AN OVERVIEW OF VARIABLE EXPONENT LEBESGUE AND SOBOLEV SPACES

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ABSTRACT. In this paper we give an overview of the development of variable exponent Lebesgue and Sobolev spaces from the beginning of the 1990's till today. We describe the basic theory and mention some applications. Additionally, we prove a new result on the density of smooth functions in variable exponent Sobolev spaces on the real line.

1. INTRODUCTION

In recent years there has been increasing interest toward variable exponent Lebesgue and Sobolev spaces. The present line of investigation goes back to a paper by O. Kováčik and J. Rákosník [32] from 1991. After this paper not much happened till the late 1990's. At this point the subject seems to have been independently (re)discovered by several researchers: S. Samko [40, 41, 42] working based on earlier Russian work (I. Sharapudinov [43] and V. Zhikov [44]), X. Fan and collaborators [15, 17, 16, 18] and G. Mingione and collaborators [1, 2, 3, 4] drawing inspiration from the study of differential equations (e.g. Marcellini [33]). Around the turn of the millennium also researchers basing their efforts directly on the work of Kováčik and Rákosník got new momentum with the investigations of D. Edmunds and collaborators [10, 11, 12, 13, 14], L. Diening [6, 7] and others [5, 8, 9, 35, 38]. Our group at the University of Helsinki started its research in the Fall of 2002. To date our efforts have yielded some publications [24, 21] and preprints [25, 26, 23, 22]. The last couple of years have seen the (partial) integration of the separate lines of investigation, but much still remains to be done. We believe that variable exponent spaces are a good entry-point requiring little prerequisites for someone who wants to start research on Sobolev spaces.

The main incentive for many of the investigators of variable exponent spaces is relaxing the coercivity conditions assumed for the solutions of a differential equation or the corresponding variational integral. Typically one would assume that the function $F\colon G \times \mathbb{R}^n \to [0, \infty)$ in the integral

$$\int_G F(x, \nabla u(x))\, dx \tag{1.1}$$

satisfies the coercivity condition

$$\tfrac{1}{c}|\nabla u(x)|^p \le F(x, \nabla u(x)) \le c|\nabla u(x)|^p. \tag{1.2}$$

In this case the minimizers of (1.1) are in $W^{1,p}(G)$, [27]. (We ignore many technical assumptions here, see the reference for details.) Now if we just replace p by $p(x)$ in (1.2),

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then we get a more general condition, but in what space will the minimizers of the variational integral lie now? Although the theory is not yet complete the tenet of variable-exponent-space-investigators is that it will be $W^{1,p(\cdot)}(G)$, the variable exponent Sobolev space (defined below).

Perhaps the next question that comes to mind is: why should we want to generalize (1.2) in the first place? The answer is again somewhat tentative – we believe that it will be useful in modeling non-homogeneous material. Till now only one such application has been investigated in greater detail, electro-rheological fluids. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. For a general account of the underlying physics confer [20] and for some technical applications [36]. The mathematical theory was presented in the monograph [39] by M. Růžička, see also [3].

The remaining part of this article strives to give a little more detailed an account of the mathematics of variable exponent spaces. In the next section we present the theory of Lebesgue spaces, in Section 3 that of Sobolev spaces. In the final section we present the proof of a new result on the density of smooth functions in Sobolev spaces in the one dimensional case. This problem is a good example of how something which is easy in classical Sobolev spaces turns out to be quite difficult in the variable case. The proof also illustrates some techniques that can be used in variable exponent spaces.

2. Lebesgue spaces

Let $p : \mathbb{R}^n \to [1,\infty)$ be a measurable function (called the *variable exponent* on $\mathbb{R}^n$). Throughout this paper the function p always denotes a variable exponent; also, we define $p^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ to consist of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\varrho_{p(\cdot)}(\lambda u) = \int_{\mathbb{R}^n} |\lambda u(x)|^{p(x)}\, dx < \infty$$

for some $\lambda > 0$. The function $\varrho_{p(\cdot)} : L^{p(\cdot)}(\mathbb{R}^n) \to [0,\infty)$ is called the *modular* of the space $L^{p(\cdot)}(\mathbb{R}^n)$. We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$||u||_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

If p is a constant function, then the variable exponent Lebesgue spaces coincides with the classical Lebesgue space and so the notation can give rise to no confusion. The variable exponent Lebesgue spaces is a special case of Orlicz-Musielak spaces treated by Musielak [34]. The basic properties of these spaces can be found from the paper of Kováčik and Rákosník [32]; many of these properties were independently established by Fan and Zhao [17].

