).$$

The compatibility conditions for this matrix are equivalent to those for any left multiple by a constant invertible matrix, a convenient choice in this case being $T(\phi_0)$. If we write $\bar{\theta} = \theta - \theta_0$ and $\bar{\phi} = \phi - \phi_0$ then the compatibility conditions simply state that the result of applying $\bar{D}_1$ to the second column of

$T(-\bar\phi)S(m_1,m_2)T(\bar\theta)$ is the same as that of applying $\overline{D}_2$ to the first column. Since $\bar\theta$ and $\bar\phi$ are both 0 at p and

$$T(-\bar\phi)S(m_1,m_2)T(\bar\theta) = \begin{bmatrix} m_1\cos\bar\phi\,\cos\bar\theta + m_2\sin\bar\phi\sin\bar\theta & m_1\cos\bar\phi\sin\bar\theta - m_2\sin\bar\phi\cos\bar\theta \\ m_1\sin\bar\phi\cos\bar\theta - m_2\cos\bar\phi\,\sin\bar\theta & m_1\sin\bar\phi\sin\bar\theta + m_2\cos\bar\phi\cos\bar\theta \end{bmatrix},$$

a trivial calculation shows that at p, and therefore at any point, the compatibility equations take the form (1.1) as desired.

Equations (1.1) constitute the nonlinear hyperbolic system alluded to at the beginning of the fourth paragraph of the introduction. We next have

Proposition 1.1. *Let D be a simply connected domain and m_1, m_2 distinct positive numbers. Then $\theta = \theta_f$ for some (m_1,m_2)-mapping f of D if and only if θ is an HP-function on D.*

Proof. Let θ and ϕ be the functions associated with an (m_1,m_2)-mapping f of D and let $abcd \subset D$ be a characteristic quadrilateral with 1-sides ab, dc. Then from (1.2) we have

$$\Delta^2\phi(abcd) = \Delta\phi(dc) - \Delta\phi(ab) = \frac{m_1}{m_2}\big(\Delta\theta(dc) - \Delta\theta(ab)\big) = \frac{m_1}{m_2}\Delta^2\theta(abcd).$$

But if we write $\Delta^2\phi(abcd)$ as $\Delta\phi(bc)-\Delta\phi(ad)$, we see that $\Delta^2\phi(abcd)$ also equals $(m_2/m_1)\Delta^2\theta(abcd)$, so that indeed $\Delta^2\theta(abcd) = 0$.

Conversely, given that θ is an HP-function, let Q be a closed characteristic quadrilateral in D and let p be an interior point of Q. Then, since $\Delta^2\theta(Q') = 0$ for all characteristic quadrilaterals $Q' \subset Q$, it is clear that once $\phi(p) = \phi_0$ has been assigned, there is a unique ϕ in Q which satisfies (1.2). This ϕ can then be extended bit by bit "from one characteristic quadrilateral to the next" to all of D so that (1.2) holds, that the resulting ϕ is single-valued follows from the simple connectedness of D via the monodromy principle. By what was established above the matrix $T(-\phi)S(m_1,m_2)T(\theta)$ is the Jacobian of an (m_1,m_2)-mapping of D, as desired. □

It is clear that given an (m_1,m_2)-mapping on a simply connected domain D, the (continuous) HP-function $\theta=\theta_f$ is uniquely determined to within an additive constant (which is a multiple of π), and that all (m_1,m_2)-mappings g of D for which $\theta_g = \theta_f$ are of the form $e^{i\alpha}f + z_0$, $\alpha\in\mathbf{R}$, $z_0\in\mathbf{C}$. The i-characteristics are the curves along which f changes arc length by a factor of m_i.

For a function $\theta\in C^2(D)$ straightforward formal calculation shows that each of the equations

$$D_jD_i\theta = (-1)^j(D_i\theta)^2 \tag{1.3}$$

when written in terms of differentiation in the x and y directions takes the form

$$\tfrac{1}{2}(\sin 2\theta)(\theta_{yy} - \theta_{xx}) + (\cos 2\theta)\theta_{xy} = (\cos 2\theta)(\theta_x^2 - \theta_y^2) - 2(\sin 2\theta)\theta_x\theta_y,$$

so that each case of (1.3) implies the other. The meaning of equations (1.3) can be expressed in the following geometric form. If $\kappa_i(z)$ denotes the unsigned curvature of the i-characteristic $C_i(z)$ through z, then the derivative of κ_i in the direction orthogonal to $C_i(z)$ towards its concave side is κ_i^2. Of key importance in what is to follow is the fact that the solution of the equation $\kappa'(t) = \kappa^2(t)$, with $\kappa(0) = \kappa_0$ is $\kappa(t) = \kappa_0/(1-\kappa_0 t)$. Equations (1.3) consequently imply that if $\kappa_i \neq 0$ it increases as we move along $C_j(z)$ towards the concave side of $C_i(z)$ and decreases as we move along $C_j(z)$ in the opposite direction. In particular the length of any j-arc emanating from z towards the concave side of $C_i(z)$ is at most $1/\kappa_i(z)$.

Proposition 1.2. *Let D be a simply connected domain and $\theta \in C^2(D)$. Then θ is an HP-function if and only if equations* (1.3) *hold on D.*

Proof. Let $\theta \in C^2(D)$. Straightforward calculations show that functions $A, B \in C^1(D)$ are of the form $A = D_1 u$ and $B = D_2 u$ for some $u \in C^2(D)$ if and only if

$$D_2 A - D_1 B = A D_1\theta + B D_2\theta \tag{1.4}$$

holds. That is, the compatibility conditions may be expressed in this form (see the discussion following (1.2)). Assume that θ is an HP-function and let m_1, m_2 be distinct positive numbers. Let ϕ be a function (whose existence was established in the preceding proposition) for which (1.1) holds. Applying (1.4) first with $A = D_1\theta$, $B = D_2\theta$ and then with

$$A = D_1\phi = \frac{m_1}{m_2} D_1\theta, \qquad B = D_2\phi = \frac{m_2}{m_1} D_2\theta,$$

we have

$$D_2 D_1\theta - D_1 D_2\theta = (D_1\theta)^2 + (D_2\theta)^2, \tag{1.5}$$

and

$$\frac{m_1}{m_2} D_2 D_1\theta - \frac{m_2}{m_1} D_1 D_2\theta = \frac{m_1}{m_2}(D_1\theta)^2 + \frac{m_2}{m_1}(D_2\theta)^2, \tag{1.6}$$

from which the equations (1.3) follow immediately.

Conversely, let $\theta \in C^2(D)$ satisfy (1.3). We define

$$P = \frac{m_1}{m_2} D_1\theta \qquad \text{and} \qquad Q = \frac{m_2}{m_1} D_2\theta.$$

Then it follows from (1.3) that

$$D_2 P - D_1 Q = P D_1\theta + Q D_2\theta,$$

so that there is a function ϕ satisfying $D_1\phi = P$ and $D_2\phi = Q$; that is, the function ϕ satisfies (1.1). Thus θ is an HP-function by the preceding proposition. ▫

In order to proceed with the development of the local analytic aspects of the theory of HP-nets and cps-mappings, and to have an important tool for the construction of such we need to introduce characteristic coordinate mappings. Let θ be an HP-function on a domain D. Let $I_i = [a_i, b_i]$, $\tau_i \in I_i$, $i = 1, 2$. Let $p \in D$ and let $z = z_i(s)$, $s \in I_i$ be an arc length parametrization of the i-arc through p with $z_i(\tau_i) = p$, $i = 1, 2$. For $(t_1, t_2) \in S = I_1 \times I_2$ let $\zeta(t_1, t_2)$ be the point common to the 1-characteristic through $z_2(t_2)$ and the 2-characteristic through $z_1(t_1)$. For $\lambda(I_1)$ and $\lambda(I_2)$ sufficiently small $\zeta\colon S \to D$ is a bi-Lipschitz homeomorphism, as follows from the Lipschitz continuity of θ and simple facts about the dependence of solutions of ordinary differential equations on initial values. Without loss of generality we can assume that $|\theta(z)| < \frac{1}{4}\pi$ in the corresponding characteristic quadrilateral $\zeta(S)$, so that in what follows all arguments lie in the interval $\left(-\frac{1}{4}\pi, \frac{3}{4}\pi\right)$. We write $\alpha_i(s) = \arg\{z_i'(s)\}$. That θ is an HP-function is equivalent to

$$\omega(t_1, t_2) = \theta\big(\zeta(t_1, t_2)\big) = \alpha_1(t_1) + \alpha_2(t_2) - \alpha_2(\tau_2). \tag{1.7}$$

Because θ is Lipschitz continuous, α_i is differentiable a.e. on I_i and α_i' is a bounded measurable function. If $\zeta = \xi + i\eta$, then the functions ξ, η satisfy the system

$$\xi_{t_1} \sin\omega - \eta_{t_1} \cos\omega = 0; \qquad \xi_{t_2} \cos\omega + \eta_{t_2} \sin\omega = 0. \tag{1.8}$$

Writing

$$v = -\xi \sin\omega + \eta \cos\omega \qquad \text{and} \qquad u = \xi \cos\omega + \eta \sin\omega,$$

that is, $u + iv = \zeta e^{-i\omega}$, the system (1.8) takes the form

$$u_{t_2} = \alpha_2'(t_2) v; \qquad v_{t_1} = -\alpha_1'(t_1) u. \tag{1.9}$$

This is a very simple hyperbolic system which becomes even more transparent when expressed in integral form

$$v(t_1, t_2) = v_0(t_2) - \int_{\tau_1}^{t_1} u(\tau, t_2)\alpha_1'(\tau)\, d\tau,$$
$$u(t_1, t_2) = u_0(t_1) + \int_{\tau_2}^{t_2} v(t_1, \tau)\alpha_2'(\tau)\, d\tau,$$

where

$$\begin{aligned} u_0(t_1) &= \mathrm{Re}\{z_1(t_1) e^{-i\alpha_1(t_1)}\}, \\ v_0(t_2) &= \mathrm{Im}\{z_2(t_2) e^{-i(\alpha_2(t_2) + \alpha_1(\tau_1) - \alpha_2(\tau_2))}\}. \end{aligned} \tag{1.10}$$

Clearly, u_0 and v_0 are Lipschitz continuous.

A straightforward and standard argument based on iteration shows that given continuous u_0 and v_0 this system of integral equations has a global solution $u, v \in C(S)$ which consequently satisfies (1.9) almost everywhere; the solution is, moreover, unique. Furthermore, if the initial data as well as the functions α_1, α_2 are C^∞, then the solution is likewise C^∞ on S; this standard regularity result stems from the fact that the derivatives of u and v satisfy a system of the same general form. Consequently, if we start with C^∞ arc length parametrizations $z = z_i(s)$, $s \in I_i$ with

$$z_1(\tau_1) = z_2(\tau_2) \qquad \text{and} \qquad z_2'(\tau_2) = i z_1'(\tau_1), \tag{1.11}$$

then we will obtain a C^∞ mapping $\zeta\colon S \to \mathbf{C}$. It is easy to see that if this mapping is one-to-one, then the images of the lines $t_i =$const form an HP-net on $\zeta(S)$; that is, that $\theta(z) = \omega\big(\zeta^{-1}(z)\big)$ will be an HP-function. In general, of course, ζ will not even be locally one-to-one, because the images of lines $t_i =$const can cross. However, if we are given an upper bound K for the curvatures of the initial curves $z_i(I_i)$, then equations (1.3) allow one to deduce that there exist $\delta = \delta(K)$ and $L = L(K)$ such that ζ will be one-to-one in $N = [\tau_1 - \delta, \tau_1 - \delta] \times [\tau_2 - \delta, \tau_2 - \delta] \cap S$, and that the corresponding θ will satisfy a Lipschitz condition with constant L in $\zeta(N)$. From this, via a simple approximation procedure and a compactness argument, one can show that the same is true if one only assumes that the functions $\arg z_i$ satisfy Lipschitz conditions with constant K (instead of the initial curves being C^∞ with curvatures bounded by this constant). Summarizing, we have the following

Proposition 1.3. *If $z = z_i(s)$, $s \in I_i$, $i = 1, 2$ are arc length parametrizations for which the $\arg\{z_i'(s)\}$ are Lipschitz continuous and satisfy* (1.11)*, then there is some neighborhood N of (τ_1, τ_2) in S on which ζ is one-to-one and such that $\theta(z) = \omega\big(\zeta^{-1}(z)\big)$ is an HP-function on $\zeta(N)$.*

If ζ is one-to-one on all of S, as will be the case when the $C_i = z_i(I_i)$ are adjacent sides of a characteristic quadrilateral of an already existing HP-function, then we shall refer to the curves $\zeta(I_1 \times \{t\})$, $t \in I_2$ as *translates* of C_1 along each $\zeta(\{t\} \times I_2)$, $t \in I_1$ and call them *parallel* arcs, and analogously when the roles of the indices 1, 2 are reversed. We shall also refer to C_i and C_j as being *perpendicular* or *orthogonal* to each other. We shall refer to the uniquely defined net given by $\theta(z) = \omega\big(\zeta^{-1}(z)\big)$ in the image of any neighborhood of (τ_1, τ_2) in which ζ is one-to-one as $\mathrm{HP}(C_1, C_2)$.

Proposition 1.4. *If, in addition to the hypotheses of Proposition* 1.3, *we assume that $\arg\{z_1'(s)\}$ is nonincreasing on $I_1^+ = \{t \in I_1 : t \geq \tau_1\}$ and $\arg\{z_2'(s)\}$ is nondecreasing on $I_2^+ = \{t \in I_2 : t \geq \tau_2\}$, then ζ is locally one-to-one on $I_1^+ \times I_2^+$ and $\theta(z) = \omega\big(\zeta^{-1}(z)\big)$ is an HP-function on $\zeta(J)$ for any open $J \subset I_1^+ \times I_2^+$ on which ζ is one-to-one.*

Proof. The hypotheses imply that $z_1(I_1^+)$ is convex toward its left-hand side (i.e., toward the side corresponding to increasing t_2) and that $z_2(I_2^+)$ is convex toward its right-hand side (i.e., toward the side corresponding to increasing t_1). In the C^∞ case this means, in light of the significance of equations (1.3) (see the paragraph immediately preceding Proposition 1.2), that both families of characteristics diverge, that is, that the curvatures of the i-characteristics decrease with increasing t_j. From this the locally one-to-one character of ζ follows immediately. (The mapping might fail to be globally one-to-one due to the possibility that ζ might not give a simple covering of $\zeta(I_1^+ \times I_2^+)$.) The general case follows by approximating the z_i by sequences $\{z_{i,n}(s)\}$ of C^∞ arc length parametrizations for which the $\arg\{z'_{i,n}(s)\}$ have the stipulated monotonicity as well as uniformly bounded derivatives which tend to $d\arg\{z'_i(s)\}/ds$ in measure. □

Next, we explain the sense in which equations (1.3) hold for general (i.e., not necessarily C^2) HP-functions θ. We define $E_i = E_i(\theta)$ to be the set of all points p such that if $z = z(s)$, $-\varepsilon < s < \varepsilon$, with $z(0) = p$ is an arc length parametrization of an i-arc of θ containing p, then $\theta\big(z(s)\big)$ is differentiable at $s = 0$. Obviously, almost all points (with respect to arc length) of each i-characteristic belong to E_i and almost all points of the domain on which θ is defined (with respect to 2-dimensional measure) belong to $E_1 \cap E_2$.

Proposition 1.5. *Let θ be an HP-function on D and let C_k, $k = 1, 2$, be the k-characteristic through $p \in E_i$. Then $C_j \subset E_i$, and equation* (1.3) *holds along C_j, when D_j is iterpreted as arc length differentiation along C_k in the direction $i^{k-1}e^{i\theta}$, $k = 1, 2$.*

Proof. Without loss of generality we assume, for definiteness, that $p \in E_1$. Let $z = z_i(s)$, $-\alpha \le s \le \alpha$, $i = 1, 2$ be arc length parametrizations of small pieces of C_1 and C_2 with $z_i(0) = p$ and with the directions of increasing s correspond to $e^{i\theta}$ and $ie^{i\theta}$, respectively. Let $\kappa_i(s) = d\arg\{z'_i(s)\}/ds$. Let ζ be the corresponding characteristic coordinate mapping and let $F_i(t_1, t_2)$ denote the translate of $z_i([0, t_i])$ along C_j from p to $z_j(t_j)$. Since $\kappa_0/(1 - t\kappa_0)$ is the solution of the initial value problem $\kappa' = \kappa^2$, $\kappa(0) = \kappa_0$, it is clearly enough to show that for each $t \in (0, \alpha]$

$$\lim_{s \to 0^+} \frac{\theta\big(\zeta(s,t)\big) - \theta\big(\zeta(0,t)\big)}{\lambda\big(F_1(s,t)\big)} = \frac{\kappa_1(0)}{1 - t\kappa_1(0)}. \tag{1.12}$$

That (1.12) also holds for negative t and as $s \to 0^-$, can be deduced with minor notational adjustments to the argument to follow. Because of trivial compactness considerations the size of $\alpha > 0$ is not important, so that we may assume that

$$\sup\{|\kappa_i(s)| : i = 1, 2,\ |s| \le \alpha\} \le \frac{1}{100\alpha}. \tag{1.13}$$

Here "sup" is to be interpreted as "essential supremum". This condition implies that the $F_i(t_1, t_2)$ are very close to being straight line segments, and in addition that if we start with $C^\infty z_i$ satisfying it, then the corresponding characteristic coordinate mapping will be one-to-one in $[-\alpha, \alpha] \times [-\alpha, \alpha]$. We fix such an α. Very simple estimates show that

$$\lim_{s \to 0} \lambda\big(F_2(s,t)\big) = t, \qquad 0 < t \leq \alpha,$$

uniformly over the class of all ζ arising from $C^\infty z_i$ satisfying (1.13).

The equation $D_2 D_1 \theta = (D_1)^2$, expressed in terms of the radius of curvature $R_1 = 1/\kappa_1$, says that $D_2 R_1 = -1$. Now, for such C^∞ initial curves, a simple calculus argument involving an appropriate Riemann sum and passage to the limiting integral, together with this differential equation for R_1 shows that

$$\lambda\big(F_1(s,t)\big) = \int_0^s 1 - \kappa(\sigma)\lambda\big(F_2(\sigma,t)\big)\, d\sigma,$$

so that

$$\begin{aligned} \lambda\big(F_1(s,t)\big) &= \int_0^s 1 - \kappa(\sigma)\big(t + o(1)\big)\, d\sigma = \int_0^s 1 - \kappa(\sigma) t\, d\sigma + o(s) \\ &= s - t\Big(\theta\big(z_1(s)\big) - \theta\big(z_1(0)\big)\Big) + o(s), \end{aligned}$$

where the "little-o" is uniform over the entire C^∞ class indicated above. From the HP-property we have that

$$\theta\big(\zeta(s,t)\big) - \theta\big(\zeta(0,t)\big) = \theta\big(z_1(s)\big) - \theta\big(z_1(0)\big).$$

Abbreviating this difference by Δ we see that the difference quotient on the left-hand side of (1.12) is equal to

$$\frac{\Delta}{s - t\Delta + o(s)} = \frac{\Delta/s}{1 - t\Delta/s + o(1)}.$$

By approximation by C^∞ functions as in the proof of the preceding proposition we have that the same holds for the original HP-net. But $\Delta/s \to \kappa_1(0)$, so that (1.12) is indeed true. □

Because of Proposition 1.5 the comments contained in the paragraph immediately preceding Proposition 1.2 are relevant in the context of general (i.e., not necessarily C^2) HP-nets and their content will play a key role in much of what follows. For convenience we formulate the next proposition, which gives a lower bound for the area of a characteristic quadrilateral, in terms of the characteristic coordinate mapping ζ discussed above.

Proposition 1.6. *Let the mapping ζ be one-to-one on all of $I_1 \times I_2$, where $I_i = [\tau_i, \sigma_i]$. If the lengths of all of the translates of $C_2 = z_2(I_2)$ along $C_1 = z_1(I_1)$ are at least m, then the area of $\zeta(I_1 \times I_2)$ is at least $\frac{1}{2} m\lambda(C_1)$.*

Proof. Again, by a straightforward approximation procedure we can reduce consideration to the case in which the arc length parametrizations z_1, z_2 are C^∞. Consider a small subarc $J = z_1([\sigma, \sigma+\delta])$ of C_1. The length of J is obviously δ and the length of its translate t units along $\zeta(\{\sigma\} \times I_2)$ is easily found (see the proof of the preceding proposition) to be $\big(1 - t\kappa_1(\sigma)\big)\delta + O(\delta^2)$, where $\kappa_1(\sigma) = d \arg z_1(\sigma)/d\sigma$. For small δ the translates of J are virtually straight line segments orthogonal to the curve $\zeta(\{\sigma\} \times I_2)$. Since the lengths of the 2-arcs of the characteristic quadrilateral in question are all at least m, equations (1.3) imply that $\kappa_1(\sigma) \le 1/m$, so that the area of $\zeta\{[\sigma, \sigma+\delta] \times I_2\}$ is then seen to be at least

$$\delta \int_0^m \Big(1 - \frac{t}{m}\Big)\, dt + O(\delta^2) = \frac{\delta m}{2} + O(\delta^2),$$

so that upon considering the appropriate Riemann sum and passing to the limiting integral, the area of $\zeta(I_1 \times I_2)$ is indeed at least $\frac{1}{2} m\lambda(C_1)$. □

We define $D_i^+\theta(p)$ to be the upper limit of $|D_i\theta(z)|$ as $z \to p$. We have the following simple consequence of Proposition 1.5.

Proposition 1.7. *Let θ be an HP-function on D.* (i) *If the j-characteristic C through p is a simple curve, then*

$$D_i^+\theta(p) \le \frac{1}{L} \le \frac{1}{\operatorname{dist}(p, \partial D)},$$

where L is the length of the shorter of the two arcs into which p divides C.

(ii) *If the j-characteristic through p is a closed curve then $D_i^+\theta(p) = 0$.*

Proof. We prove (i), the proof of (ii) involving only minor variations. For definiteness and without loss of generality we assume that $i = 1$. We can also assume that $D_i^+\theta(p) > 0$, since otherwise there is nothing to prove. From the definition of $D_i^+\theta(p)$ it follows that for any $\varepsilon > 0$ there is a point $q \in E_1(\theta)$ within ε of p for which $|D_1\theta(q)| > D_i^+\theta(p) - \varepsilon$ and such that there are 2-arcs C_+ and C_- emanating from q in the directions $ie^{i\theta(q)}$ and $-ie^{i\theta(q)}$, respectively, which have length greater than $L - \varepsilon$. Assume that $D_1\theta(q)$ is positive. Then applying Proposition 1.5 at the point $z(s)$ which lies s units from q along C_+ we have that

$$D_1\theta\big(z(s)\big) = \frac{D_1\theta(q)}{1 - sD_1\theta(q)}.$$

Thus

$$L - \varepsilon < \lambda(C_+) \le \frac{1}{D_1\theta(q)} < \frac{1}{D_i^+\theta(p) - \varepsilon};$$

that is, that $D_i^+\theta(p) - \varepsilon < 1/(L-\varepsilon)$, which establishes the desired bound. If $D_1\theta(q) < 0$ then one arrives at the same conclusion by following C_- instead of C_+. □

As an immediate corollary of Proposition 1.7 we have

Proposition 1.8. *If θ is an HP-function on all of $\mathbf{C}$, then $D_i^+\theta(p) = 0$ for all $p \in \mathbf{C}$, $i = 1, 2$, so that θ is a constant.*

Proposition 1.9 (Compactness principle). *For any domain D the family $\{e^{2i\theta} : \theta \in \mathrm{HP}(D)\}$ is compact in the topology of uniform convergence on compact subsets of D.*

Proof. This follows via elementary arguments, since Proposition 1.7 implies that if $U \subset D$ is a closed disk, then θ satisfies a Lipschitz condition with constant at most $1/\operatorname{dist}(U, \partial D)$. □

We end this subsection with the discussion of an important limiting case of the characteristic coordinate mapping construction we have been using, namely that in which one of the initial curves degenerates to a point. Let $z = z(s)$, $s \in I_1 = [0, \tau_1]$ be an arc length parametrization with Lipschitz continuous derivative, and let I_2 be an interval one of whose endpoints is 0 and whose length is less than 2π. Let $\omega(t_1, t_2) = \arg\{z'(t_1)\} + t_2$. We consider the same system (1.8) of differential equations as before, but with initial conditions corresponding to $\zeta(t_1, 0) = z(t_1)$ and $\zeta(0, t_2) = z(0)$, that is, for the functions u, v defined by $u + iv = \zeta e^{-i\omega}$, the equations are

$$u_{t_2} = v \qquad \text{and} \qquad v_{t_1} = -\alpha'(t_1)u,$$

where $\alpha(t) = \arg\{z_1(t)\}$ with the corresponding initial conditions

$$u_0(t_1) = \operatorname{Re}\{z(t_1)e^{-i\alpha(t_1)}\} \qquad \text{and} \qquad v_0(t_2) = \operatorname{Im}\{z(0)e^{-i\alpha(0)-t_2}\}.$$

This characteristic initial value problem, in light of the discussion preceding Proposition 1.4, is well-posed.

Proposition 1.10. *The mapping ζ defined immediately above exists on $I_1 \times I_2$ and is one-to-one on $J = [0, \varepsilon] \times I_2$ for some $\varepsilon > 0$. Moreover, the function $\theta(z) = \omega\bigl(\zeta^{-1}(z)\bigr)$ is an HP-function on the interior of $\zeta(J)$.*

Proof. This may be proved in a fashion directly analogous to that in which Proposition 1.2 was justified, or alternatively by applying the compactness principle to the family of HP-functions resulting from the original (i.e., nondegenerate) characteristic coordinate mapping construction with $z_1(s) = z(s)$, $s \in I_1$ and

$$z_2(s) = z(0) + \delta(e^{is/\delta} - 1)z'(0), \qquad s \in \delta I_2 = \{\delta s : s \in I_2\} = I_2(\delta).$$

The curve given by z_2 is an arc of a circle of radius δ orthogonal to $z(I_1)$ at $z(0)$ for which $z'(0)$ is an outward pointing normal. One then lets $\delta \to 0$ and obtains the desired result by the compactness principle together with the convexity of the curve $z_2\bigl(I_2(\delta)\bigr)$. □

It is to be noted that the characteristic arcs $\zeta(\{t_1\}\times I_2)$, $t_1\in(0,\varepsilon]$ are convex (with their concave side towards $z(0)$). The family of orthogonal arcs $\zeta((0,\varepsilon]\times\{t_2\})$, $t_2\in I_2$ is a *fan* of characteristic arcs which are confluent at $z(0)$. If ζ is one-to-one in all of the rectangle, and the curve parametrized by z is C, then we will denote the resulting uniquely defined net by $\mathrm{Fan}(C,I_2)$.

1.3. Isolated singularities of HP-functions and cps-mappings. Henceforth the r-neighborhood of a point $p\in\mathbf{C}$ will be denoted by $N(p,r)$, and $N(p,r)\backslash\{p\}$ will be denoted by $N'(p,r)$. If p is a point of the domain D, an HP-function θ on $D\backslash\{p\}$ is said to have an *isolated singularity* at p. The point p will be called a *true singularity* of θ if θ cannot be extended to an HP-function in D. We use the terms "singularity" and "true singularity" for HP-nets also. An HP-function (HP-net) on $D\backslash A$ for some set A of isolated points of D will be called an HP*-function (HP*-net) on D; $\mathrm{HP}^*(D)$ will denote both the class of HP*-functions and that of HP*-nets on D. Furthermore, we shall denote by $\mathrm{cps}^*(D)$ (by $\mathrm{cps}^*(D,m_1,m_2)$) the set of all cps-mappings ((m_1,m_2)-mappings) which are defined on a set of the form $D\backslash A$, where A is a set of isolated points of D, and whose continuous extensions to D are local homeomorphisms.

Proposition 1.11. *If θ has a true singularity at p, then the essential supremum of $\nabla\theta$ is not finite in $N'(p,\varepsilon)$, for any $\varepsilon>0$.*

Proof. If the essential supremum of $\nabla\theta$ is finite in some such punctured neighborhood $N'(p,\varepsilon)$, then θ is single valued and Lipschitz continuous there, and consequently can be extended by continuity to all of $N(p,\varepsilon)$ with the same Lipschitz constant. If $abcd$ is any characteristic quadrilateral whose closure lies in the punctured neighborhood, then $\Delta^2\theta(abcd)=0$. By a trivial limit argument it then follows that this is true even if p is on the boundary of the quadrilateral. If $abcd$ is a characteristic quadrilateral of θ which contains p in its interior, then there are points $a'\in ab$, $b'\in bc$, $c'\in cd$, and $d'\in da$ such that $a'c'$ and $b'd'$ are characteristic arcs of θ passing through p. But then

$$\Delta^2\theta(abcd)=\Delta^2\theta(a'bb'p)+\Delta^2\theta(pb'cc')+\Delta^2\theta(d'pc'd)+\Delta^2\theta(aa'pd')=0,$$

so that θ is an HP-function in $N(p,\varepsilon)$, contrary to the hypothesis. □

Proposition 1.12. *Let θ be an HP-function which has a true singularity. Then there is a characteristic arc of θ of finite length one of whose endpoints is p.*

Proof. Let θ be defined in $N'(p,\varepsilon)$. From the preceding proposition it follows that there are points $q\neq p$ in $N(p,\varepsilon/2)$ such that $D_i^+\theta(q)>2/\varepsilon$ for at least one of $i=1$ or 2. But then it follows from Proposition 1.7 that there is a j-arc of length at most $\varepsilon/2$ which joins q to p. □

Proposition 1.12 suggests the following classification of true singularities.

Definition 1.2. A true singularity of an HP-function θ is said to be a *singularity of type R* or a *Riemann singularity* if θ is bounded on some characteristic arc of finite length which terminates at p. Otherwise it is said to be of *type S* or a *spiral singularity.*

The reason we have chosen the names "Riemann" and "spiral" for the two kinds of singularities will be made clear in what follows (see (i) and (ii) in the following subsection, and Theorems 2.1 and 3.1). Here again, we apply these terms to HP-nets also.

1.4. HP-nets with one or two singularities. In this paragraph we define four families of HP-nets; one of the main goals of this paper is to prove that apart from the trivial nets corresponding to constant θ, these are the only nets in $\mathrm{HP}^*(\mathbf{C})$.

(i) *Spiral nets.* For $p \in \mathbf{C}$, $-\frac{1}{2}\pi \le \alpha \le \frac{1}{2}\pi$, we define

$$\sigma_{p,\alpha}(p + re^{i\phi}) = \phi + \alpha.$$

To see that $\theta = \sigma_{\rho,\alpha}$ is an HP-function, let $r = |z - p|$. Simple trigonometry shows that $D_1\theta(z) = \sin\alpha/r$ and $D_2 r = -\sin\alpha$, from which it follows that $D_2 D_1 \theta(z) = (\sin\alpha/r)^2 = \big(D_1\theta(z)\big)^2$, so that $\sigma_{\rho,\alpha}$ is indeed an HP-function by Proposition 1.2. It is equally easy to see that for $0 < |\alpha| < \frac{1}{2}\pi$ we have that $\sigma_{p,\alpha}(z) \to \pm\infty$ as $z \to p$ along any i-characteristic ($-\infty$, if $0 < \alpha$ and $i = 1$ or $\alpha < 0$ and $i = 2$; $+\infty$ otherwise), so that the characteristics spiral around p. Quite specifically, for $p = 0$ the polar equations of the 1- and 2-characteristics through the point $r_0 e^{i\theta_0}$ are

$$r = r_0 e^{(\phi - \theta_0)\cot\alpha} \quad \text{and} \quad r = r_0 e^{-(\phi-\theta_0)\tan\alpha}, \tag{1.14}$$

respectively. The values $\alpha = 0, \pm\frac{1}{2}\pi$ give rise to the degenerate case in which the families of 1-characteristics consist of rays emanating from p (when $\alpha = 0$) and circles centered at p (when $\alpha = \pm\frac{1}{2}\pi$). The net corresponding to $\sigma_{p,\alpha}$ will be denoted by $\mathscr{S}_{p,\alpha}$.

Let $f \in \mathrm{cps}^*(\mathbf{C}, m_1, m_2)$ for which $\theta_f = \sigma_{p,\alpha}$. Then the curves $f(C)$ are congruent for all i-characteristics C, from which it follows that $f\big(N(p,\delta)\big)$ is a disk $N\big(f(p),\delta'\big)$. Simple trigonometry implies that f changes arc length on circles $\partial N(p,\delta)$ by a factor of $\sqrt{m_1^2\sin^2\alpha + m_2^2\cos^2\alpha}$. Since f changes area by a factor of $m_1 m_2$ we have

$$m_1 m_2 \pi\delta^2 = \pi\Big(\delta\sqrt{m_1^2\sin^2\alpha + m_2^2\cos^2\alpha}\Big)^2,$$

so that if $m_1/m_2 = \mu$, we have

$$\mu\sin^2\alpha + \frac{1}{\mu}\cos^2\alpha = 1,$$

from which it follows that $\mu = \cot^2 \alpha$. Since $\mu \neq 0, 1$, we see that in order for f to exist, $|\alpha|$ must be in $(0, \frac{1}{2}\pi)\backslash\{\frac{1}{4}\pi\}$. The argument just given is easily seen to be reversible, that is, for all such α there is an (essentially unique) (m_1, m_2)-mapping for which $\theta_f = \sigma_{p,\alpha}$, provided that $m_1/m_2 = \cot^2 \alpha$.

If for $0 < |\alpha| < \frac{1}{2}\pi$ we follow the two characteristics of $\sigma_{p,\alpha}$ through $z_0 = p + re^{i\theta_0}$ as they move toward p, they cross infinitely often. For convenience let $0 < \alpha < \frac{1}{2}\pi$. Then a very simple calculation based on (1.14) shows that they meet for the first time at the point $a = p + z_0 e^{T(i - \tan\alpha)}$, where $T = 2\pi\cos^2\alpha$. Let C_i denote the i-arc joining z_0 to a. Arcs C_1 and C_2 together form a simple closed curve encircling p and are both convex toward the outside of this curve. The interior angles at z_0 and a are seen to be $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, respectively, so that if β_i is the (unsigned) change in θ along C_i we have

$$\beta_1 + \beta_2 = 2\pi. \tag{1.15}$$

(ii) *Riemann nets.* For $p \in \mathbf{C}$, α real and $0 \leq \beta \leq \pi$, we define

$$\rho_{p,\alpha,\beta}(p + re^{i\phi}) = \begin{cases} \alpha, & \alpha \leq \phi \leq \alpha + \frac{1}{2}\pi, \\ \phi - \frac{1}{2}\pi, & \alpha + \frac{1}{2}\pi \leq \phi \leq \alpha + \beta + \frac{1}{2}\pi, \\ \alpha + \beta, & \alpha + \beta + \frac{1}{2}\pi \leq \phi \leq \alpha + \beta + \pi, \\ \phi - \pi, & \alpha + \beta + \pi \leq \phi < \alpha + 2\pi, \end{cases}$$

and then define $\rho_{p,\alpha,\beta}(p + re^{i(\phi + 2\pi n)})$ to be $\rho_{p,\alpha,\beta}(p + re^{i\phi}) - n\pi$, for $n = \pm 1, \pm 2, \ldots$. It is easy to verify that the multivalued function $\rho_{p,\alpha,\beta}$ is indeed an HP-function in $\mathbf{C}\backslash\{p\}$; in each of the four sectors it coincides with one of the degenerate cases of $\sigma_{p,\alpha}$ or is constant, and it is continuous (modulo π). Quite specifically, in the four sectors on the right-hand side of the definition of $\rho_{p,\alpha,\beta}$ the 1-characteristics are, respectively, the rays $\{p + sie^{i\alpha} + te^{i\alpha} : t \geq 0\}$, $s \geq 0$, circular arcs with center at p, rays $\{p + sie^{i(\alpha+\beta)} - te^{i(\alpha+\beta)} : t \geq 0\}$, $s \geq 0$, and rays $\{p + te^{i\phi} : t \geq 0\}$, $\alpha + \beta + \pi \leq \phi \leq \alpha + 2\pi$. We also note that $\rho_{p,\alpha,\beta}$ increases by π along any simple closed curve which goes around p in the positive direction and that $\nabla\rho_{p,\alpha,\beta}$ has jumps along the rays that separate the four sectors (because along these rays characteristics of one or the other of the families change from straight lines to circular arcs). The net corresponding to $\rho_{p,\alpha,\beta}$ will be denoted by $\mathscr{R}_{p,\alpha,\beta}$. We have chosen the term "Riemann nets", because their restrictions to half-planes arise in connection with certain "Riemann problems" for the hyperbolic system (1.1). A trivial calculation based on (1.1) shows that $\rho_{p,\alpha,\beta}$ is θ_f for some $f \in \mathrm{cps}^*(\mathbf{C}, m_1, m_2)$ if and only if $(m_1/m_2)\beta + (m_2/m_1)(\pi - \beta) = \pi$ (since otherwise $f\big(N(p,\delta)\big)$ would not give a simple covering of a neighborhood of $f(p)$).

The remaining two nets are special cases of the following general construction; no simple formula for the corresponding HP-functions would appear to be available. Let $I_k = [0, t_k]$ and $z_k \colon I_k \to \mathbf{C}$ be arc length parametrizations of the curves C_k with Lipschitz continuous derivatives for which $z_1(0) = z_2(0)$ and

$z_1(t_1) = z_2(t_2)$. Assume furthermore that $\arg\{z_1'(s)\}$ and $\arg\{z_2'(s)\}$ are nonincreasing and nondecreasing, respectively, that $z_1(0)$, $z_1(t_1)$ are the only two points that these curves have in common, and that

$$z_2'(0) = iz_1'(0) \qquad \text{and} \qquad z_2'(t_2) = iz_1'(t_1).$$

One sees that the simple closed curve $C_1 \cup C_2$ is the boundary of a "heart-shaped" domain whose "point" is at $z_1(t_1) = z_2(t_2)$. Now apply the characteristic coordinate construction of Section 1.2. By Proposition 1.4 and simple geometry it follows that the characteristic coordinate mapping is one-to-one on $[0,t_1) \times [0,t_2)$. Let $E(C_1,C_2) = \zeta\big([0,t_1) \times [0,t_2)\big)$, $Z(C_1) = \zeta([0,t_1] \times \{t_2\})$, $Z(C_2) = \zeta(\{t_1\} \times [0,t_1])$. From the convexity of the original curves C_1 and C_2 together with the assumption that they meet at right angles at both endpoints, it easily follows that $E(C_1,C_2)$ lies in the complement of the interior of the simple closed curve $C_1 \cup C_2$ and that $Z(C_1)$ and $Z(C_2)$ satisfy exactly the same conditions as the original curves C_1 and C_2 did. We inductively define $C_i^{(0)} = C_i$, $C_i^{(k+1)} = Z(C_i^{(k)})$, $i = 1,2$, $k = 0,1,2,\ldots$. It is then easy to see that the HP-nets so defined in the interiors of the $E(C_1^{(k)},C_2^{(k)})$, $k = 0,1,2,\ldots$, fit together to form a single HP-net in the interior of their union, that is, in the doubly connected domain which constitutes the exterior of the original simple closed curve $C_1 \cup C_2$.

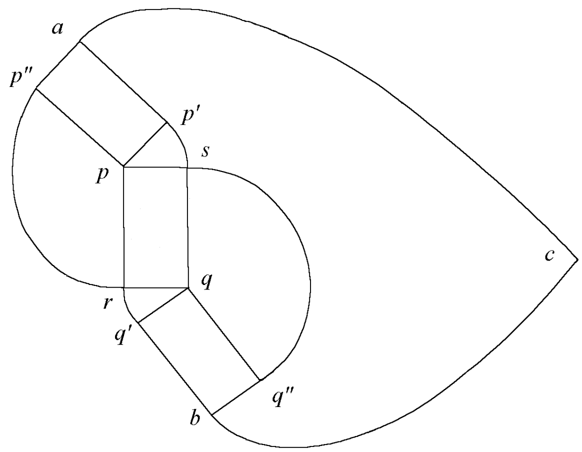

Figure 1.