Basic properties. Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects – they are Banach spaces [32, Theorem 2.5], the Hölder inequality holds [32, Theorem 2.1], they are reflexive if and only if $1 < p^- \le p^+ < \infty$ [32, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [32, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [32, Theorem 2.8]: if $0 < |\Omega| < \infty$ and p, q are variable exponent so that $p(x) \le q(x)$ almost everywhere in Ω then there exists an imbedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ whose norm does not exceed $|\Omega| + 1$.

One problem in variable exponent Lebesgue spaces is the relation between the modular and the norm. If $p^+ < \infty$ then

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \le \varrho_{p(\cdot)}(u) \le \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\}$$

[17, Theorem 1.3]. It follows that if $p^+ < \infty$ and (f_i) is a sequence of functions in $L^{p(\cdot)}$, then $\|f_i\|_{p(\cdot)} \to 0$ if and only if $\varrho_{p(\cdot)}(f_i) \to 0$ [32, Theorem 2.4]. However, if $p^+ = \infty$ the topology defined by the norm is not the same as that defined by the modular. Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [10].

Variable exponent Lebesgue spaces do not have the mean continuity property. If p is continuous and non-constant function in an open ball B, then there exists a function $u \in L^{p(\cdot)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for $h \in \mathbb{R}^n$ with arbitrary small norm [32, Theorem 2.10]. In [19] Fiorenza has studied weak version of mean continuity in variable exponent Sobolev spaces.

The Hardy-Littlewood maximal operator. Finally we give an example of a classical property which is now understood quite well also in the variable exponent case. Assume that $1 < p^- \le p^+ < \infty$ and there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|} \tag{2.1}$$

for every $x, y \in \mathbb{R}^n$, $|x-y| \le \frac{1}{2}$ and

$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}$$

for every $x, y \in \mathbb{R}^n$, $|y| \ge |x|$. Under these assumptions on p, Cruz-Uribe, Fiorenza and Neugebauer proved that the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself [5, Theorem 1.5]. This was an improvement of earlier work by Diening [6] and Nekvinda [35]. The assumptions on p are more or less sharp: Pick and Růžička showed that if (2.1) does not hold then the Hardy-Littlewood maximal operator is not bounded [37] while Cruz-Uribe, Fiorenza and Neugebauer proved that if p is upper semicontinuous and $p^- = 1$ then the operator is not bounded [5, Theorem 1.7]. Cruz-Uribe, Fiorenza and Neugebauer proved a weak type estimate to the Hardy-Littlewood maximal operator provided that $1/p \in RH_\infty$ [5, Theorem 1.8]. This assumption is true for example if $p^+ < \infty$. Maximal operators have also been studied in weighted $L^{p(\cdot)}$ spaces by Kokilashvili and Samko [31].

Samko [40, 41], Kokilasvili and Samko [28, 29, 30] and Diening and Růžička [8, 9] have also studied other kinds of integral operators in variable exponent and weighted variable exponent spaces.

3. Sobolev spaces

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\mathbb{R}^n)$ is the subspace of functions $L^{p(\cdot)}(\mathbb{R}^n)$ whose distributional gradient exists almost everywhere and lies in $L^{p(\cdot)}(\mathbb{R}^n)$. The function $\varrho_{1,p(\cdot)} : W^{1,p(\cdot)}(\mathbb{R}^n) \to [0,\infty)$ is defined by $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(|\nabla u|)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space, separable if $p^+ < \infty$ and reflexive if $1 < p^- \le p^+ < \infty$ [32, Theorem 3.1]. Variable exponent Sobolev spaces have close connection to the study of partial differential equation and variational integrals with non-standard growth, see for example [1, 2, 4, 16, 44]

Density of smooth functions. Samko [42] and Fan and Zhao [16] have proved independently that if p satisfies (2.1) then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Edmunds and Rákosník showed that if $\Omega \subset \mathbb{R}^n$ is bounded with Lipschitz boundary and the exponent p satisfies condition similar to (2.1) then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ [13, Theorem 3.2], see also Diening's version of this [6, Theorem 3.7].

Edmunds and Rákosník derived the same conclusion under a local monotony condition on p [12, Theorem 1]. In Section 4 of this paper we show that on the real line C_0^∞-functions are dense in $W^{1,p(\cdot)}(\mathbb{R})$ provided only that $p^+ < \infty$. An example constructed by Zhikov [44, p. 36] shows that if the exponent is discontinuous then smooth functions need not be dense in variable exponent Sobolev space if the dimension is at least 2.