(iii) *Double Riemann nets.* We shall define nets $\mathscr{D}_{p,q,\alpha,\beta,\gamma}$ where $p \neq q$ are points, $0 \leq \beta$, $\gamma \leq \pi$, α real, and $0 < \arg\big(e^{-i\alpha}(q-p)\big) < \frac{1}{2}\pi$. They are called double Riemann nets because in neighborhoods of p and q they coincide with $\mathscr{R}_{p,\alpha,\beta}$ and $\mathscr{R}_{q,\alpha+\pi,\gamma}$, respectively. The construction is facilitated by reference to Figure 1, in which $\alpha = -\frac{1}{2}\pi$, and $0 < \beta, \gamma < \pi$. Changing α simply requires a rotation of the figure; the cases in which β or γ is 0 or π require self-evident modifications which are left to the reader. This picture, in which some characteristic arcs have been drawn, is merely descriptive and is not meant to show exactly what these arcs look like, but only their general form.

Since the nets $\mathscr{R}_{p,\alpha,\beta}$ and $\mathscr{R}_{p,\alpha+\pi,\gamma}$ coincide in the rectangle $prqs$, the two together give an HP-net in the curvilinear polygon whose sides (listed in positive order) are the 2-arc ap'', the 2-arc $p''r$, the 1-arc rq', the 1-arc $q'b$, the 2-arc bq'', the 2-arc $q''s$, the 1-arc sp', and the 1-arc $p'a$ (except, of course, at the singularities p and q). Note that the only jumps in the argument of the tangent when this simple closed curve is traversed in the positive direction are jumps of $\frac{1}{2}\pi$ at a and b and $-\frac{1}{2}\pi$ at r and s. Using the characteristic coordinate mapping construction and Proposition 1.4, it is clear that this net can be extended to the heart shaped region bounded by the union of the curves C_1 and C_2 which are made up of the arcs rp'', $p''a$, ac, and rq', $q'b$, bc, respectively. The construction described in the paragraph immediately preceding this discussion of double Riemann nets can then be used to extend this net to the entire complement of $\{p,q\}$. It is obvious that the only singularities of this net are the ones of type R at p and q.

(iv) *Degenerate double Riemann nets.* We shall define nets $\mathscr{F}_{p,q,\sigma,\beta,\gamma}$ where $p \neq q$ are points, $0 \leq \beta, \gamma \leq \pi$, and $\sigma = +$ or $-$. These nets arise as limiting cases of double Riemann nets as $\arg\bigl(e^{-i\alpha}(q-p)\bigr)$ tends to 0 or $\frac{1}{2}\pi$. The description is again facilitated by reference to the corresponding Figure 2 in which $\sigma = +$, and $0 < \beta$, $\gamma < \pi$. The value of σ indicates the sign of $\arg\bigl((p'-p)/(q-p)\bigr) \in [-\pi, \pi]$. The cases in which β or γ is 0 or π require self-evident modifications which are left to the reader, and similarly for the case $\sigma = -$. The net is defined initially as $\mathscr{R}_{p,\arg(q-p)-\pi/2,\beta}$ and $\mathscr{R}_{q,\arg(p-q)-\pi/2,\gamma}$ inside the circular sectors pqp' and qpq', respectively. One then uses the fan construction summed up in Proposition 1.10 to define $\mathrm{Fan}(pq', [\beta-\pi, 0])$ and $\mathrm{Fan}(qp', [\gamma-\pi, 0])$, which gives an HP-net in the interior of the curvilinear polygon $pwq'qup'p$. One then extends this net by tacking on $\mathrm{HP}(qq'w, qu)$ (that is, the characteristic quadrilateral $qwvu$), so that the net is now defined in the interior of the heart-shaped region bounded by the mutually orthogonal characteristics $pp' \cup p'u \cup uv$ and $pw \cup wv$ (except at the singularities p and q). Finally, the construction given just before the discussion of double Riemann nets is applied to define the net in the entire complement of $\{p,q\}$. The case $\sigma = -$ is the same except that $(p'-p)/(q-p)$ and $(q'-q)/(p-q)$ lie in the lower half-plane.

2. Riemann singularities

We shall establish a series of lemmas which lead to the complete description of singularities of type R given in Theorem 2.1. The reader is reminded that λ denotes 1-dimensional measure and that the term "translate" is used in the sense explained in the paragraph immediately following the statement of Proposition 1.3. We begin with

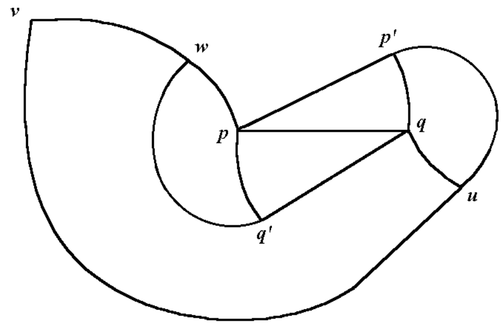

Figure 2.

Lemma 2.1. *Let θ have a singularity of type R at p. Then there is a characteristic arc of θ, one of whose endpoints is p and along which $\theta(z)$ has a limit as z tends to p.*

Proof. From the definition of this type of singularity it follows that we can assume that θ is regular in $N = \overline{N(p,\delta)}\backslash\{p\}$, and that there is an i-arc C of θ of finite length which joins a point lying outside of $\overline{N(p,\delta)}$ to p and on which θ is bounded. Let $w = w(s)$, $0 \le s \le \delta$, with $w(\delta) = p$ be the arc length parametrization of a final segment of C. Let $\theta(s) = \theta\big(w(s)\big)$. The curvature $\kappa(s) = \theta'(s)$, exists for $s \in A$, where $\lambda(A) = \delta$. We may assume that $\lim_{s\to\delta}$ $\theta(s)$ does not exist, since otherwise C is itself a characteristic arc of the kind we are seeking. For any $\kappa_0 > 0$ and $0 \le \delta' < \delta$ we have

$$\lambda(\{s \in (\delta', \delta) \cap A : \kappa(s) > \kappa_0\}) > 0$$

and

$$\lambda(\{s \in (\delta', \delta) \cap A : \kappa(s) < -\kappa_0\}) > 0,$$

since if either of these sets had measure zero for any such κ_0 and δ', the boundedness of $\theta(s)$ on $(0,\delta)$ would imply the existence of $\lim_{s\to\delta}\theta(s)$. From this it is easy to see that there is an $s_1 \in (\delta/2, \delta)$ such that $\kappa(s_1) > 2/\delta$, and s_1 is a density point of $\{s \in A : \kappa(s) > 0\}$. The part C_1 of the j-characteristic emanating from $w(s_1)$ to the left of C (as it is traversed in the direction of increasing s) has length at most $\delta/2$ (by Proposition 1.5), and so lies entirely in the punctured neighborhood N and terminates at p. Similarly, there is an $s_2 \in (s_1, \delta)$ for which $\kappa(s_2) < -2/\delta$, and which is a density point of $\{s \in A : \kappa(s) < 0\}$. The part C_2 of the j-characteristic emanating from $w(s_2)$ to the right of C again has length at most $\delta/2$, and so lies entirely in the punctured neighborhood N and terminates at p. Also, the opposite signs of the curvature of C at the points $w(s_1)$ and $w(s_2)$, together with simple topological considerations show that $C_1 \cap C_2 = \{p\}$. Now, we do not know *a priori* that C_1 does not intersect C at more than one point, so we let $s_1' < s_2$ be the greatest number less than s_2 such that $w(s_1') \in C_1$ and we let C_1' denote that part of C_1 which connects $w(s_1')$ to p. (The arc C_1' might, of course, be all of C_1.) It follows from the fact that p is the only point common to C_1 and C_2 and the fact that the lengths of these curves as well as that of C are all less than δ, that $C_1' \cup w([s_1', s_2]) \cup C_2$ is a simple closed curve whose interior D lies entirely in the punctured neighborhood N of p; the exterior of D is easily seen to lie to the right as $w([s_1', s_2])$ is traversed from s_1' to s_2. We denote by E the part of the j-characteristic emanating to the left of C from $w(s_2)$; obviously, $E \subset \bar{D}$. Let $v = v(s)$, $0 \le s < L$ be the arc length parametrization of E, with $v(0) = w(s_2)$. (We are not *a priori* excluding the possibility that $L = \infty$.) Since we chose s_2 to be a density point of $\{s \in A : \kappa(s) < 0\}$, there is a positive $\varepsilon < \delta - s_2$, $s_2 - s_1'$ such that

$$\lambda(\{s \in I \cap A : \kappa(s) < 0\}) > \tfrac{1}{2}\varepsilon, \tag{2.1}$$

both when $I = I_1$ and $I = I_2$, where I_1 and I_2 are the left and right halves of the interval $J = [s_2 - \varepsilon, s_2 + \varepsilon]$. Let $L' > 0$ denote the supremum of all $s \in (0, L)$ for which the translates of $w(J)$ along E down to $v(s)$ exist. (Here again we are not assuming *a priori* that L' is finite, although we shall show shortly that this must be the case.) These translates of $w(J)$ are all contained in D. For $t \in [-\varepsilon, \varepsilon]$ and $s \in [0, L')$ we let $v(t,s)$ be the point which lies both on the translate of $w(J)$ through $v(s)$ and on the translate of an initial arc of E through $w(s_2 + t)$. Now, it follows from (2.1) that for $s \in [0, L')$ the lengths of the translates of $w(I_1)$ and $w(I_2)$ down to $v(s)$, are both at least $\varepsilon/2$, which together with Proposition 1.6 implies that $\frac{1}{2}\varepsilon L' \le \pi\delta^2$. Thus, L' is finite. Also, Proposition 1.7 implies that $D_j^+\big(v(s)\big) \le 2/\varepsilon$, for $s \in (0, L')$. This, together with the finiteness of L' means that $\lim_{s\to L'} \theta\big(v(s)\big)$ exists, so that by the defining property of HP-nets, $\lim_{s\to L'} \theta\big(v(t,s)\big)$ exists for each $t \in J$. This then says that $\lim_{s\to L'} v(t,s)$ exists for each $t \in J$. But the definition of L' implies that for some $t = t_0 \in J$ this limit is p. The j-arc parametrized by $v(t_0, s)$, $0 \le s < L'$, can therefore be taken as the desired arc. ▫

Lemma 2.2. *Let θ be regular in $N = N'(p,\delta)$. Let C_k be a k-arc of θ lying in N and with arc length parametrization $z_k(s)$, $0 \le s < L_k$, $k = 1, 2$. Let $\lim_{s\to L_i} z_i(s) = p$ and $\lim_{s\to L_i} \theta\big(z(s)\big)$ exist. Let $z_i(0) = z_j(0)$. Then either* (i) *for some $\varepsilon > 0$ all translates of C_i along $z_j([0,\varepsilon])$ lie in N and terminate at p, or* (ii) *for some $\varepsilon > 0$ the translate of $z_j\big((0,\varepsilon]\big)$ along C_i provides a j-arc terminating at p, along which $\theta(z)$ has a limit as z tends to p and which is orthogonal to C_i at p.*

Proof. By replacing C_i by a sufficiently small final subarc, and δ by a smaller number if necessary, we may assume that the values of θ on C_i lie in some small interval, say of length $1/100$. Let $\eta > 0$ be so small that the values of $\theta\big(z_j(s)\big)$ on $[0,\eta]$ also lie in an interval of length $1/100$ and all translates of $z_j([0,\eta])$ along C_i lie in N. Let E_s denote the translate of C_i with initial point $z_j(s)$, and let $p(s)$ denote its terminal point. If for some $s_0 \in (0,\eta]$, $p(s_0) = p$, then the small variability of θ on $z_j([0,\eta])$ implies that the length of the translates of $z_j([0,s_0])$ down to $z_i(s)$ tend to 0 as s tends to L_i. This means that (i) holds with $\varepsilon = s_0$. In the opposite case $p(s)$, $0 < s \le \eta$, parametrizes a j-arc for which $\lim_{s\to 0}\theta\big(p(s)\big)$ exists. Now, because of the small variability of θ on C_i, $\sup\{\lambda(E_s) : 0 \le s < L_i\} = L < 3\delta$. It is easy to show by C^∞ approximation and simple calculus that the length of all translates of $z_j([0,s])$ along C_i is at most $s + L\xi(s)$, where $\xi(s)$ is the total variation of θ on $z_j([0,s])$. Since $s + L\xi(s) \to 0$ as $s \to 0$, it follows that $p(s) \to p$ as $s \to 0$, so that (ii) is true with $\varepsilon = \eta$. The orthogonality of $p([0,\eta])$ to C_i at p follows immediately from the fact that $\xi(s) \to 0$. ▫

Lemma 2.3. *Let θ be regular in $N = N'(p,\delta)$. Let $z(s)$, $0 \le s < L$ be an arc length parametrization of an i-arc C of θ lying in N. Let $\lim_{s\to L} z(s) = p$ and let $\lim_{s\to L}\theta\big(z(s)\big)$ exist. Then one of the following must happen*:

(i) *There is an open j-arc containing some point of C such that none of the translates of C along this arc terminates at p.*

(ii) *Passing through some point of C there is a closed j-arc C' of positive length such all translates of C along C' terminate at p, but on one side of each end point of C' no nearby translates of C terminate at p. Furthermore, the former translates are mutually nontangential at p.*

(iii) *$\theta(z)$ is given by $\arg(z-p)$ or $\arg(z-p) \pm \frac{1}{2}\pi$ (to within an integral multiple of π) in a punctured neighborhood of p.*

Proof. By replacing δ by a smaller number if necessary, we may assume that θ is regular in $N'(p,6\delta)$, that C joins $\partial N(p,6\delta)$ to p and that the values of θ on C all lie in a very small interval, of length $1/100$, say. Let q denote the point of C for which $|q-p| = \delta$. To facilitate the exposition we assume that $j = 1$ and that at q the unit tangent to C pointing towards p is $ie^{i\theta(q)}$. Let $w(s)$, $K < s < K'$, be the arc length parametrization of the j-characteristic in $N'(p,3\delta)$ for which $w(0) = q$ and such $w'(s) = e^{i\theta(w(s))}$. (It is possible that this j-characteristic is a simple closed curve or that it has infinite length; to cover these possibilities we allow either one or both of K, K' to be infinite.) Assume that (i) does not hold. Let $\mathscr{I}$ denote the set of all subintervals I of (K,K') which contain 0 and are such that for all s in I, $w(s)$ is joined to p by an i-arc C_s in $N'(p,2\delta)$ (which is just a translate of the part of C which joins q to p). By Lemma 2.2, $\mathscr{I}$ contains intervals of positive length. Let $I \in \mathscr{I}$. By the defining property of HP-nets, the set $\{\theta(z) : z \in C_s\}$ is the same for all $s \in I$, and by assumption it is an interval of length less than $1/100$. From this it follows that

$$\left|\arg \frac{w'(s)}{i\big(w(s)-p\big)}\right| < 1/100, \qquad s \in I, \tag{2.2}$$

and that

$$|w(s)-p| \le \lambda(C_s) \le 2|w(s)-p|, \qquad s \in I. \tag{2.3}$$

(The constant 2 is, of course, much larger than necessary.) It furthermore follows from the fact that all of the C_s, $s \in I$, meet at p, together with Proposition 1.5 that,

$$d\theta\big(w(s)\big)/ds = 1/\lambda(C_s), \qquad \text{a.e. on } I. \tag{2.4}$$

Indeed, were this not the case, a simple argument shows that there would be a positive distance between two of the arcs C_s, which is not consistent with their all meeting at p. Let $\phi(s)$ denote the argument of the tangent (the one pointing away from p, to be definite) to C_s at p. Note that by the defining property of HP-nets all of the C_s have tangents at p because the original C did. A straightforward argument based on this property shows that $\phi'(s) = d\theta\big(w(s)\big)/ds$. From this, together with (2.3) and (2.4) it follows that

$$\phi'(s) = d\theta\big(w(s)\big)/ds \ge 1/\big(2|w(s)-p|\big) \qquad \text{a.e. on } I. \tag{2.5}$$

If $\lambda\Big(\theta\big(w(I)\big)\Big) \le 2\pi$, it follows from (2.2) and the fact that $|q-p| = \delta$ that

$$\delta/2 < |w(s)-p| < 2\delta, \qquad s \in I, \tag{2.6}$$

and in particular that $\overline{w(I)} \subset N'(p, 3\delta)$.

First, assume that $\sup\Big\{\lambda\Big(\theta\big(w(I)\big)\Big) : I \in \mathscr{I}\Big\} < 2\pi$. Then it is easy to see that $\cup\{I : I \in \mathscr{I}\} = I_0 \in \mathscr{I}$ is closed. This gives the first sentence in (ii); the second sentence follows from the lower bound for $\phi'(s)$ in (2.5).

To complete the proof it therefore suffices to assume that $\sup\Big\{\lambda\big(\phi(I)\big) : I \in \mathscr{I}\Big\} \ge 2\pi$. If this is the case, then there is an $I = [\alpha, \beta] \in \mathscr{I}$, such that for all s in I, $w(s)$ is joined to p by an i-characteristic arc C_s in $N'(p, 3\delta)$ and such that $\phi(\beta) = \phi(\alpha) + 2\pi$. But then it follows that either C_α and C_β coincide or one is a proper subarc of the other, since distinct characteristics cannot be tangent at p, as follows immediately from the positiveness of $\phi'(s)$. First we consider the case that they coincide, that is, that $w([\alpha, \beta])$ is a simple closed convex curve (recall that $\arg\big(w'(s)\big)$ is nondecreasing) containing p in its interior. Let $w(\gamma)$ be the point on this curve at maximum distance from p. Then $D_j^+\theta\big(w(\gamma)\big) \ge 1/|w(\gamma)-p|$, so that by Proposition 1.7, C_γ must be a straight line segment, from which it follows that all of C_s are straight line segments. This means that there are straight line j-characteristics terminating at p at all angles ϕ, $0 \le \phi \le 2\pi$, so that (iii) holds. The other possibility cannot, in fact occur. To see this, we assume for definiteness that C_β is a proper subarc of C_α. Thus $w([\alpha,\beta])$ starts at a point $w(\alpha)$ on C_α, goes around p once and ends up at a point $w(\beta)$ on C_α between $w(\alpha)$ and p. But then it is easy to see that on $[\alpha, K')$, $w(s)$ winds around p infinitely many times in the counterclockwise direction, crossing C_α successively at points $w(s_n)$, $n = 0,1,2,\ldots$, each of which lies farther along C_α towards p than its predecessor. Since the arcs $w(s_n)w(s_{n+1})$ are translates of each other, if there were any variation of θ on the j-arc $w(s_0)w(s_1)$, $\theta(z)$ would not have a limit as $z \to p$ along C_α, an obvious contradiction, since C_α is parallel to a terminal segment of our original i-arc C. But then the arcs $w(s_n)w(s_{n+1})$ are all line segments of the same length, which is obviously impossible. □

Definition 2.1. In the case that (ii) of the preceding lemma holds, the family $F(C)$ of translates of the i-arc C joining the points of the j-arc C' to p will be called an *i-fan* of the net at p, and the characteristics passing through the endpoints of C' are called the *bounding characteristics* of $F(C)$. The angle formed at p by the bounding characteristics is called the *angle* of the fan. In the case that (i) holds we regard C alone as constituting a (degenerate) fan with angle 0 and both bounding characteristics coinciding with C.

In dealing with such fans we will usually restrict attention to a small neighborhood of p so that on C, θ takes values in a small interval (taken to have length less than $1/100$, to be specific). In this way all of the i-arcs making up an i-fan are virtually straight line segments and join p to the boundary of the disk about p in which we are working. These i-arcs are connected by almost circular j-arcs whose curvature increases uniformly to infinity as they approach p.

With these preliminaries out of the way we are in a position to derive a description of singularities of type R. For the sake of descriptive simplicity we assume, without loss of generality, that $p = 0$. We apply Lemma 2.1 and then Lemma 2.3 and assume that we are not in case (iii) of the latter. There is therefore an i-fan of characteristic arcs terminating at 0 (which might consist of a single arc—case (i) of Lemma 2.3), which, without loss of generality, we assume to be "symmetric" with respect to the positive x-axis, that is, that the bounding characteristics C_i^+ and C_i^- have outgoing tangents which make angles of $\pm\frac{1}{2}\phi$, respectively, with the positive x-axis. As we saw in the proof of Lemma 2.3, $0 \le \phi < 2\pi$ (since we are not in case (iii)). By choosing an appropriate $\delta > 0$, we can assume that θ is regular in $N'(0, 2\delta)$, and that along each of the characteristics of the fan the values of θ lie in an interval of length $1/100$, say. From the definition of fan in conjunction with Lemma 2.2, we see that there are distinct j-characteristics C_j^+ and C_j^- emanating from 0, whose outgoing tangents at 0 form angles of $\pm\frac{1}{2}\pi$ with those of C_i^+ and C_i^-, respectively. By picking a smaller δ, if necessary, we can assume that along each of these j-characteristics the values of θ lie in an interval of length $1/100$. We note the following.

(1) $\phi \le \pi$. This is clear since were $\pi < \phi < 2\pi$, nearby translates of C_j^+ would cross nearby translates of C_j^- in $N'(0,2\delta)$ and form an angle strictly between 0 and π, a patent impossibility.

(2) If $\phi = 0$, the fans $F(C_j^+)$ and $F(C_j^-)$ cannot both consist of a single characteristic. To see that this is so, assume otherwise. Then there would be an i-characteristic, E, tangent to the x-axis and emanating to the left of 0. We can assume as before that along E the values of θ lie in an interval of length $1/100$. But this means that through each point of $C_i^+ = C_i^-$, C_j^+, C_j^- and E there passes a perpendicular characteristic which joins two points of $\partial N(0,2\delta)$ without passing through p. This, in turn, implies that $|\nabla\theta(z)| \le 2/\delta$, a.e. in $N'(0,\delta)$, which, by Proposition 1.11, is impossible, because we are assuming that the net has a true singularity at 0.

(3) $F(C_j^+) = F(C_j^-)$. This is obvious if $\phi = \pi$, since then C_j^+ and C_j^- are tangent at 0 and consequently coincide (see (ii) of Lemma 2.3), and the corresponding fan is degenerate. To handle the case in which $0 \le \phi < \pi$, we assume to the contrary that these fans are disjoint. Let E^+ and E^- be the other bounding j-characteristics of $F(C_j^+)$ and $F(C_j^-)$, respectively. Then it follows from (2), or the condition $0 < \phi < \pi$, whichever is applicable, that the angle between these distinct j-characteristics is strictly between 0 and π. But then by the definition of fan there are i-characteristics perpendicular to E^+ and E^-, respectively, which intersect in $N'(0,\delta)$ and form an angle strictly between 0 and π, which is obviously absurd.

We are now in a position to state in the form of several theorems the basic facts about Riemann singularities which we shall use in what follows.

Theorem 2.1. *Let p be a Riemann singularity of an HP-function θ. Then there exists a $\delta > 0$ such that one of the following holds.*

(A) *$\theta(z)$ is given by $\arg(z-p)$ or $\arg(z-p) \pm \frac{1}{2}\pi$ (to within an integral multiple of π) in $N'(p,\delta)$.*

(B) *There are four arcs C_i^+, C_j^+, C_j^-, C_i^- with the following properties:*

(1) *Each one joins $\partial N(p,\delta)$ to p.*

(2) *The arc length parametrization each of them has a Lipschitz continuous derivative.*

(3) *C_i^+ and C_j^+ have their single common point at p, where they meet at right angles, and similarly for C_i^- and C_j^-.*

(4) *C_k^+ and C_k^- either have their single common point at p or coincide, $k = 1,2$.*

(5) *The four arcs occur in the indicated order in the counterclockwise sense.*

(6) *If $\alpha_k \in [0,\pi]$ denotes the (unsigned) angle between C_k^- and C_k^+, $k = 1,2$, then in the smaller curvilinear sector of $N(p,\frac{1}{2}\delta)$ between C_i^- and C_i^+ the characteristics of the net coincide with those of $\mathrm{Fan}\,(C_i^-,[0,\alpha_i])$ and in the smaller curvilinear sector of $N(p,\frac{1}{2}\delta)$ between C_j^+ and C_j^- they coincide with those of $\mathrm{Fan}\,(C_j^+,[0,\alpha_j])$.*

(7) *In the two parts into which* $N(0,\frac{1}{2}\delta)$ *is divided by the removal of these fans, the characteristic arcs of the net coincide with those of* $\mathrm{HP}(C_i^+,C_j^+)$ *and* $\mathrm{HP}(C_j^-,C_i^-)$, *respectively.*

(8) $\alpha_i+\alpha_j=\pi$.

Proof. If (A) (which is (iii) of Lemma 2.3) does not occur, then it follows from the preceding discussion that there is one fan of i-arcs and one of j-arcs, and that at most one of these fans can be degenerate. If, as above, we assume that θ is regular in $N'(0,2\delta)$, and that along each of the curves belonging to these fans the values of θ lie in a small interval (say of length $1/100$), then, by Proposition 1.7, the curvature on the parts of these curves lying in $N'(0,\delta)$ is bounded above by $1/\delta$, since for any point on any these curves there is an orthogonal characteristic arc of length at least δ emanating to either side and lying in $N'(0,2\delta)$; this proves point (2). Points (3) and (4) follow from the preceding discussion and point (5) is merely of a notational nature. Point (6) follows from Proposition 1.10 and the discussion preceding it. It follows from point (2) and the discussion preceding Proposition 1.4 that in each of the two parts into which $N(0,\delta)$ is divided by the removal of the fans, θ is the solution of the characteristic initial value problem corresponding to bounding characteristics of the two fans; that is, in those regions the HP-net is given by $\mathrm{HP}(C_i^+,C_j^+)$ and $\mathrm{HP}(C_j^-,C_i^-)$, respectively. That the angles of the fans sum to π, follows from the preceding discussion. ◻

Definition 2.2. In case (A) of the preceding theorem the singularity p will be called a *degenerate spiral singularity*; in case (B) it will be called a *nondegenerate Riemann singularity.*

The following theorem is essentially the converse of Theorem 2.1; its proof is an immediate consequence of the constructions of HP-nets given in Section 1.2.

Theorem 2.2. *Let* C_1, C_2 *be closed arcs with bounded curvature* (*i.e., whose arc length parametrizations have Lipschitz continuous first derivatives*) *with an endpoint* p *in common and which form a right angle at* p. *Let* $\phi_1,\phi_2\geq 0$ *and* $\phi_1+\phi_2=\pi$. *Then in some punctured neighborhood* N *of* p *there is a unique HP-net which coincides with* $\mathrm{HP}(C_1,C_2)$ *in the part of* N *in the smaller curvilinear sector determined by these curves and which has fans* $F(C_1)$ *and* $F(C_2)$ *with angles* ϕ_1, ϕ_2, *respectively.*

As an immediate consequence of the foregoing discussion we also have the following

Theorem 2.3. *Let* p *be a Riemann singularity of an* HP*-net. Let* C_i *be an* i*-characteristic arc which terminates at* p *and let* C_j *be an open* j*-arc which intersects* C_i. *If all of the translates of* C_i *along* C_j *exist and terminate at* p, *then* C_j *is strictly concave towards* p.

Here we mean, of course, that if $z(s)$, $a<s<b$, is an arc length parametrization of C_j for which $iz'(s)$ points in the direction along i-arcs towards p, then

the essential infimum of $d\arg z'(s)/ds$ is strictly positive. The following theorem follows immediately from Theorem 2.1 together with Proposition 1.5.

Theorem 2.4. *Let p be a Riemann singularity of an* HP*-net for which the angle of the i-fan is π and let E_1 and E_2 be the bounding characteristic arcs of the i-fan at p. Then E_1 and E_2 cannot be both concave or both convex towards the* (*unique*) *j-characteristic terminating at p.*

We end this section with

Theorem 2.5. *If $\mathscr{H}$ is an* HP*-net regular in* **C** *except for a single nondegenerate Riemann singularity at p, then $\mathscr{H}$ is an $\mathscr{R}_{p,\alpha,\beta}$.*

Proof. It is clearly sufficient to prove that the fans at p consist entirely of straight lines. If this were not so, then one, and hence all of the characteristics in the i-fan, say, would have nonvanishing curvature on some set of positive measure. Thus, there would be a j-arc C_j which joins a point q on a bounding characteristic C_i of the i-fan to p. The i-arc E joining q to p (including the endpoints) together with C_j forms a simple closed curve. Simple topological considerations show that either C_j is an initial arc of one of the bounding characteristics of the j-fan at p, or one of these bounding characteristics has an initial arc lying in the interior of the simple closed curve $E\cup C_j$ and joining p to some point q' on E. But then consideration of nearby translates of E produces an i-arc joining two points of a j-characteristic, yielding an i-arc and a j-arc joining a pair of distinct points and thereby forming a simple closed curve on and inside of which the net is regular, a patent absurdity. □

3. Spiral singularities

Now that we have completely described Riemann (type R) singularities, we deal with the remaining possibility, spiral or type S singularities. In this section we shall consequently assume that p is an isolated singularity of an HP-function such that if C is a characteristic arc joining $q\neq p$ to p, then either C has infinite length or $\theta(z)$ is unbounded as z tends to p along C.

Lemma 3.1. *Let θ be regular in $N'(p,2\delta)$ and have a spiral singularity at p. Let C be an i-arc of θ lying in $N'(p,\delta)$, one of whose endpoints is p. Then C is not tangent to any of the concentric circles $\partial N(p,d)$, $0<d<\delta$.*

Proof. Without loss of generality we may assume that $p=0$. Assume, to the contrary, that $z(s)$, $0\le s<L$, is the arc length parametrization of a subarc of C' of C with $\lim_{s\to L} z(s)=0$, such that $z'(0)\cdot z(0)=0$, where for simplicity we regard complex numbers as vectors and use the dot notation for inner product. Let $w(s)=z'(s)\cdot z(s)$. Since $z(s)$ tends to 0 as s tends to L, $\lim_{s\to L} w(s)=0$. Thus there is a point $\sigma\in(0,L)$ at which $|w|$ attains its maximum. Arbitrarily close to σ there are numbers $s_1<s_2$ such that

$$0=w(s_2)-w(s_1)=\bigl(z'(s_2)-z'(s_1)\bigr)\cdot z(s_2)+z'(s_1)\cdot\bigl(z(s_2)-z(s_1)\bigr).$$

Dividing by $s_2 - s_1$, and taking into account that z' is continuous and $|z'(\sigma)| = 1$, we see that $|z(\sigma)|D_i^+\big(z(\sigma)\big) \geq 1$. But then it follows from Proposition 1.7 that $z(\sigma)$ is joined to p by a j-arc which is a straight line segment. But this contradicts the hypothesis that p is a spiral singularity. □

The following lemma is similar to Lemma 2.2; we have chosen not to combine the two in a single more general lemma, since to do so requires some discussion not necessary for the purposes at hand.

Lemma 3.2. *Let θ be regular in $N'(p, 2\delta)$ and have a spiral singularity at p. Let C_k be a k-arc of θ lying in $N'(p,\delta)$ with arc length parametrization $z_k(s)$, $0 \leq s < L_k$, $k = 1,2$. Let $\lim_{s\to L_i} z_i(s) = p$. Let $z_i(0) = z_j(0)$. Then for some $\varepsilon > 0$ each point of $z_j([0,\varepsilon])$ is joined to p by an i-arc lying in $N(p,\delta)$.*

Proof. We begin with the following observation. Let $w_n(s)$, $0 \leq s \leq \alpha_n$, $n = 1,2,\ldots$, be arc length parametrizations of j-arcs $E_n \subset N'(p,\delta)$ of θ for which $\lambda(E_n) \geq \xi > 0$ and $\lambda\big(\theta(E_n)\big) < 1/100$. Because of the latter condition the E_n are close to being line segments, so that in particular $\xi \leq \lambda(E_n) \leq 3\delta$ for all $n \geq 1$. Thus, the sequence of functions $v_n(t) = w_n\big(\lambda(E_n)t\big)$, $0 \leq t \leq 1$, is uniformly bounded and equicontinuous, and consequently has a subsequence which converges uniformly on $[0,1]$ to some function v. A straightforward argument shows that either $v(t)$, $0 \leq t \leq 1$, gives a parametrization of a j-arc of θ in $\overline{N(p,\delta)}\backslash\{p\}$ with one endpoint possibly coinciding with p, or there is a $t_0 \in (0,1)$ such that v, when restricted to each of the intervals $[0,t_0]$ and $[t_0,1]$, parametrizes such a j-arc. Furthermore, the values of θ on any of these arcs parametrized by v lie in an interval of length at most $1/100$.

By replacing C_j with an initial subarc, if necessary, we may assume that $\lambda\big(\theta(C_j)\big) < 1/100$ and that C_j intersects C_i only at $z_i(0) = z_j(0)$. For $s \in (0, L_i)$ and $\eta \in (0, L_j]$ let $C(s,\eta)$ denote the translate of the subarc $z_j([0,\eta])$ of C_j along C_i down to $z_i(s)$, provided it exists and lies in $N'(p,\delta)$. For each $\eta \in (0, L_j]$ let $L(\eta)$ be the supremum of all $s \in [0, L_i)$ such that $C(s,\eta)$ exists. Obviously, $L(\eta) > 0$ for all $\eta \in (0, L_j]$. We claim that there is some $\eta > 0$ for which $L(\eta) = L_i$. This could only fail to be true if for each $\eta \in (0, L_j]$ there is an increasing sequence $\{s_m(\eta)\}$ in $\big[0, L(\eta)\big)$ which tends to some $\sigma(\eta) < L_i$ such that one of the following happens:

(i) $\operatorname{dist}\big(C\big(s_m(\eta),\eta\big), \partial N(p,\delta)\big) \to 0$

or

(ii) $\operatorname{dist}\big(C\big(s_m(\eta),\eta\big), p\big) \to 0$.

First we show that (i) cannot happen for arbitrarily small η. Say, to the contrary, that (i) actually happens for $\eta = \eta_n$, where $\eta_n \to 0$. Then, since $\lambda\big(\theta\big(z_j\big([0,\eta]\big)\big)\big) \to 0$ as $\eta \to 0$, the observation of the first paragraph of the proof will produce a straight line j-arc E joining $\partial N(p,\delta)$ to some point of $C_1 \cup \{p\}$. But this contradicts the hypothesis that p is a spiral singularity, since there would then be a straight line j-arc (a translate of an initial segment of E) of length at least $\operatorname{dist}\big(C_1, \partial N(p,\delta)\big) > 0$ emanating from each point of $C_1 \cup \{p\}$. Thus there is some $\eta_0 > 0$ such that (i) cannot happen if $\eta \le \eta_0$. Now let $\eta \le \eta_0$ and assume that (ii) happens. But then, again by the initial observation, there will be a j-arc J joining $z_i\big(\sigma(\eta)\big)$ to p for which $\lambda\big(\theta(J)\big) < 1/100$, which again contradicts the hypothesis that p is a spiral singularity. Thus, indeed, there is some $\varepsilon > 0$ such that $C(s,\varepsilon)$ is well defined for all $s \in [0, L_i)$. This, of course, is the same as saying that all the translates of C_i along $z_j([0,\varepsilon])$ exist and lie in $N'(p,\delta)$. To finish we only have to show that $\lambda\big(C(s,\varepsilon)\big) \to 0$ as $s \to L_i$. But if this were not true we would have a sequence $\{s_m\}$ in $[0, L_i)$ tending to L_i for which $\lambda\big(C(s_m,\varepsilon)\big) \ge \xi > 0$, and the observation with which we began would give us a j-arc J of length at least ξ terminating at $p = \lim_{m\to\infty} z_i(s_m)$ for which $\lambda\big(\theta(J)\big) \le 1/100$, which again contradicts the assumption that θ has a spiral singularity at p. □

Lemma 3.3. *Let θ be a* HP*-function with a type S singularity at p. Then there exists some $\delta_1 > 0$ such that θ is regular in $N'(p, 2\delta_1)$ and such that each point q in $N'(p,\delta_1)$ is joined to p in $N(p, |q-p|)$ by characteristic arcs from both families neither of which is tangent to $\partial N(p, |q-p|)$.*

Proof. Let θ be regular in $N(p,\delta)$, and let C be an i-arc lying in $N(p,\delta)$, with arc length parametrization $z(s)$, $0 \le s < L$, and which terminates at p. We begin by observing that (the measurable function) $|d\theta\big(z(s)\big)/ds|$ must be unbounded on $[0, L)$. Assume, to the contrary, that it is bounded. Then L must be ∞, since if $L < \infty$, $\lim_{s\to L} \theta(s)$ would exist, which would contradict the hypothesis that p is a spiral singularity. But if $|d\theta\big(z(s)\big)/ds|$ is bounded and $L = \infty$, then clearly $\lim_{s\to L} z(s)$ cannot exist. Thus indeed $|d\theta\big(z(s)\big)/ds|$ is unbounded on $[0, L)$. Proposition 1.7 then implies that there exists a δ_1 in $(0, \frac{1}{2}\delta)$ such that some point q of $\partial N(p,\delta_1) \cap C$ is joined by a j-arc lying in $N(p, \frac{1}{2}\delta)$. A simple application of Lemma 3.1 shows that we can assume in addition that this j-arc as well as a subarc C' of C join q to p inside $\overline{N(p,\delta_1)}\backslash\{p\}$. For $0 < \varepsilon \le \delta_1$ let $A_i(\varepsilon)$ be the set of $z \in \partial N(p,\varepsilon)$ which can be joined to p by an i-arc lying in $\overline{N(p,\varepsilon)}\backslash\{p\}$. From what we have shown it follows that $A_i(\varepsilon) \ne \emptyset$, $i = 1, 2$, $0 < \varepsilon \le \delta_1$.

We show that $A_i(\varepsilon)$ is open in $\partial N(p,\varepsilon)$. If there is an i-arc which joins a point z of the circle $\partial N(p,\varepsilon)$ to p in $\overline{N(p,\varepsilon)}$, then it follows immediately from Lemma 3.2 that there is a neighborhood S of z on this circle such that all w in S can also be joined to p by an i-arc lying in $N(p, \frac{1}{2}\delta)$. Moreover, it follows from Lemma 3.1 that these i-arcs are not tangent to the circle and that they lie in $\overline{N(p,\varepsilon)}$.

Next we show that $\partial N(p,\varepsilon)\backslash A_i(\varepsilon)$ is also open in $\partial N(p,\varepsilon)$. If $z \in \partial N(p,\varepsilon)$ is not joined to p by an i-arc lying in $\overline{N(p,\varepsilon)}$, then we claim that if we proceed along the i-characteristic through z in either direction, we will be led outside of $\overline{N(p,\varepsilon)}$. Assume, to the contrary, that we can proceed indefinitely along an i-arc $C \subset \overline{N(p,\varepsilon)}$, beginning at z but without coming to p. Then C has infinite length, and if $w(s)$, $0 \leq s < \infty$, is the arc length parametrization of C, then $\lim_{s\to\infty} w(s)$ is not p. In consequence, if ξ is the infimum of all η such that $N(p,\eta)$ contains $w\big((s,\infty)\big)$ for some s, then $0 < \xi \leq \varepsilon$. Let $\delta > \xi' > \xi > \xi'' > 0$. Then for some s', $w\big((s',\infty)\big)$ lies in $N(p,\xi')$, but for arbitrarily large s, $w(s)$ lies outside of $N(p,\xi'')$. From this it easily follows that there are arbitrarily large s for which $w(s)$ lies in $N(p,\xi')\backslash N(p,\xi'')$, but such that $D_i^+\theta\big(w(s)\big) > 1/\xi'$. However, for such s there is a j-arc of length less than ξ' which joins $w(s)$ to p, by Proposition 1.7. Obviously, the length of such an arc must be at least ξ'', so that these arcs come closer and closer to straight line segments as ξ' and ξ'' approach ξ. By a simple limit argument (see the first paragraph of the proof of Lemma 3.2) we conclude that there is a straight line j-arc joining a point of $\partial N(p,\xi)$ to p. But this is impossible, since p is a spiral singularity. Thus indeed, by following the i-characteristic through z in either direction we will be led outside $\overline{N(p,\varepsilon)}$. From this it follows that there is a neighborhood S of z on $\partial\Delta(p,\varepsilon)$ such that no w in S can be joined to p by an i-arc lying in $\overline{N(p,\varepsilon)}$; that is, $\partial N(p,\varepsilon)\backslash A_i(\varepsilon)$ is open in $\partial N(p,\varepsilon)$. From the connectedness of $\partial N(p,\varepsilon)$ it now follows that $A_i(\varepsilon) = \partial N(p,\varepsilon)$, which is to say that each point of $\partial N(p,\varepsilon)$ is joined to p by characteristic arcs of both families lying inside this circle. That these arcs are not tangent to this circle follows from Lemma 3.1 □

We are now in a position to completely characterize type S singularities.