The Sobolev and Poincaré inequalities. Diening proved that if $1 < p^- \le p^+ < n$, (2.1) holds and p is a constant outside some large ball then there exists a continuous imbedding $W^{1,p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^n)$ [7, Theorem 5.2]. Here $p^*(x) = np(x)/(n - p(x))$. He also has version of this in bounded domain with Lipschitz boundary [7, Corollary 5.3]. Fan, Shen and Zhao have proved the Sobolev imbedding for larger class of domains than Diening but under stricter condition on p [15]. Edmunds and Rákosník assumed that p is Lipschitz continuous in the closure of an open set Ω and there exists real numbers q, r such that $1 < r \le p(x) \le q < \min\{n, r^*\}$ and showed

$$\|u\|_{p^*(\cdot)} \le C\|\nabla u\|_{p(\cdot)}$$

for every $u \in W^{1,p(\cdot)}(\Omega)$ with spt $u \subset \Omega$ [13, Theorem 3.1]. Kováčik and Rákosník proved an example which shows that the Sobolev imbedding does not hold if the exponent is discontinuous [32, Example 3.2].

The situation changes if we are ready to sacrifice an $\varepsilon > 0$ from one of the norm involved. Kováčik and Rákosník [32, Theorem 3.3] assumed that $p : \overline{\Omega} \to [1, n)$ is continuous and showed that for every small $\varepsilon > 0$ and every measurable function $1 \le q(x) \le p^*(x) - \varepsilon$ the inequality

$$\|u\|_{q(\cdot)} \le C\|\nabla u\|_{p(\cdot)}, \; u \in C_0^\infty(\Omega)$$

holds. Edmunds and Rakosnik also have Sobolev type inequalities in variable exponent spaces with weights [14]. More discussion about Sobolev inequality can be found in [38]. The Sobolev imbedding for $p(x) > n$ has been studied in [21, Section 3 and 4] by Harjulehto and Hästö.

It is interesting to note that the Poincaré inequality holds under a much weaker assumption on p than the Sobolev inequality and imbedding, namely if the exponent is not too discontinuous [21, Section 2]. For instance, if the exponent is continuous up to the boundary in a bounded convex domain, then we have a Poincaré inequality [21, Corollary 2.16]. Harjulehto and Hästö also gave an example which shows that the condition for p is sharp [21, Example 2.6].

Sobolev capacity and its applications. In [24] Harjulehto, Hästö, Koskenoja and Varonen introduced a Sobolev capacity in the variable exponent Sobolev space, which is defined as follows. Suppose that E is an arbitrary subset of $\mathbb{R}^n$. The *Sobolev $p(\cdot)$-capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\right) dx,$$

where infimum is taken over those $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ which are at least one in some open set containing E. If $1 < p^- \leq p^+ < \infty$, then the Sobolev $p(\cdot)$-capacity is an outer measure and Choquet capacity [24, Corollaries 3.3 and 3.4]. It is possible to relax the conditions of p by assuming $p^- > 1$ locally [26]. As in the fixed exponent case the capacity is a finer measure than the n-dimensional Lebesgue measure [24, Section 4]. If continuous functions are dense in variable exponent Sobolev spaces and if $1 < p^- \leq p^+ < \infty$ then every Sobolev function has a quasicontinuous representative [24, Theorem 5.2].

Using the Sobolev $p(\cdot)$-capacity Harjulehto, Hästö, Koskenoja and Varonen defined Sobolev spaces with zero boundary values as in metric measure spaces. This definition coincides with that based on the closure of C_0^∞-functions if the smooth functions are dense [24, Chapter 3]. This allowed them to study the existence of a minimizer of the Dirichlet energy integral in variable exponent spaces with Sobolev boundary values [25]. This problem is not trivial even in the one dimensional case, see [23]. The Sobolev $p(\cdot)$-capacity also seems to be the right one for studying Lebesgue points of Sobolev functions. Harjulehto and Hästö proved in [22] that quasievery point is a Lebesgue point of a Sobolev function if p satisfies (2.1). It is also possible to define a capacity using the norm instead of the modular [21, Theorem 4.2] however this does not yield a Choquet capacity.

Summary. We have seen that condition (2.1) is a very common assumption on the theory of variable exponent spaces. This condition, sometimes strengthened with some technical assumptions, guaranties that many classical properties hold. Moreover, it seems that in many instances this is exactly the right condition. However, there are also many cases in which there is also a gap between the necessary and sufficient conditions that we cannot bridge.

Under the weaker assumptions $p^+ < \infty$ or $1 < p^- \leq p^+ < \infty$ variable exponent spaces still have many interesting properties. However, in this case many classical techniques (e.g. convolution) are not available and new methods must be devised to prove standard results. For $p^+ = \infty$ only few properties have been established for variable exponent spaces.

4. Density of smooth functions in real line

In this section we show that the density problem, which remains open in higher dimensions despite much effort, is essentially trivial in the one dimensional case. As for the condition $p^+ < \infty$, note that for $p^+ = \infty$ we do not even know if $C^\infty(I) \cap L^{p(\cdot)}(I)$ is dense in $L^{p(\cdot)}(I)$. Here I is the open interval $(-1, 1)$.

4.1**. Theorem.** *If $p^+ < \infty$, then $C^\infty(I) \cap W^{1,p(\cdot)}(I)$ is dense in $W^{1,p(\cdot)}(I)$.*

Proof. Let $u \in W^{1,p(\cdot)}(I)$ and fix $\epsilon > 0$. By [32, Theorem 3.4] there exists a function $g \in C^\infty(I) \cap L^{p(\cdot)}(I)$ such that $\|u' - g\|_{p(\cdot)} \leqslant \epsilon$. Fix $z \in I$. Define

$$v(y) = u(z) + \int_z^y g(x)dx$$

for $y \in I$, so that $v' = g$. We then have

$$|v(y) - u(y)| \leqslant \left|u(z) + \int_z^y g(x)dx - u(z) - \int_z^y u'(x)dx\right| \leqslant \int_z^y |g(x) - u'(x)|dx \leq \|u' - g\|_1.$$

By [32, Theorem 2.8] it follows that $\|u' - g\|_1 \leqslant 2\|u' - g\|_{p(\cdot)} < 2\epsilon$. By the same theorem and the above inequality, $\|v - u\|_{p(\cdot)} \leqslant 2\,\|v - u\|_\infty < 4\epsilon$, and so

$$\|v - u\|_{1,p(\cdot)} = \|v - u\|_{p(\cdot)} + \|u' - g\|_{p(\cdot)} < 5\epsilon.$$

Since $v \in C^\infty(I)$ and since ϵ can be arbitrarily small, we have found our C^∞ approximation and so we are done. □

4.2. **Theorem.** *If $p^+ < \infty$, then $C_0^\infty(\mathbb{R})$ is dense in $W^{1,p(\cdot)}(\mathbb{R})$.*

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R})$ and fix $\epsilon > 0$. Denote $I_x = (x-1, x+1)$ for $x \in \mathbb{R}$. Let $\phi(x) \in C_0^\infty([-1,1])$ (extended by 0 outside $[-1,1]$) be an L–Lipschitz function such that $\sum_{k\in\mathbb{Z}} \phi(x-k)$ is identically equal to 1.

For $k \in \mathbb{Z}$ we choose a $C^\infty(I_k)$ function v_k such that $\|u - v_k\|_{W^{1,p(\cdot)}(I_k)} < \epsilon/2^{|k|}$, using Theorem 4.1. Define $v(x) = \sum_{k\in\mathbb{Z}} v_k(x)\phi(x-k)$. Then

$$\begin{aligned}
\|u - v\|_{1,p(\cdot)} &= \Big\| \sum_{k\in\mathbb{Z}} (u(x) - v_k(x))\phi(x-k) \Big\|_{1,p(x)} \\
&\leqslant \sum_{k\in\mathbb{Z}} \Big\| (u(x) - v_k(x))\phi(x-k) \Big\|_{1,p(x)} \\
&\leqslant \sum_{k\in\mathbb{Z}} \|u - v_k\|_{p(\cdot)} + \Big\| \{(u(x) - v_k(x))\phi(x-k)\}' \Big\|_{p(x)} \\
&\leqslant \sum_{k\in\mathbb{Z}} (1+L)\|u - v_k\|_{p(\cdot)} + \|(u - v_k)'\|_{p(\cdot)} \\
&\leqslant (1+L) \sum_{k\in\mathbb{Z}} \|u - v_k\|_{1,p(\cdot)} \\
&\leqslant (1+L)\epsilon \sum_{k\in\mathbb{Z}} 2^{-|k|} = 3(1+L)\epsilon.
\end{aligned}$$

Since

$$\lim_{l\to\infty} \Big\| u(x) - \sum_{k=-l,\dots,l} v_k(x)\phi(x-k) \Big\|_{1,p(x)} \leqslant 3(1+L)\epsilon,$$

it follows that there exists an integer l such that

$$\Big\| u(x) - \sum_{k=-l,\dots,l} v_k(x)\phi(x-k) \Big\|_{1,p(x)} \leqslant 4(1+L)\epsilon.$$

But clearly $\sum_{k=-l,\dots,l} v_k(x)\phi(x-k) \in C_0^\infty(\mathbb{R})$, so we are done. □

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