Theorem 3.1. *If the* HP*-function* θ *has a singularity of type S at* p*, then in some neighborhood of* p*,* $\theta(z) = \arg(z-p) + \alpha$*, where* α *is a constant.*

Proof. In this proof we use U to denote the unit punctured disk, $N'(0,1)$. Without loss of generality we may assume that $p = 0$. Let δ_1 be as in the conclusion of Lemma 3.3. By replacing θ with the HP-function $\theta(\delta_1 z)$ we may assume that $\delta_1 = 1$. It follows from Lemma 3.3 and the continuity of θ that $e^{i\theta(z)}/z$ must lie in one of the four open quadrants for all z in $\bar{U}\backslash\{0\}$ and on a compact subset thereof on each concentric circle $r\partial U$, $0 < r \leq 1$; to be specific, we shall assume that it is the second quadrant, so that the vectors $e^{i\theta(z)}$ and $ie^{i\theta(z)}$ point in the direction of movement towards 0 along the 1- and 2-characteristics, respectively. From the confluence of characteristics at 0 implied by Lemma 3.2, it follows that

$$(-1)^{i+1} D_i\theta(z) \geq 0 \qquad \text{a.e. in } U,\ i = 1,2, \tag{3.1}$$

so that θ varies monotonically on characteristics in U. Because we have a singularity of type S at 0 it follows that θ tends monotonically to ∞ $(-\infty)$ as we move along 1-characteristics (2-characteristics) towards 0. The construction given just before the discussion of double Riemann nets in Section 1.4 shows that θ has a unique extension to an HP-function in $\mathbf{C}\backslash\{0\}$, so that for convenience we may regard it as being defined in the entire punctured plane. In light of that construction it follows that θ tends monotonically to $-\infty$ (∞) as we move along 1-characteristics (2-characteristics) away from 0.

Let $L_i(z)$ denote the distance along the i-characteristic from z to 0. From the convexity properties of the characteristics implied by (3.1) it follows that $L_i(z)$ is nondecreasing as z moves along a j-characteristic away from 0. From Proposition 1.7 it follows that $D_i^+\theta(z) \leq 1/L_j(z)$. This means that $L_i(z)$ is never infinite, since if it were infinite at some point, we would have straight line j-arcs, and by translating them along i-arcs down to 0 we would obtain straight line characteristic arcs terminating at 0, a contradiction. From this, in turn, it follows that $|D_i\theta(z)| = 1/L_j(z)$ a.e. on each i-characteristic, since otherwise a simple argument shows that the j-characteristics would not be confluent at 0. Since the function L_j is obviously continuous in the punctured plane, we must actually have that this last equation holds everywhere there; more precisely we have $D_i\theta(z) = (-1)^{i+1}/L_j(z)$, $z \neq 0$. Thus the continuous branches of θ are C^1 functions, and $|D_i\theta(z)| > 0$ on $\mathbf{C} - \{0\}$, $i = 1, 2$.

From the spiral nature of the characteristics, it follows that for any a in the punctured plane there are i-arcs $C_i(a)$, $i = 1, 2$, which both join a to a point $b = b(a)$ closer to 0, and whose only points in common are a and b. These arcs are, moreover, uniquely determined by these conditions. Let $\tau_i(a)$ denote $\Delta\theta(ab)$ along $C_i(a)$, so that $\tau_1(a) > 0$ and $\tau_2(a) < 0$. Simple geometry implies that

$$\tau_1(a) - \tau_2(a) = 2\pi. \tag{3.2}$$

If we replace a by a point a' near it on the 2-characteristic through it, then b will be replaced by $b' = b(a')$. Consideration of the characteristic quadrilateral with vertices a, a', b, b' shows that $\tau_1(a') = \tau_1(a)$, which in light of (3.2) implies that $\tau_2(a) = \tau_2(a')$, also. Similarly, these functions are unchanged by small movements along 1-characteristics. This means that τ_1 and τ_2 are constants. In other words if, starting from any point a in the punctured plane, we move along the i-characteristic towards 0 in such a way that θ changes by τ_i we come to the same point independently of whether i is 1 or 2.

For a given characteristic C and two points $a, b \in C$ we let $\Delta_C\theta(a, b)$ denote the change in θ as one moves along C from a to b, i.e., $\Delta_C\theta(a, b) = \bar{\theta}(b) - \bar{\theta}(a)$, where $\bar{\theta}$ denotes any branch of θ which is continuous on C. (This is the same "$\Delta\theta$" notation we have been using up to now, with the subscript indicating the characteristic added to avoid possible confusion.) Let F_i be the i-characteristic through $1 \in \mathbf{C}$. For $i = 1, 2$ we define a mapping of $\mathbf{R}^2$ onto $\mathbf{C}\backslash\{0\}$ as follows. For $(t_1, t_2) \in \mathbf{R}^2$, let q be the point on F_i for which $\Delta_{F_i}\theta(1q) = \tau_i t_i/\pi$ and let $z = \zeta_i(t_1, t_2)$ be the point on the j-characteristic C through q for which $\Delta_C\theta(qz) = \tau_j t_j/\pi$. It follows immediately from the defining property of HP-nets that if $\zeta_1(t_1, t_2) = \zeta_2(t_1, t_2)$, then there is a $\delta > 0$ such that $\zeta_1(t_1', t_2) = \zeta_2(t_1', t_2)$ and $\zeta_1(t_1, t_2') = \zeta_2(t_1, t_2')$ for all t_1', t_2' such that $|t_1' - t_1| < \delta$ and $|t_2' - t_2| < \delta$. Since $\zeta_1(0, 0) = 1 = \zeta_2(0, 0)$, the continuity of ζ_1 and ζ_2 then implies that

$\zeta_1(t_1,t_2) = \zeta_2(t_1,t_2)$ for all $t_1,t_2 \in \mathbf{R}$; their common value will be denoted by $\zeta(t_1,t_2)$. Let $(t_1,t_2) \in \mathbf{R}^2$ and let $q,r \in F_1$ be such that

$$\Delta_{F_1}\theta(1q) = \frac{\tau_1 t_1}{\pi} \qquad \text{and} \qquad \Delta_{F_1}\theta(1r) = \frac{\tau_1(t_1+\pi)}{\pi} = \frac{\tau_1 t_1}{\pi} + \tau_1.$$

Then by what was said in the preceding paragraph $\Delta_C\theta(qr) = \tau_2$, where C is the 2-characteristic through q. From this it follows immediately that $\zeta_1(t_1+\pi,t_2) = \zeta_1(t_1,t_2+\pi)$, that is, that

$$\zeta(t_1+\pi,t_2) = \zeta(t_1,t_2+\pi) \qquad \text{for all } t_1,t_2 \in \mathbf{R}. \tag{3.3}$$

One easily calculates that

$$\frac{\partial\zeta}{\partial t_k} = i^{k-1}\frac{\tau_k}{\pi}\frac{e^{i\theta(\zeta)}}{D_k\theta(\zeta)}, \qquad k = 1,2,$$

(using the fact that $\zeta(t_1,t_2) = \zeta_l(t_1,t_2)$, where $\{k,l\} = \{1,2\}$), so that since continuous branches of θ are C^1 functions, ζ is also of class C^1. If we define

$$\omega(t_1,t_2) = \frac{\tau_1 t_1 + \tau_2 t_2}{\pi} + \theta(1), \tag{3.4}$$

then it is clear that

$$\arg\left\{\frac{\partial\zeta}{\partial t_1}\right\} = \omega(t_1,t_2) \qquad \text{and} \qquad \arg\left\{\frac{\partial\zeta}{\partial t_2}\right\} = \omega(t_1,t_2) + \frac{\pi}{2}. \tag{3.5}$$

We also point out that for $n \in \mathbf{Z}$, $T \in \mathbf{R}$, $i = 1,2$,

$$\zeta\big(\{(t_1,t_2) : n\pi \le t_i < (n+1)\pi,\ t_j \ge T\}\big) \supset N'(0,\delta), \tag{3.6}$$

for some $\delta > 0$.

For each $z \in U$ let $p_i(z)$ be the point on the i-characteristic through z for which $\Delta\theta\big(zp_i(z)\big) = (-1)^{i+1}2\pi$, that is, the point on this characteristic which we arrive at by moving along it towards 0 from z in such a way that θ changes by 2π in the case $i = 1$ and by -2π in the case $i = 2$. We claim that $|p_i(z)/z|$, which is clearly less than 1 for all z in U, is actually bounded away from 1 there. If it were not, then there would be a sequence $\{z_k\}$ of points in U tending to 0 and such that $|p_i(z_k)/z_k|$ approaches 1. Application of the compactness principle (Proposition 1.9) to the family of nets corresponding to the HP-functions $\theta(|z_k|z)$ would then yield a subsequence converging to a net $\mathscr{H}$ in the entire punctured plane for which $\partial\overline{U}$ is an i-arc. But then it is clear that $\mathscr{H}$ is the degenerate spiral net $\mathscr{S}_{0,\pi/2}$. This in turn means that the original net would have to have i-arcs arbitrarily close to a full circle centered at 0, which contradicts the fact that

$|\tau_i| < 2\pi$. Thus indeed, the $|p_i(z)/z|$ are bounded away from 1 in U. In terms of our characteristic coordinates this means that $|\zeta(t_1,t_2)|$ decreases exponentially as either of the t_i tends to ∞; more precisely it means that there exist $A, K > 0$, such that

$$|\zeta(t_1,t_2)| < Ae^{-K(t_1+t_2)}, \qquad \text{for } t_1,t_2 \geq 0. \tag{3.7}$$

In light of (3.5) and (3.6) we must show that if $u(t_1,t_2) = \zeta(t_1,t_2)\cdot e^{i\omega}$, $v(t_1,t_2) = \zeta(t_1,t_2)\cdot ie^{i\omega}$, then $u/|\zeta|$, $v/|\zeta|$ are constant in some half-strip $n\pi \leq t_i < (n+1)\pi$, $t_j \geq T$. (Here again we are using the dot to denote inner product.) In terms of the real and imaginary parts x and y of ζ, the equations (3.5) may be written as (see equations (1.8))

$$\cos\omega y_{t_1} - \sin\omega x_{t_1} = 0 \qquad \text{and} \qquad \sin\omega y_{t_2} + \cos\omega x_{t_2} = 0,$$

which in turn imply

$$v_{t_1} = -\frac{\tau_1}{\pi}u \qquad \text{and} \qquad u_{t_2} = \frac{\tau_2}{\pi}v. \tag{3.8}$$

For $w = u$ or v we let $\bar{w}(s,t)$ stand for $w\big(\frac{1}{2}(t+s), \frac{1}{2}(t-s)\big)$, so that by (3.2), (3.3) and (3.4), $\bar{u}(s,t)$ and $\bar{v}(s,t)$ are periodic with period 2π in s.

Let $\gamma^2 = -\tau_1\tau_2/\pi^2$, $\gamma > 0$. Since $0 < \tau_1, -\tau_2 < 2\pi$ and $\tau_1 - \tau_2 = 2\pi$, it follows that $0 < \gamma \leq 1$. If we write equations (3.8) in integral form and take the bound (3.7) into account, we see that both u and v satisfy the equation

$$w(t_1,t_2) = \gamma^2 \int_{t_1}^{\infty}\int_{t_2}^{\infty} w(s_1,s_2)\,ds_2\,ds_1.$$

A standard argument based on this equation, the bound (3.7), and the continuity of u and v shows that they are both C^∞-functions on $\mathbf{R}^2$. The integral equation written in differential form is simply the well-known telegraph equation

$$w_{t_1t_2} - \gamma^2 w = 0,$$

so that in terms of the variables s and t, $\bar{u}$ and $\bar{v}$ satisfy

$$w_{tt} - w_{ss} = \gamma^2 w,$$

are periodic with period 2π in s and tend uniformly to 0 as $t \to \infty$. It is easily established by separation of variables that the solutions of the telegraph equation for which $\bar{w}(s+2\pi,t) = \bar{w}(s,t)$, with smooth periodic Cauchy data on $t=0$ are given in the upper half-plane $t \geq 0$ by

$$\bar{w}(s,t) = \sum_{k=0}^{\infty} F_k(s,t),$$

where

$$F_0(s,t) = A_0 e^{\gamma t} + B_0 e^{-\gamma t}$$

and for $k \geq 1$

$$F_k(s,t) = (A_k \cos ks + B_k \sin ks) \cos\big((k^2-\gamma^2)^{1/2}t\big) + (C_k \cos ks + D_k \sin ks) \sin\big((k^2-\gamma^2)^{1/2}t\big),$$

with the exception that when $\gamma = 1$

$$F_1(s,t) = A_1 \cos s + B_1 \sin s + (C_1 \cos s + D_1 \sin s)t.$$

When $\gamma < 1$ the uniform exponential decay of $\bar{w}(s,t)$ to 0 as $t \to \infty$ implies that there is a $K > 0$ such that for $t \geq K$, $k \geq 1$,

$$\big|A_k \cos\big((k^2-\gamma^2)^{1/2}t\big) + C_k \sin\big((k^2-\gamma^2)^{1/2}t\big)\big| = \left|\frac{1}{\pi}\int_0^{2\pi} \bar{w}(s,t)\cos ks\, ds\right| \leq \varepsilon.$$

This clearly implies that A_k and C_k are 0 for $k \geq 1$. Similarly, B_k and D_k are 0 for $k \geq 1$. The case $\gamma = 1$ is handled in a similar way. Thus $\bar{u}$ and $\bar{v}$ are of the form $A_0 e^{\gamma t} + B_0 e^{-\gamma t}$. But since $\bar{u}(s,t) \to 0$ as $t \to \infty$, we must have $A_0 = 0$, so that $\bar{u}(s,t) = B_0 e^{-\gamma t}$. Similarly, $\bar{v}(s,t) = B'_0 e^{-\gamma t}$, for some other constant B'_0. But then $|\zeta| = (B_0^2 + {B'}_0^2)^{1/2} e^{-\gamma t}$, so that $\bar{u}/|\zeta| = B_0/(B_0^2 + {B'}_0^2)^{1/2}$ and $\bar{v}/|\zeta| = B'_0/(B_0^2 + {B'}_0^2)^{1/2}$ are constants in the upper half-plane, and therefore $u/|\zeta|$ and $v/|\zeta|$ are indeed constant in some half-strip $n\pi \leq t_i < (n+1)\pi$, $t_j \geq T$. □

Theorem 3.2. *If $\mathscr{H} \in \mathrm{HP}^*(\mathbf{C})$ has a spiral or degenerate spiral singularity, then $\mathscr{H}$ is one of the $\mathscr{S}_{p,\alpha}$.*

Proof. Let $\mathscr{H}$ as in the hypothesis have a spiral singularity at p and let q be another singularity of $\mathscr{H}$ at minimum distance from p. The convexity of the characteristics emanating from p shows that q is a Riemann singularity. But then Theorem 2.4 shows that there is, in fact, no singularity at q. Thus p is the only singularity of $\mathscr{H}$. The conclusion follows from Theorem 3.3 together with the uniqueness of the nets resulting from the construction process described just before the discussion of double Riemann nets in Section 1.4. The case of a degenerate spiral singularity is handled similarly. □

Theorem 3.2 together with Theorem 2.5 gives us the following complete description of HP-nets with a single singularity.

Theorem 3.3. *If $\mathscr{H}$ is an* HP*-net regular in* $\mathbf{C}$ *except for a single singularity at p, then $\mathscr{H}$ is an $\mathscr{R}_{p,\alpha,\beta}$ or an $\mathscr{S}_{p,\alpha}$.*

4. More than one singularity

We begin with the following

Lemma 4.1. *Let D be a Jordan domain with C^1 boundary. Let θ be an* HP**-function on D with singularity set A. Let R denote the set of nondegenerate Riemann singularities of θ. Then*

$$\sum_{a\in R}\operatorname{dist}(a,\partial D)\le\frac{2\lambda(\partial D)}{\pi},\tag{4.1}$$

and

$$\sum_{a\in A\setminus R}\operatorname{dist}(a,\partial D)\le\frac{\lambda(\partial D)}{\pi}.\tag{4.2}$$

Proof. We begin with the following observation. Let Q be a characteristic quadrilateral of θ in $D\setminus A$ one of whose j-sides is $C=ab$, and let $E\subset Q$ be a simple arc joining the i-sides of Q. Assume, furthermore, that C is everywhere convex toward the inside of Q. Then

$$\lambda(E)\ge|\Delta\theta(ab)|\min\{|p-e|:p\in C,\ e\in E\}.\tag{4.3}$$

To see this we recall that if $p\in C$ is a point at which $D_i\theta$ exists and $z=z(s)$, $0\le s\le\sigma$, is the arc length parametrization of the j-arc F_C joining p to the other j-side of Q, then the (unsigned) radius of curvature $R(s)=1/|D_j\theta\big(z(s)\big)|$ is differentiable and satisfies $R'(s)=1$. Since $D_j\theta(p)$ exists for almost all $p\in C$ and the length of the i-arc joining p to a point $e\in E$ in $Q\setminus E$ is at least $|p-e|$, a simple calculus argument shows that (4.3) indeed holds.

It is clearly sufficient to show that if D' is any smoothly bounded subdomain of D for which $A\cap\partial D'=\emptyset$, then

$$\sum_{a\in R\cap D'}\operatorname{dist}(a,\partial D')\le\frac{2\lambda(\partial D')}{\pi}\quad\text{and}\quad\sum_{a\in(A\setminus R)\cap D'}\operatorname{dist}(a,\partial D')\le\frac{\lambda(\partial D')}{\pi}.\tag{4.4}$$

It follows from Theorems 2.3 and 2.4 that each nonbounding characteristic of the i-fan $F_{a,i}$ of $a\in R$ has a unique subarc which joins a to some point $\partial D'$. Denote the set of all such points of $\partial D'$ by $E_{a,i}$. It is clear that for each i all of the sets $E_{a,i}$ are disjoint. Let $\varepsilon>0$ and let G be an open j-arc in $N(a,\varepsilon)$ which joins the bounding characteristics of $F_{a,i}$. Then each point $p\in G$ is contained in an open subarc C of G which is the side of a characteristic quadrilateral in $D\setminus A$ for which the other j-side lies outside of $\overline{D'}$. By the initial observation the length of the part of $E_{a,i}$ in this quadrilateral is at least $\delta\big(\operatorname{dist}(a,\partial D')-\varepsilon\big)$, where δ is the change in θ on C. From this it follows that

$$\lambda(E_{a,i})\ge\gamma_{a,i}\operatorname{dist}(a,\partial D'),$$

$\gamma_{a,i}$ being the i-fan angle at a. Thus

$$\sum_{i=1}^{2} \sum_{a \in R \cap D'} \gamma_{a,i} \operatorname{dist}(a, \partial D') \leq 2\lambda(\partial D'),$$

so that the first bound of (4.4) follows upon taking into account that $\gamma_{a,1}+\gamma_{a,2} = \pi$.

The second inequality of (4.4) follows in essentially the same way. For a true spiral singularity a, by Theorem 3.1, θ coincides with $\sigma_{a,\alpha}$ for some α. As explained at the end of the discussion of spiral nets in Section 1.4, there are k-arcs C_k, $k = 1, 2$, contained in $N(a, \varepsilon)$ along which θ has change $\beta_{\alpha,k} > 0$ with $\beta_{a,1} + \beta_{a,2} = 2\pi$. If we replace the i-fan used above in the case of Riemann singularities with the set of j-arcs issuing from C_i and moving away from a, the argument proceeds just as in that case and we obtain that

$$\lambda(E_{a,i}) \geq \beta_{a,i} \operatorname{dist}(a, \partial D').$$

The same holds in the case of degenerate spiral singularities, with the difference that one of the $\beta_{a,k}$ vanishes. In any event, the second bound of (4.4) now follows immediately. □

Corollary 4.1. *Let D be a Jordan domain with C^1 boundary. Let θ be an* HP**-function on D with singularity set S. Then*

$$\sum_{a \in S} \operatorname{dist}(a, \partial D) \leq \frac{3\lambda(\partial D)}{\pi}.$$

Corollary 4.2. *An* HP**-function θ on* **C** *can have at most four singularities.*

Proof. Let $p_1, \ldots, p_n$ be distinct singularities of θ. If any of them is a spiral or degenerate spiral singularity then, by Theorem 3.2, $n = 1$. Thus we assume that they are all nondegenerate Riemann singularities. If $N(0, R)$ is a disk containing all of these singularities and such that θ is regular on $\partial N(0, R)$, then by Lemma 4.1

$$\sum_{k=1}^{n} (R - |p_k|) \leq 4R,$$

from which the desired conclusion follows upon letting $R \to \infty$. □

For each $p \in D$, a very similar argument shows that there is a neighborhood $U = U(D, p)$ of p such that any $\theta \in \mathrm{HP}^*(D)$ can have at most four singularities in U. This, together with the compactness principle for $\mathrm{HP}(D)$ (Proposition 1.9) and a straightforward diagonal argument, gives the following

Corollary 4.3 (Compactness principle for $\mathrm{HP}^*(D)$). *For any sequence $\{\theta_n\}$ of functions in $\mathrm{HP}^*(D)$ there exists a subsequence $\{\theta_{n'}\}$ and a $\theta \in \mathrm{HP}^*(D)$ such that $e^{2i\theta_{n'}}$ converges to $e^{2i\theta}$ in the topology of uniform convergence on compact subsets of $D\backslash A$, where A is the singularity set of θ.*

The remainder of this section is devoted to proving the following

Theorem 4.1. *An HP^*-net in all of $\mathbf{C}$ can have at most two singularities. If it has no singularities, then the corresponding θ is a constant. If it has one singularity then it is either one of the Riemann nets $\mathscr{R}_{p,\alpha,\beta}$ or one of the spiral nets $\mathscr{S}_{p,\alpha}$. If it has two singularities then it is either one of the double Riemann nets $\mathscr{D}_{p,q,\alpha,\beta,\gamma}$ or one of the degenerate double Riemann nets $\mathscr{F}_{p,q,\sigma,\beta,\gamma}$.*

The case of zero or one singularity is covered by Proposition 1.8 and Theorem 3.3. In light of Theorem 3.2 any HP*-net in $\mathbf{C}$ which has a spiral or a degenerate spiral singularity is an $\mathscr{S}_{p,\alpha}$. Thus, for the remainder of this section we shall assume that $\mathscr{H}$ is an HP-net which is regular in all of $\mathbf{C}$ except for distinct nondegenerate Riemann singularities $p_1, p_2, \ldots, p_n$, $2 \le n \le 4$. We shall show that the corresponding net is one of the double Riemann nets $\mathscr{D}_{p,q,\alpha,\beta,\gamma}$ or one of the degenerate double Riemann nets $\mathscr{F}_{p,q,\sigma,\beta,\gamma}$ described in Section 1.4. We begin with a number of preliminary observations and the introduction of some further notation and terminology that will facilitate the discussion.

It is easy to see that no singularity can be joined to itself by a characteristic. Indeed, if C is an i-characteristic which both begins and ends at one of the singularities p, then what we know about the i-fan of p implies that there is a subarc E of C whose endpoints are also joined by a j-arc E' and such that the domain between E and E' contains no singularities, an obvious impossibility. Thus, each characteristic either emanates from some singularity and has infinite length, extends from one singularity to another in finite length or extends infinitely in both directions. If p is a singularity, the fan of i-characteristics emanating from p will be denoted by $F_i(p)$. In addition, $R_i(p)$ and $L_i(p)$ will denote the bounding characteristics of $F_i(p)$ on the right and left sides of this fan (with respect to movement away from p). For each pair (p,i), where p is a singularity and the angle of $F_i(p)$ is positive, we choose a fixed nonbounding characteristic $M_i(p)$ of $F_i(p)$ and let $z(s)$, $0 < s < L$, be the arc length parametrization of $M_i(p)$. For all sufficiently small s there is an open j-arc $C_j(p,s)$ containing $z(s)$ and joining a point of $R_i(p)$ to a point of $L_i(p)$. This arc is everywhere concave towards p. Because of this, in light of Theorems 2.3 and 2.4, all characteristics of $F_i(p)$, with the possible exception of $R_i(p)$ and $L_i(p)$, can be extended indefinitely away from p (without terminating at any other singularity), so that $L = \infty$ and $C_j(p,s)$ is well defined for all $s > 0$. (The arbitrariness of the choice of $M_i(p)$ will be of no consequence.)

Lemma 4.2. *Let p be a singularity of an HP-function θ on $\mathbf{C}$ which has only nondegenerate Riemann singularities. Let C be an i-arc of θ with arc length* *parametrization* $z(s)$, $0 < s < L < \infty$. *Then there exist* $0 = s_0 < s_1 < \cdots < s_m = L$ *such that for* $1 \leq i \leq m-2$, $z\big((s_i, s_{i+1})\big)$ *is either a straight line segment or* $C_j(a,s)$ *for some singularity* a *and some* $s > 0$. *For* $i = 0$ *or* $m-1$, $z\big((s_i, s_{i+1})\big)$ *is either a straight line segment or a subarc of some such* $C_j(a,s)$.

Proof. The only way this could fail to be so would be for there to be infinitely many disjoint arcs of the form $C_j(a,s)$ on C. But, since the associated fan angles have at most four different positive values, this is inconsistent with the fact that $\theta(z)$ is continuous on C and possesses limits as z approaches the endpoints of C. □

The proof that the only possible nets are the (nondegenerate and degenerate) double Riemann nets is broken down into two main cases according to whether or not some pair of (distinct) singularities are joined by a characteristic; each of these cases is further broken down into subcases. In each instance the proof reduces to showing that there are only two singularities and that a certain characteristic arc is a straight line segment.

Case I. There are singularities p, q joined by a characteristic i-arc C. We begin by noting the following simple consequences of this hypothesis.

(1) There is a neighborhood N of C such that θ is regular in $N - \{p,q\}$.

This is obvious from the definition of isolated singularity and the regularity of θ along C.

(2) C is the only i-arc joining p to q.

Assume, to the contrary, that there is another one C'. Then from what we know about the structure of Riemann singularities, C and C' both belong to both $F_i(p)$ and $F_i(q)$. But then the orthogonal characteristics connecting C and C' inside these fans must have nonvanishing curvature, and have their concave side towards both p and q, which is impossible.

(3) $C \in \{R_i(p), L_i(p)\} \cap \{R_i(q), L_i(q)\}$.

If C were neither $R_i(p)$ nor $L_i(p)$, then from the structure of Riemann singularities it follows that there are j-characteristic arcs forming an angle of π at q and which are strictly concave towards p. But this contradicts Theorem 2.4.

(4) C is $R_i(p)$ and $R_i(q)$ or $L_i(p)$ and $L_i(q)$.

If either of the fans $F_i(p)$ or $F_i(q)$ is made up of a single characteristic then there is nothing to prove, so we assume that this is not the case. Say that C is $R_i(p)$. If it were $L_i(q)$, then it is easy to see that there would be two distinct i-characteristics joining p and q, which contradicts point (1) above.

We now begin the analysis of Case I. For the sake of definiteness we assume C is $R_i(p)$ and $R_i(q)$. We also assume for the time being that angles β, γ of the fans $F_i(p)$ and $F_i(q)$ satisfy $0 < \beta, \gamma < \pi$; we will discuss the degenerate cases separately. The accompanying Figure 3 has been included to facilitate understanding the following discussion. We shall show that the net is $\mathscr{F}_{p,q,+,\beta,\gamma}$, so that this figure is to be compared to Figure 2. Let $e \in \{p,q\}$. We consider that $C_j(e,s)$ is oriented so that movement in the positive direction along it coincides

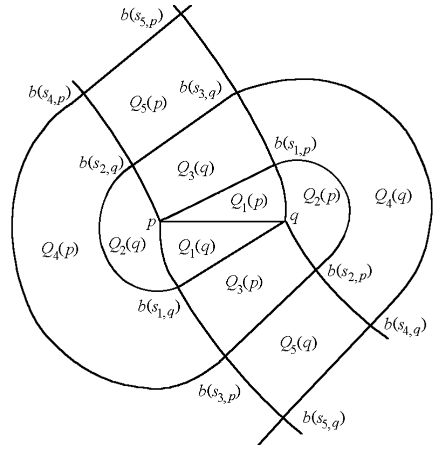

Figure 3.

with increasing θ; the initial and terminal points of its closure will be denoted by $a_e(s)$ and $b_e(s)$; for completeness we set $a_e(0) = b_e(0) = e$. For small s, $a_e(s)$ lies on the curve C. Let $\{e, f\} = \{p, q\}$. We follow the movement of $C_j(e, s)$ as s varies from 0 to ∞. On an initial interval $[s_{0,e}, s_{1,e}]$, $s_{0,e} = 0$, $a_e(s)$ traces out C from e to f, while $b_e(s)$ traces out an arc of $L_i(e)$. Next, on an interval $[s_{1,e}, s_{2,e}]$, $a_e(s)$ is stable at f while $b_e(s)$ traces out a j-arc which is concave towards f. Then, on an interval $[s_{2,e}, s_{3,e}]$, $a_e(s)$ runs along the arc $b_f([s_{0,f}, s_{1,f}])$. Continuing in this manner we see that there are two increasing sequences $\{s_{k,e}\}$, $e = p, q$, such that on $[s_{k+2,e}, s_{k+3,e}]$, $a_e(s)$ runs along the arc $b_f([s_{k,f}, s_{k+1,f}])$ of $L_i(f)$. The arcs $C_j(e, s)$, $s \in [s_{k+2,e}, s_{k+3,e}]$ and $C_j(f, s)$, $s \in [s_{k,f}, s_{k+1,f}]$ lie on opposite sides of $L_i(f)$. In addition, they are both concave towards f. From this, together with Theorem 2.3 it follows that $L_i(p)$ and $L_i(q)$ are both infinite (that is, as we move out from p and q along them we never encounter singularities.

For $k \geq 1$, let $Q_k(e)$ denote the (closed) characteristic quadrilateral with vertices $b_e(s_{k-1,e})$, $a_e(s_{k-1,e})$, $a_e(s_{k,e})$ and $b_e(s_{k,e})$. These are all *bona fide* characteristic quadrilaterals with the exception of $Q_1(e)$, $Q_2(e)$ which have degenerate sides at ee and ff, respectively. The fact that there are no singularities on any of the $C_j(e, s)$, $s > 0$, nor on the $L_i(e)$ means that the only singularities in any of the $Q_k(e)$ are the points p and q. It is easy to see that $|b_e(s) - a_e(s)| \geq K\lambda\big(C_j(p, s)\big) \to \infty$ so that the union of the Q_k is the entire plane. Thus indeed the only singularities are p and q. Finally, since none of the points of C is joined to a singularity by a j-characteristic, C is a straight line segment. Comparison with the construction of the degenerate double Riemann nets now shows that this net is indeed $\mathscr{F}_{p,q,+,\beta,\gamma}$.

We still have to discuss those degenerate situations in which the angles of one or both of the fans $F_i(p)$ and $F_i(q)$ is 0 or π. By interchanging p and q, if necessary, it is sufficient to consider the following possibilities:

(A) $0 < \beta < \pi$, $\gamma = 0$.
(B) $0 < \beta < \pi$, $\gamma = \pi$.
(C) $\beta = \gamma = \pi$.
(D) $\beta = 0$, $\gamma = \pi$.
(E) $\beta = \gamma = 0$.

Cases A–D. Cases A–D are very similar to the case in which $0 < \beta, \gamma < \pi$, which we have just treated, the only difference being that some of the quadrilaterals $Q_k(p)$, $Q_k(q)$ degenerate into characteristic arcs or do not even appear, and the proof proceeds with minor alterations, the details of which we leave to the reader. In Case A none of the $Q_k(q)$ appear, because the fact that $\gamma = 0$ causes the $C_j(q,s)$, in effect, to degenerate to a point. Nonetheless, the $Q_k(p)$ cover the plane, and allow us to reach the desired conclusion as before. In Case B the degeneracy of $C_i(q,s)$ makes $Q_2(p)$, $Q_4(q)$, $Q_6(p)$, $Q_8(p), \ldots$ degenerate into j-arcs, but again the remaining quadrilaterals cover the plane in the necessary fashion. In Case C it is all the even numbered quadrilaterals that degenerate into j-arcs because of the degeneracy of both $C_i(p,s)$ and $C_i(q,s)$, but once again the remaining quadrilaterals cover the plane in the necessary manner. Finally, in Case D (see Figure 4) all of the $Q_k(p)$ degenerate (more precisely, $Q_2(p)$ degenerates to the point q and all the other $Q_k(p)$ degenerate to arcs) because the $C_j(p,s)$ and the $C_i(q,s)$ degenerate to points; again the $Q_k(q)$ cover the plane. In all four of these cases the only singularities are p and q and the i-arc C is a straight line, so that the net is seen to be $\mathscr{F}_{p,q,+,\beta,\gamma}$.

Case E. The treatment of this case differs substantially from that of the other four, since none of the quadrilaterals actually appears because of the degeneracy of both $C_j(p,s)$ and $C_j(q,s)$. Let M_e be any fixed nonbounding characteristic of $F_j(e)$, and let $z_e(s)$, $0 < s < \infty$ be an arc length parametrization of M_e. (The characteristics M_e are infinite by Theorems 2.3 and 2.4 as pointed out above.) We apply the decomposition of Lemma 4.2 to M_e. The first subarc must be a straight line segment, since otherwise it would have to be a subarc of a $C_j(f,s)$ or $C_j(e,s)$, which is inconsistent with the hypothesis that the angle of the j-fan at both e and f is 0. Let $s_1 > 0$ be such that $G = z_e\big((0,s_1)\big)$ is a straight line segment. It follows that for small positive s, there is a single i-characteristic N_s, which is a simple closed curve consisting of semicircles of radius s (which are $C_j(p,s')$, $C_j(q,s')$ for appropriate values of s') joined by translates of C. Let σ be a fixed such s. Let $w(t)$, $0 \le t \le L$, be an arc length parametrization of N_σ traversed once starting at and ending on G. Let G_t denote the associated translates of G along N_σ. We claim that θ is regular on all of these G_t. If not, let σ', where $\sigma < \sigma' < s_1$, be the smallest number for which a singularity is encountered on a translate of $z_e((0,\sigma'])$. In the event that there is more than one such singularity we consider the one lying on the G_t with the smallest possible value of $t \in (0,L)$. It is clear that for all $s < \sigma'$, there is an N_s as described above.

It is also clear that for this singularity the angle of the j-fan is π, since it lies in the interior of one of the straight j-arcs G_t. But this means that points $z_e(s)$, for $s < \sigma'$ near σ' are joined by i-characteristics to points $z_e(s)$, for $s > \sigma'$ near σ', which is inconsistent with the existence of the simple closed i-characteristics N_s, of the kind described above, for $s < \sigma'$. Now assume that in the decomposition of M_e according to Lemma 4.2 we have a second interval, that is, $z_e\big((0,s_1)\big)$ is a straight line segment, but $z_e\big((s_1,s_2)\big) = C_j(c,\tau)$, for some singularity c, and some $\tau > 0$. But then the argument we used before will again show that there can be no singularities on the translates of $z_e\big((0,s_2)\big)$ around N_σ away from c, that is, in the direction of the side towards which $z_e\big((s_1,s_2)\big)$ is convex. But this contradicts the existence of the singularity c. Thus no such s_1 exists, and the M_e are both straight lines on all of whose translates θ is regular. From this it follows the j-characteristic through each point of C is a straight line (infinite in both directions) so that C is itself straight. From the straightness of the M_e it follows that the net in question is $\mathscr{F}_{p,q,+,0,0} = \mathscr{F}_{p,q,-,0,0}$.

Case II. No two singularities are joined by a characteristic arc.

We assume that there are at least two singularities and that no two of them are joined by a characteristic. Let p be a singularity. It follows that the bounding characteristics of the fans $F_i(p)$, $i = 1, 2$, are all free of singularities, so that when we apply the decomposition of Lemma 4.2 to each of them the initial piece is always a line segment. It is easy to see that all of these initial segments must be finite. Assume, to the contrary, that all the characteristics making up $F_i(p)$ are rays. It is enough to show that all translates of $R_i(p)$ along $L_j(p)$ (and of $L_i(p)$ along $R_j(p)$) are free of singularities, since then $L_j(p)$ and $R_j(p)$ will be rays also, from which the desired conclusion follows immediately. Say, to the contrary, that as we translate $R_i(p)$ along $L_j(p)$ we encounter a first singularity at q. It is clear that the angle of the j-fan at q must be 0. Then $L_i(q)$ is a ray. But this cannot be because $L_i(q)$ contains $C_i(p,\tau)$ for some τ, and these curves are nowhere straight. Thus, indeed, all of the initial segments are finite.

We examine translates of these initial segments; to be specific we consider the initial line segment S_R of $R_i(p)$. It is clear that the initial straight line segment S_L of $L_i(p)$ has the same length σ as S_R. Let $\{T,U\} = \{R,L\}$. Let $z_U(s)$, $0 < s < \infty$, be the arc length parametrization of $U_j(p)$. Let s_T be the smallest $s > 0$ for which the translate $S_T(s)$ of S_T along $U_j(p)$ to $z_U(s)$ has a singularity, where $s_T = \infty$ if no such s exists. If s_T is finite it is clear that the singularity encountered must be at distance $\frac{1}{2}\sigma$ from $U_j(p)$, and that the angle of its j-fan must be 0 with this j-fan consisting of a single characteristic extending back towards $F_i(p)$ at distance $\frac{1}{2}\sigma$ from $U_j(p)$. From this it follows that we cannot have both s_R and s_L finite, since we would then have two singularities joined by a characteristic. We bear these preliminary comments in mind as we deal with the various subcases into which Case II is divided.

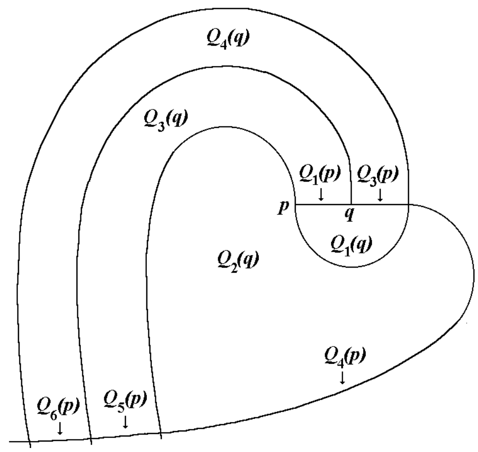

Figure 4.

Case II-1. There are no singularities with fan angles 0 and π.

It follows from the preceding comments that all four initial segments are of finite length and that all their translates are free of singularities. Consider the initial segments S_R and S_L on $R_i(p)$ and $L_i(p)$, respectively. In reference to the decomposition of Lemma 4.2, the piece of $R_i(p)$ which follows the straight line segment S_R is of the form $C_i(q,\sigma)$ for some singularity q. In light of the underlying assumption the angles of both fans at q are strictly between 0 and π, we can assume this arc curves away from the fan $F_i(p)$ (that is, that as we move away from p along this piece of $R_i(p)$, the argument of the tangent decreases), since if it had the opposite sign we could work with S_L and $L_i(p)$ instead of S_R and $R_i(p)$ and have it curve away from the fan. The remainder of the discussion of this case is very similar to that of the nondegenerate Case I. Indeed, let e be p or q. For $0 < s < \infty$, we let $E(e,s)$ be the open i-arc made up of the $C_i(e,s)$, its endpoint $c(s)$ on $L_j(e)$ and the translate of S_R one of whose endpoints is $c(s)$. Furthermore, we let $E(e,0)$ be the translate of S_R one of whose endpoints is e and which lies to the left of $L_j(e)$ (when movement along this characteristic is away from e), so that $E(e,s)$ is continuous for $s \geq 0$. (Note that $E(p,0) = S_R$.) The desired conclusion is obtained by following the movement of $E(p,s)$ and $E(q,s)$ as s varies from 0 to ∞. Let $\{e,f\} = \{p,q\}$. We consider that $E(e,s)$ is oriented so that the positive direction along it coincides with that of nondecreasing θ and we denote the initial and terminal points of its closure by $a_e(s)$ and $b_e(s)$. We let $s_{0,e} = 0$. Then on an interval $[s_{0,e}, s_{1,e}]$, $b_e(s)$ traces out an arc of $L_j(f)$, which ends at f. Then we define $s_{k,e}$, $k \geq 2$, inductively by $b_e(s_{e,k}) = a_e(s_{k-2,f})$. Note that $b_e(s) = f$, for $s \in [s_{1,e}, s_{2,e}]$; on this interval the arcs $E(e,s)$ are initial arcs of the characteristics of the i-fan at f.

For $k \geq 1$, let $Q_k(e)$ denote the (closed) characteristic quadrilateral with vertices $b(s_{k-1,e})$, $a(s_{k-1,e})$, $a(s_{k,e})$ and $b(s_{k,e})$. These are all *bona fide* characteristic quadrilaterals, except for $Q_2(e)$ which has the degenerate side ff. The fact that there are no singularities on any of the $E(e,s)$, $s>0$, nor on the $R_j(e)$, $L_j(e)$ means that the only singularities in any of the $Q_k(e)$ are the points p and q. It is easy to see that $|b_e(s)-a_e(s)| \geq K\lambda\big(E(p,s)\big) \to \infty$ so that the union of the Q_k is the entire plane. Thus indeed the only singularities are p and q. Finally, it is clear that none of the points of $c([s_{0,e}, s_{1,e}])$ is joined to a singularity by an i-characteristic, so that this arc is a straight line segment. Comparison with the construction of the double Riemann nets in Section 1.4 now shows that in this case the net is $\mathscr{D}_{p,q,\beta,\gamma,\alpha}$, where β is the angle of $F_i(q)$, γ is the angle of $F_i(p)$ and α is the inclination of the initial segment of $L_j(p)$. This finishes the discussion of Case II-1.

Case II-2. There is a singularity with fan angles 0 and π at p. This case is further divided into the following subcases, where the angles of $F_i(p)$ and $F_j(p)$ are 0 and π, respectively.

(A) All translates of the initial segments at p are free of singularities.

(B) The midpoint of some translate of an initial segment of $L_i(p)=R_i(p)$ is a singularity of θ.

(C) The midpoint of some translate of the initial segments of $L_j(p)$ and $R_j(p)$ is a singularity of θ.

Case A. The following discussion is illustrated in Figure 5. As in Case II-1, all initial segments are of finite length and all their translates are free of singularities. The piece of $R_j(p)$ which follows its initial line segment S_R is of the form $C_j(q,\sigma)$, $\sigma>0$, for some singularity q. We may assume that as one moves along $R_j(p)$ away from p, θ decreases; if this is not the case, one works with $L_j(p)$ instead of $R_j(p)$. We assume that the angles of the fans at q are strictly between 0 and π; the minor modifications required if this is not so (i.e., the angle of $F_i(q)$ is π) are left to the reader. By assumption $R_i(p)=L_i(p)$ is free of singularities. Let $l>0$ be the length of S_R. There are translates of S_R (which are all free of singularities) with endpoints on $R_i(p)$ extending outward on both sides of $R_i(p)$. Let $z(s)$, $s\geq 0$, be the arc length parametrization of $R_i(p)$, and let $E(s)$ denote the open line segment j-arc of length $2l$ and with midpoint $z(s)$. We see that starting with $E(0)$, $E(s)$ moves towards q with one endpoint on $L_i(q)$, and then on some interval $[\tau,\tau']$ has an endpoint at q while it rotates around q in the counterclockwise direction forming initial segments of the characteristics in $F_j(q)$, until $E(\tau')$ forms the initial segment of $L_j(q)$. Thereafter it has one of its endpoints on $R_i(q)$. We now define $E'(s)$, $s>\tau'$, to be the j-arc consisting of $E(s)$, its endpoint on $R_i(q)$ together with the $C_j(q,t)$ which shares this endpoint. For $s=\tau'$ we let $E'(s)$ consist of $E(\tau')$ alone (since the corresponding $C_j(q,t)$ tends to the single point q as t tends to 0). It is clear that there are no singularities on any of the $E'(s)$ since there are none on the $C_j(q,t)$, on $R_i(q)$ or on the $E(s)$. It follows from the definition of the $E'(s)$ that both of their endpoints are on $L_i(q)$. This, together with the fact that $L_i(q)$ is free of singularities, implies that

$$L_i(q) \cup \left(\bigcup_{s>\tau'} E'(s) \right) \cup \left(\bigcup_{0 \le s \le \tau'} E(s) \right) \cup A = \mathbf{C},$$

where A denotes the semidisk consisting of those translates of S_R which are the initial segments of the characteristics of $F_j(p)$. But then the only singularities of θ are p and q, so that in particular there are no singularities on any of the translates of $z\big((0,\tau)\big)$, which means that this arc is a straight line segment. Hence, this net is $\mathscr{D}_{p,q,\pi,\beta,\alpha}$, where β is the angle of $F_j(q)$, and α is the inclination of the initial segment of $L_i(q)$.

Case B. It is easy to see that this case is identical to the situation which arises in the foregoing Case A when the angle of $F_i(q)$ is π if we interchange the roles of p and q, and i and j.

Case C. To be specific, assume that the midpoint m of the translate of the initial segment S_R of $R_j(p)$ along $L_i(q) = R_i(q)$ is the singularity q. The line joining m to q is the initial arc of $L_i(q) = R_i(q)$; by the defining assumption of Case II one encounters no singularities as one moves along $R_i(q)$ away from q. Let $z(s)$, $s \ge 0$, be the arc length parametrization $R_i(q)$, and let $E(s)$ denote that translate of S_R whose midpoint is $z(s)$. Let $\tau > 0$ be such that $E(\tau) = S_R$. We assign the (continuously varying) initial point $a(s)$ of $E(s)$ so that $a(\tau) = p$. Then $a(s)$ moves back along $R_i(p)$ from $a(0)$ to p, remains at p while $E(s)$ rotates through an angle of π about p in the counterclockwise direction then moves along $R_i(p)$ for $s > \tau' = \tau + |p - m|\pi$. As $s > \tau'$ increases $a(s)$ traverses a translate of $z\big((0,\tau)\big)$, a semicircle $C_j(q,t)$, another translate of $z\big((0,\tau)\big)$, a semicircle $C_j(p,t')$, and so on. From this it is easy to see that the $E(s)$, $s \ge 0$, together with $R_j(p)$ (including p itself) make up all of $\mathbf{C}$. But then once again the only singularities of θ are p and q, so that in particular none of the points of $z\big((0,\tau)\big)$ is joined to a singularity by a j-characteristic, and therefore this arc is a line segment. Hence, the net is $\mathscr{D}_{p,q,\pi,\pi,\alpha}$, where α is the inclination of the initial segment of $R_i(q)$.

5. Boundary limits and constant principal strain mappings

We begin with the following

Lemma 5.1. *Let θ be a nonconstant* HP*-function on a simply connected domain D. Then there is an open interval I such that for all $M \in I$ the $(M,1)$-mappings corresponding to θ are not one-to-one in D.*

Proof. It follows from the hypothesis that there is a point z_0 at which θ is differentiable and at which its gradient does not vanish. Assume for the moment that $\kappa_0 = D_1\theta(z_0) \neq 0$. Without loss of generality we may also assume that $\theta(z_0) = 0$. Let $z(s)$, $|s| < s_0$, $z(0) = z_0$, be an arc length parametrization of an open arc of the 1-characteristic of θ through z_0, so that $z'(s) = e^{\tau(s)i}$ with $\tau(s) = \theta\bigl(z(s)\bigr)$. Then $\tau(s) = \kappa_0 s + o(s)$, as $s \to 0$. Let f be an $(M,1)$-mapping corresponding to the HP-function θ, and let $w(s) = f\bigl(z(s)\bigr)$. Again without loss of generality we can assume that $w'(0) > 0$. It then follows from the compatibility equations (1.1) and the fact that f is an $(M,1)$-mapping that

$$w'(s) = Me^{M\tau(s)i} = Me^{M\kappa_0 si + Mo(s)} = Me^{M\kappa_0 si}\bigl(1 + Mo(s)\bigr),$$

so that for $|t| \le 4\pi/M|\kappa_0|$

$$\begin{aligned} w(t) - w(0) &= \int_0^t Me^{M\kappa_0 si}\bigl(1 + Mo(s)\bigr)\,ds = \frac{(e^{M\kappa_0 ti} - 1)}{\kappa_0 i} + M^2 o\biggl(\Bigl(\frac{4\pi}{M\kappa_0}\Bigr)^2\biggr) \\ &= \frac{(e^{M\kappa_0 ti} - 1)}{\kappa_0 i} + o(1), \qquad \text{as } M \to \infty. \end{aligned}$$

From this together with the fact that on 2-arcs f does not alter arc length and changes curvature by a factor of $1/M$ (as follows from the second equation in (1.1)) it is clear that for sufficiently large M, f is not one-to-one in D. If we had $D_2\theta(z_0) \ne 0$, the analogous procedure would give us noninvertible $(1,M)$-mappings f for all sufficiently large M; the desired mappings are then given by $(1/M)f$. □

Definition 5.2. We say that θ_0 is the *nontangential limit* of $\theta \in \mathrm{HP}^*(D)$ at $p \in \partial D$ if there exists a $\xi \in \mathbf{R}$ such that given any $\varepsilon > 0$ there is a $\delta > 0$ with the property that the set

$$U = \bigl\{p + z : |\arg\{z/e^{i\xi}\}| < \tfrac{1}{2}\pi - \varepsilon\bigr\} \cap N'(p,\delta)$$

consists entirely of regular points of θ in D and the limit as z tends to p of some single-valued branch of $\theta(z)$ in U is θ_0.

Theorem 5.1. *Let D be a domain bounded by a C^1 Jordan curve. If θ is an* HP**-function on D, then θ has nontangential limits at almost all points of ∂D.*

Proof. Let θ be as in the hypothesis and let S denote the set of singularities of θ. We shall use $(M,1)$-mappings f for which $\theta_f = \theta$. To do so, however, we must work with a simply connected subdomain of D, which we obtain by removing from D small pieces which join the points of S to ∂D. Every point of ∂D is contained in an almost straight closed subarc C such that there is a circular arc C' lying in D which joins the end points of C and contains no point of S. Obviously, it is enough to prove the theorem for the domain bounded by $C \cup C'$, so that we shall assume that D has this form. Clearly, there is a bi-Lipschitz homeomorphism Λ of $\bar{D}$ onto the rectangle R with vertices -2, 2,

$2+2i$, $-2+2i$ such that $\Lambda(S)$ is contained in the rectangle $\frac{1}{2}R$ and such that the sum of the imaginary parts of all of the points in $\Lambda(S)$ is at most 1. That the latter is possible follows immediately from Corollary 4.1. Let $z_k = x_k + y_k i$ be an enumeration of the points of $\Lambda(S)$ with k starting at 1. Let $\varepsilon > 0$ and let $q_{k,\varepsilon}$ be the real function on $[-2,2]$ whose graph consists of the segments $[-2, x_k - \varepsilon/2^k]$, $[x_k - \varepsilon/2^k, z_k]$, $[z_k, x + \varepsilon/2^k]$, $[x + \varepsilon/2^k, 2]$, and let q_ε be the sum of all of the $q_{k,\varepsilon}$. We have $0 \le q_\varepsilon(x) \le 1$, $x \in [-2,2]$. The domain R_ε bounded by the graph of q_ε and $\partial R \backslash [-2,2]$ is a Jordan domain contained in $R \backslash \Lambda(S)$ whose boundary contains all of ∂R except for a set of 1-dimensional measure at most ε. Let $\delta > 0$. Since Λ is bi-Lipschitz, for sufficiently small ε, $D^* = \Lambda^{-1}(R_\varepsilon) \subset D \backslash S$ is a Jordan domain whose boundary is rectifiable and contains all of ∂D except for a set of 1-dimensional measure less than δ. Since $\delta > 0$ is arbitrary it is sufficient to show that (any continuous branch of) θ on D^* has nontangential boundary limits a.e. on ∂D^*. (Implicit in this construction of D^* was the assumption that $S \ne \emptyset$; if $S = \emptyset$ we let $D^* = D$.)

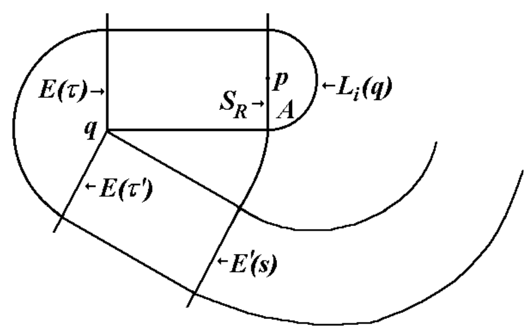

Figure 5.

For each positive $M \ne 1$, let f_M be an $(M,1)$-mapping of D^* corresponding to θ. Let $v(s)$, $0 \le s \le L$, $v(0) = v(L)$ be a positively oriented arc length parametrization of the simple closed curve ∂D^*. Obviously, v is differentiable on a subset B of $(0,L)$ of measure L. It follows from the manner in which D^* was constructed that there is a K such that given any $\eta > 0$, for any two $\xi_1, \xi_2 \in (0,L)$, $v(\xi_1)$ and $v(\xi_2)$ can be joined in D^* by an arc of length at most $K|\xi_2 - \xi_1| + \eta$. (Note that the domain R_ε used to construct D^* in the preceding paragraph has this property with $K = 1$.) Because of this and the fact that f_M is Lipschitz, f_M has a unique continuous extension to $\overline{D^*}$. Clearly, $\psi_M(s) = f_M\big(v(s)\big)$ is Lipschitz continuous with constant $K \max\{M, 1/M\}$, so that it is differentiable on a set $A_M \subset (0,L)$ of measure L. Let P denote the set of positive rationals different from 1, and let $A = \bigcap_{M \in P} A_M$. Let $\xi \in B \cap A$. It is enough to show that the nontangential limit of θ exists at $v(\xi)$ since $\lambda\big(\partial D \backslash v(B \cap A)\big) = 0$. Without loss of generality we may make the normalizing assumption that $v(\xi) = 0$

and $\arg v'(\xi) = 0$. By appropriately choosing the constants of integration we may also stipulate that $\psi_M(\xi) = 0$.

Let H denote the open upper half-plane. All $(M,1)$-mappings in H have a unique continuous extension to $\overline{H}$ which is Lipschitz continuous with constant $\max\{M, 1/M\}$, and in dealing with such mappings we implicitly consider that they have been so extended. Although $\theta_t(z) = \theta(tz)$ is not defined in all of H, for any compact $X \subset H$, θ_t is defined in X for all sufficiently small t, so that it is meaningful to talk about local uniform convergence in H of θ_t as $t \to 0$. Indeed, we will have established the desired conclusion if we can show that θ_t converges in H to a constant in this sense. To do so we show first that if for some $\{t_k\}$ approaching 0, θ_{t_k} tends locally uniformly to χ, then χ must be a constant χ_0 which satisfies

$$4\cos^2\chi_0 + \sin^2\chi_0 = \bigl(\psi_2'(\xi)\bigr)^2. \tag{5.2}$$

Let $\{t_k\}$ be such a sequence. By replacing it with a subsequence, if necessary, we may assume that for each $M \in P$ the $(M,1)$-mappings $f_M(t_k z)/t_k$ converge locally uniformly to an $(M,1)$-mapping g_M corresponding to χ. Since $\xi \in B \cap A$, it follows that the g_M are linear on $\partial H = \mathbf{R}$. Assume that χ were not a constant. By Lemma 5.1 there is a $\mu \in P$ such that g_μ is not one-to-one on H. For convenience we assume that $\mu > 1$; the opposite case is dealt with in the same way apart from minor notational differences. By general injectivity criteria for quasi-isometries developed by John [J3],

$$\tau = \inf\{M \in P \cap (1,\infty) : g_M \text{ is noninjective on } H\} > 1.$$

Let $\{M_k\}$ be a sequence in $P \cap (\tau, \tau+1)$ approaching τ and such that g_{M_k} is noninjective in H. That is, for each k there are distinct points $a_k, b_k \in H$ for which $g_{M_k}(a_k) = g_{M_k}(b_k)$.

It also follows from results of John [J3] that if g is an $(M,1)$-mapping in $N(a,r)$, then g is one-to-one in $N(a, r/M)$. Thus $\operatorname{Im} a_k$, $\operatorname{Im} b_k \le (2+\tau)|b_k - a_k|$, since if, for example, $\operatorname{Im} b_k > (2+\tau)|b_k - a_k|$, then g_{M_k} would be one-to-one in $N\bigl(b_k, (2+\tau)|b_k - a_k|/(1+\tau)\bigr)$, which contains both a_k and b_k. Let $a_k' = (a_k - \operatorname{Re} a_k)/|b_k - a_k|$ and $b_k' = (b_k - \operatorname{Re} a_k)/|b_k - a_k|$. Then $\{a_k'\}$ and $\{b_k'\}$ are bounded. By replacing $\{M_k\}$ by an appropriate subsequence we may assume that a_k' and b_k' converge to a and b, respectively. Clearly, $|b - a| = 1$. Let

$$h_k(z) = \bigl(g_{M_k}(|b_k - a_k|z + \operatorname{Re} a_k) - g_{M_k}(\operatorname{Re} a_k)\bigr)/|b_k - a_k|.$$

Then h_k is an $(M_k,1)$-mapping of H associated with the HP-function $\chi_k(z) = \chi(|b_k - a_k|z + \operatorname{Re} a_k)$ for which $h_k(a_k') = h_k(b_k')$ and $h_k(0) = 0$. Again, by replacing $\{M_k\}$ with an appropriate subsequence we may assume that there is an HP-function χ' on H such that $e^{2i\chi_k}$ converges to $e^{2i\chi'}$ locally uniformly on

H and that the h_k converge to a $(\tau,1)$-mapping h of H associated with χ'; the convergence of the h_k is locally uniform on $\overline{H}$. But since for $M \in (1,\tau)$, $(M,1)$-mappings associated with χ are injective, so are those associated with χ_k and therefore also those associated with χ'. If l_M denotes the $(M,1)$-mapping associated with χ' for which $l_M(i) = h(i)$ and for which $\theta_{l_M}(i) = \theta_h(i)$ and $\phi_{l_M}(i) = \phi_h(i)$, then $l_M \to h$ as $M \to \tau$ uniformly on compact subsets of H. Since the l_M are injective on H for $M \in (1,\tau)$, so is h. On the other hand, $h(a) = h(b)$, since $h_k(a'_k) = h_k(b'_k)$. Thus, $a, b \in \partial H$. However, since each h_k is linear on ∂H, so is h. Since $a \neq b$, this means that h is a constant on ∂H. But this cannot be since it would imply that the image under h of the strip $0 < \operatorname{Im} z < 1$, which obviously has infinite area, is contained in a disk of radius τ. Our assumption that χ is not a constant has therefore led to a contradiction, so that χ is indeed a constant χ_0. That χ_0 satisfies equation (6.2), follows from the fact that g_2 is a linear $(2,1)$-mapping of H.

Finally, we show that the θ_t do in fact converge as t tends to 0. If they did not, the compactness principle (Proposition 1.9) together with the preceding paragraphs, would imply that there are two sequences $\{t_k\}$ and $\{t'_k\}$ for which the corresponding sequences $\{\theta_{t_k}\}$ and $\{\theta_{t'_k}\}$ converge to constants $\chi_1 < \chi_2$, both of which satisfy equation (5.2) above. Let $\chi_3 \in (\chi_1, \chi_2)$ be a number which does not satisfy that equation. Let $z_0 \in H$. Since $\theta(t_k z_0)$ and $\theta(t'_k z_0)$ tend to χ_1 and χ_2, respectively, it follows from the intermediate value theorem that for all sufficiently large k there is an s_k between t_k and t'_k for which $\theta(s_k z_0) = \chi_3$. But then by the compactness principle $\{\theta_{s_k}\}$ has a subsequence which converges to an HP-function χ for which $\chi(z_0) = \chi_3$. This contradicts what was established in the preceding paragraph. □

We now relate the class $\mathrm{cps}^*(D, m_1, m_2)$ to $\mathrm{HP}^*(D)$. We define the *dilatation* $\mu(p) = \mu_H(p)$ of an HP-net H (for which the two families of characteristics have been numbered) as follows. If p is a Riemann singularity with k-fan angles α_k, $k = 1, 2$, then $\mu(p) = \alpha_2/\alpha_1$, when $\alpha_1, \alpha_2 \neq 0, \frac{1}{2}\pi, \pi$, and is undefined otherwise. If p is a spiral singularity and α_1 is the positive acute angle between the 1-characteristics and the rays emanating from p, then $\mu(p) = \cot^2 \alpha_1$, when $\alpha_1 \neq 0, \frac{1}{2}\pi, \pi$, and is undefined otherwise. There is an (m_1, m_2)-mapping f in a neighborhood of an isolated singularity if and only if $\mu(p) = m_1/m_2$. Indeed, the relevant calculations for the spiral nets $\mathscr{S}_{p,\alpha}$ were made in (i) of Section 1.4. The case of Riemann singularities follows immediately from the observation that if θ has a Riemann singularity at p with fan angles α_1 and α_2, then a necessary and sufficient condition that there be a one-to-one single-valued (m_1, m_2)-mapping f in a neighborhood of p is that the images of these fans have angles which sum to π. Since these angles are given by $(m_j/m_i)\alpha_i$ the desired conclusion follows immediately. A standard argument then shows that in order for an HP^*-net in a simply connected domain D to correspond to some (m_1, m_2)-mapping with the same set of singularities, it is necessary and sufficient that for all singularities p

of $\mathscr{H}$, $\mu_{\mathscr{H}}(p) = m_1/m_2$. Note also that from this we have the obvious corollary to Theorem 4.1: any $f \in \text{cps}^*(\mathbf{C})$ can have at most 2 singularities; those with no singularities are linear, those with a single singularity correspond to $\mathscr{S}_{p,\alpha}$, $0 < |\alpha| < \frac{1}{4}\pi$ and $\mathscr{R}_{p,\alpha,\beta}$, $0 < |\beta - \frac{1}{2}\pi| < \frac{1}{2}\pi$, and those with two singularities correspond to the double Riemann and degenerate double Riemann nets $\mathscr{D}_{p,q,\alpha,\beta,\beta}$ and $\mathscr{F}_{p,q,\sigma,\beta,\beta}$ with $0 < |\beta - \frac{1}{2}\pi| < \frac{1}{2}\pi$. All questions about the distribution of singularities of HP*-functions therefore have a direct but more restrictive counterpart for cps*-functions.

We next present a construction which shows that for any smoothly bounded Jordan domain D there are mappings in cps*(D) which have infinitely many singularities, but before going into the details, we make a few relevant comments. If D is a simply connected domain and if $\mathscr{H} \in \text{HP}^*(D)$ is known to have a singularity at some point $p \in D$ then there can be no singularity at any point which is joined to p by a nonbounding characteristic of either of the fans associated with p in the case of a Riemann singularity or by any characteristic in the case of a spiral singularity. For this reason it is much easier for an HP-net to have many singularities if ∂D is highly contorted, since pockets formed by ∂D will allow characteristics emanating from p to hit the boundary before covering too much of D. For this reason it is easy to construct HP*-nets and cps*-mappings with infinitely many singularities (even with infinitely many spiral singularities), but such constructions will result in domains for which the boundary is not smooth. It is therefore of interest to see that the appearance of infinitely many singularities is possible even when the boundary is smooth; this follows immediately from the following

Theorem 5.2. *There is a number M_0 such that for all $M > M_0$ there is an $f \in \text{cps}^*(H,1,M)$, with Riemann singularities at $2^k z_0$, $-\infty < k < \infty$, for some $z_0 \in H = \{z : \operatorname{Im} z > 0\}$.*

Proof. For (small) $\varepsilon > 0$ let $T = T(\varepsilon)$ denote the circular sector

$$\left\{1 + i\tan^{-1}(\tfrac{1}{2}\varepsilon) + \rho e^{i\tau} : 0 < \rho \le 1,\ \pi + \tfrac{1}{2}\varepsilon \le \tau \le 2\pi - \tfrac{1}{2}\varepsilon\right\}.$$

We begin by applying the characteristic coordinate constructions of Section 1.2 to obtain an HP*-net with an isolated singularity at $p = 1 + i\tan^{-1}(\frac{1}{2}\varepsilon)$ with $C_1 = \{|p|e^{it} : \frac{1}{2}\varepsilon \le t \le \pi + \varepsilon\}$ and $C_2 = [0,p]$ and fan angles ε and $\pi - \varepsilon$, respectively. For the resulting net the bounding characteristics of the 2-fan are the sides $[p,0]$ and $[p,2]$ of the sector $T + 1 = \{z + 1 : z \in T\}$ and the bounding characteristics of the 1-fan are C_1 and a convex arc C' which joins p to a point b in the third quadrant (whose distance from $e^{i(\pi+\varepsilon)}$ is $O(\varepsilon)$). Let F_0 be the translate of C' along the 2-arc formed by the segment $[p,2]$; F_0 is also a convex curve and joins 2 to some point c_0 in the third quadrant. Note that the line segment 2-arc $[b,c_0]$ is horizontal. The HP*-net so constructed is defined in the domain bounded by the 1-arc $\{p + e^{i\tau} : \pi + \frac{1}{2}\varepsilon \le \tau \le 2\pi - \frac{1}{2}\varepsilon\} \cup F_0$ and the 2-arc

$[c_0, b] \cup bq \cup [q, 0]$, where $q = -e^{i\varepsilon}$. We now extend this net outward to larger and larger parts of H.

Assume inductively that we have a convex curve F_k, $k \geq 0$, with the following properties:

(i) F_k starts at $a_k = 6 \cdot 2^k - 4$ on the positive real axis and joins this point to a point c_k in the third quadrant.

(ii) The tangents to F_k at a_k and c_k have argument $\frac{1}{2}(\pi - \varepsilon)$ and $\frac{3}{2}\pi$, respectively.

(iii) There is an HP*-net in a domain D_k which contains the region bounded by the part of F_k lying in the upper half-plane and a segment of the real line and in which the only singularities of the net are at the points $5 \cdot 2^m - 4 + i2^m \tan^{-1}(\frac{1}{2}\varepsilon)$, $0 \leq m \leq k$.

(iv) F_k is a 1-characteristic of this net.

(v) Each point of F_k other than a_k is joined to 0 by a 2-characteristic of the net of (iii) which consists of a line segment $[0, e^{i\tau}]$, $\frac{1}{2}\varepsilon < \tau < \pi + \varepsilon$, followed by a curve along which the argument of the tangent lies in the interval $[\tau - \frac{1}{2}\varepsilon, \tau + \frac{1}{2}\varepsilon]$.

The arc F_0 satisfies (i)–(v). A simple compactness argument shows that (v) implies that there is an ε_0, $0 < \varepsilon_0 < 1/100$, such that if $\varepsilon < \varepsilon_0$, then $\lambda(F_k) < 2\pi a_k$. (The number ε_0 is an absolute constant which does not depend on k.) Henceforth we assume that $\varepsilon < \varepsilon_0$.

Let E_k be the circle of radius $2^k \csc(\frac{1}{2}\varepsilon)$ centered at the point $7 \cdot 2^k - 4 + i2^k \cot(\frac{1}{2}\varepsilon)$ which joins the point a_k to $8 \cdot 2^k - 4$ in the lower half-plane. Note that the tangent to this arc at $8 \cdot 2^k - 4$ has argument $\frac{1}{2}\varepsilon$. We apply the usual characteristic quadrilateral HP-construction (Proposition 1.3) with F_k as the 1-arc and

$$G_k = E_k \cup [8 \cdot 2^k - 4, 10 \cdot 2^k - 4 + i2^{k+1} \tan^{-1}(\tfrac{1}{2}\varepsilon)]$$

as the 2-arc. It is clear from the definition of ε_0 that we have an *a priori* bound of $2\pi \cdot 10 \cdot 2^k = 5\pi \cdot 2^{k+2}$ on the length of the translates of F_k along the 2-arc E_k, so that the concavity of E_k causes no singularity formation (that is, the mapping ζ of Proposition 1.3 is one-to-one in the whole rectangle $I_1 \times I_2$) since the curvature of the curved part of this arc is $2^{-k} \sin(\frac{1}{2}\varepsilon)$, and

$$2^{-k} \sin(\tfrac{1}{2}\varepsilon) \cdot 5\pi \cdot 2^{k+2} < 10\pi\varepsilon < 10\pi\varepsilon_0 < \tfrac{1}{10}\pi < 1.$$

Denote by F_k' the translate of F_k along G_k. We then put a singularity with 1- and 2-fan angles ε and $\pi - \varepsilon$, respectively, at $10 \cdot 2^k - 4 + i2^{k+1} \tan^{-1}(\frac{1}{2}\varepsilon)$, constructed in accordance with the procedures of Section 1.2, with 1-arc F_k' and 2-arc G_k. The 2-fan contains the entire sector $T_{k+1} = 2^{k+1}T + 5 \cdot 2^{k+1} - 4$. Let F_k'' be the other bounding arc of the 1-fan, and let F_{k+1} be the translate of F_k'' along the right-hand segment of ∂T_{k+1}. One sees that F_{k+1} starts at $6 \cdot 2^{k+1} - 4$ and has the correct tangent angle there. For D_{k+1} we take $D_k \cup T_{k+1}$ together with the region covered by the translates of F_k along G_k, the 1-fan with bounding characteristics F_k' and F_k'' and the translates of F_k'' along the right-hand segment of ∂T_{k+1}. The 2-arcs lying in this new region and joining points of F_k and F_{k+1} consist of two line segments and two curved arcs which have total curvature (when reckoned as one moves away from the origin) ε and $-\varepsilon$, respectively. From this it follows that the terminal point of F_{k+1} indeed lies in the third quadrant and that its tangent there has argument $\frac{3}{2}\pi$. It is also clear that (v) holds for $k+1$. The result of the entire inductive process is an HP*-net in all of H having singularities with dilatation $(\pi-\varepsilon)/\varepsilon$ at the points $2^k\bigl(5+i\tan^{-1}(\frac{1}{2}\varepsilon)\bigr)-4$, $k\geq 0$. Consideration of a convergent subsequence of $\{\theta(2^n z): n\geq 0\}$, where θ is a corresponding HP-function, yields a HP*-net with singularities of this kind at the points $2^k\bigl(5+i\tan^{-1}(\frac{1}{2}\varepsilon)\bigr)$, $-\infty<k<\infty$, which, in turn gives the desired cps*-mapping. □

The discussions of this and the preceding sections allow us to give some justification for identifying microscopic flaws in cryptocrystalline laminae with isolated singularities of the corresponding cps-mappings, or to be more precise, with isolated singularities of the HP-function θ corresponding to the inverse of the cps-mapping giving the deformation. Assume that the original uncrystallized lamina is represented by $D\subset \mathbf{C}$ and that the deformation produced by the crystallization is given by a homeomorphism $f\colon D\to\mathbf{C}$ which is a cps-mapping on $D\backslash S$. Here S is a closed set with components S_k, $k\in I\subset\mathbf{N}$. The flaws will be the $f(S_k)$. In the first place, because small flaws in close proximity to one another are apt to be perceived as a single larger flaw, it is reasonable to consider that a truly tiny flaw should be isolated as well as minute, that is, that for some $T>0$, $\operatorname{dist}\bigl(f(S_k),f(S_l)\bigr)\geq T$ for all distinct $k,l\in I$ and $\operatorname{diam}\bigl(f(S_k)\bigr)<\varepsilon$, $k\in I$, where $\varepsilon>0$ is some suitably small number. A simple compactness argument shows that for given $T,\delta_1,\delta_2>0$, there is an $\varepsilon>0$ such that if $\theta\in\mathrm{HP}\bigl(N(p,T)\backslash F\bigr)$ with $F\subset N(p,\varepsilon)$, then there is a $\theta_0\in\mathrm{HP}\bigl(N'(p,T)\bigr)$ such that

$$\sup\{|\theta(z)-\theta_0(z)|:\delta_1<|z-p|<T\}<\delta_2.$$

In a neighborhood of $f(S_k)$, $\theta_{f^{-1}}$ is therefore either close to an HP-function θ_0 with an isolated singularity at some $p\in f(S_k)$ or to one which is regular in that neighborhood, these two possibilities being mutually exclusive since, as our analysis shows, the θ_0 with a true singularity at p are not approximable in any reasonable sense by functions in $\mathrm{HP}\bigl(N(p,T)\bigr)$. In the case that θ_0 is regular we discard the flaw $f(S_k)$ from consideration as such since its presence would produce no macroscopic effects. It would therefore appear that cps*-mappings provide a reasonable mathematical description of deformations induced by the solidification of planar liquid films which result in laminae known only to have microscopic flaws.

6. Further questions

In line with the geometric function theory focus outlined in the fourth paragraph of the introduction, the most immediate issues raised by the foregoing involve extremal questions about the possible distribution of the singularities of

$\theta \in \mathrm{HP}^*(D)$ and $f \in$cps*(D), and more specifically $f \in$cps*(D, m_1, m_2). The cps-cases of this question have clear-cut significance in physical situations involving deformations with constant principal strains.

Since Corollary 4.1 says that for any Jordan domain D with smooth boundary the sum of the distances to the boundary of the singularities of any $\theta \in \mathrm{HP}^*(D)$ is bounded by $3\lambda(\partial D)/\pi$, it is most reasonable to ask about the smallest possible upper bounds for this sum, and for its analogues for cps*(D) and cps*(D, m_1, m_2). Although in general it is probably not possible to give an explicit answer to this question in terms of transparent geometric parameters of the domain D, it may well be in the case of disks; it is not clear, however, whether these problems have corresponding extremal functions. Somewhat more accessible, perhaps, is the determination of the greatest number of singularities that can appear in a given compact subset K of D; again the disk case, with K a concentric disk, is the most interesting instance. Theorem 5.2 tells us that given m_1, m_2 with $m_1/m_2 > 1$ sufficiently large, cps*(D, m_1, m_2) contains mappings with infinitely many singularities. One wonders if the condition that m_1/m_2 be sufficiently large can be removed. Moreover, given the interpretation of cps-mappings in the context of cryptocrystalline laminae, it would be of considerable interest to determine if there always exist *injective* $f \in$cps*(D, m_1, m_2) with infinitely many singularities, since noninjective f correspond to laminae which overlap themselves upon solidification. We believe that it should not be too difficult to resolve this issue. In a less quantitative direction we mention the possibility that if $\theta \in \mathrm{HP}(D\backslash S)$, where S is a "sufficiently small set" (linear measure 0 is perhaps small enough), then, in fact, θ can be extended to an HP-function in $D\backslash S'$, where $S' \subset D$ consists entirely of isolated points of D.

The distribution of isolated singularities is only one of the numerous issues regarding HP-nets and cps-mappings which invite investigation. In addition to the distortion questions touched on in [G1], specific mapping problems offer another possibility. Here the analogy with conformal mappings becomes somewhat thin, since as will be shown in the sequel [G4] to this paper (see also [G3]), given any smoothly bounded Jordan domain D there is another such domain E such that for no $f \in$cps(D) is $f(D) = E$. This raises the problem of finding general (but not necessarily exhaustive) intrinsic conditions on domains D and E which imply the existence of cps-homeomorphisms of one onto the other; particularly interesting is the question of when a domain has cps-self-homeomorphisms. An important tool for studying such mapping questions is the fact that if D and E are smoothly bounded domains and if f is a cps-homeomorphism of D onto E, then characteristics of the associated HP-function θ which meet ∂D do so at a well defined angle. This by no means trivial fact, to be proved in [G4], implies that the boundary values of θ are well defined (in an appropriate sense) and as a result allows one to relate such mapping questions to Cauchy problems. This, in turn, makes it possible to give simple descriptions of all cps-self-homeomorphisms of the half-plane and the exterior of a disk. All of these mapping questions have corresponding cps*-versions and obvious interpretations in the context of cryptocrystalline laminae.

Constant principal strain mappings between 2-manifolds are treated in [ChG] and one can ask if it is possible to say something about the distribution of singularities in that context. In that paper a complete description of the infinite dimensional family of cps-self-homeomorphisms of the hyperbolic plane $\mathbf{H}^2$ is given, so that it would be interesting to find the $\mathbf{H}^2$-analogue of Theorem 4.1. One might also ask if it is possible to derive some global results for higher dimensional mappings with constant principal strains, which are governed by a much more complex nonlinear hyperbolic system (see [G2]) or whether the results on HP-nets contained in this paper can be generalized to families of nets associated with other 2×2 genuinely nonlinear hyperbolic systems.

References

[C] Collins, I.F.: Boundary value problems in plane strain plasticity. - In: Mechanics of Solids: The Rodney Hill 60th Anniversary Volume, edited by H.G. Hopkins and M.J. Sewell, Pergamon Press, Oxford, 1982, pp. 135–184.

[ChG] Chuaqui, M., and J. Gevirtz: Constant principal strain mappings on 2-manifolds. - Preprint.

[CS] Carathéodory, C., and E. Schmidt: Über die Hencky–Prandtlschen Kurven. - Z. Angew. Math. Mech. 3, 1923, 468–475.

[G1] Gevirtz, J.: On planar mappings with prescribed principal strains. - Arch. Rational Mech. Anal. 117, 1992, 295–320.

[G2] Gevirtz, J.: A diagonal hyperbolic system for mappings with prescribed principal strains. - J. Math. Anal. Appl. 176, 1993, 390–403.

[G3] Gevirtz, J.: Constant principal strain mappings of plane domains. - Amer. Math. Soc. Abstracts J. 15, 1994, 73.

[G4] Gevirtz, J.: Boundary behavior and the transformation problem for planar mappings with constant principal strains. - In preparation.

[Hem] Hemp, W.S.: Optimum Structures. - Clarendon Press, Oxford, 1973.

[Hen] Hencky, H.: Über einige statisch bestimmte Fälle des Gleichgewichts in plastischen Körpern. - Z. Angew. Math. Mech. 3, 1923, 241–251.

[Hi] Hill, R.: The Mathematical Theory of Plasticity. - Clarendon Press, Oxford, 1964.

[J1] John, F.: Rotation and strain. - Comm. Pure Appl. Math. 14, 1961, 391–413.

[J2] John, F.: On quasi-isometric mappings, I. - Comm. Pure Appl. Math. 21, 1968, 77–110.

[J3] John, F.: On quasi-isometric mappings, II. - Comm. Pure Appl. Math. 22, 1969, 265–278.

[Pra] Prandtl, L.: Über die Eindringungsfestigkeit (Härte) plastische Baustoffe und die Festigkeit von Schneiden. - Z. Angew. Math. Mech. 1, 1921, 15–20.

[Pri] Priwalow, I.I.: Randeigenschaften Analytischer Funktionen. - VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

[SK] Strang, G., and R.V. Kohn: Hencky–Prandtl nets and constrained Michell trusses. - Comput. Methods Appl. Mech. Engrg. 36, 1983, 207–222.

[Y] Yin, W.-L.: Two families of finite deformations with constant strain invariants. - Mech. Res. Comm. 10, 1983, 127–132.

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