HARDY'S INEQUALITY IN VARIABLE EXPONENT SOBOLEV SPACE

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ABSTRACT. We show that a norm version of Hardy's inequality holds in variable exponent Sobolev space provided the maximal operator is bounded. Our proof uses recent local versions of the inequality for fixed exponent. We give an example to show that our assumptions on the exponent are essentially sharp. In the one-dimensional case we derive a necessary and a sufficient condition for Hardy's inequality to hold.

1. Introduction

The classical Hardy inequality for fixed exponent $p \in (1, \infty)$ is

$$\int_0^\infty |u(x)|^p x^{-p+a}\, dx \le \left(\frac{p}{p-1-a}\right)^p \int_0^\infty |u'(x)|^p x^a\, dx,$$

where $a < p-1$, u is absolutely continuous on $[0,\infty)$, and $u(0) = 0$. Nečas [20, 8.8] proved a version of this inequality in higher dimensions for bounded domains $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and functions $u \in C_0^\infty(\Omega)$. Now the inequality reads

$$\int_\Omega |u(x)|^p \delta(x)^{-p+a}\, dx \le C \int_\Omega |\nabla u(x)|^p \delta(x)^a\, dx, \tag{1.1}$$

where a and p are as before and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. This inequality in the space setting was further generalized by Kufner [15] to domains with Hölder boundary, and then by Wannebo [26] to domains satisfying a generalized Hölder boundary condition.

Lewis [16] and Wannebo [25] gave independent proofs that the Hardy inequality (1.1) holds in a proper open subset Ω of $\mathbb{R}^n$ provided its complement is fat enough with respect to a certain capacity. Hajłasz [5] and Kinnunen and Martio [11] obtained the pointwise inequality

$$|u(x)| \le \delta(x)\mathcal{M}|\nabla u|(x),$$

where on the right hand side is a kind of maximal function depending on the distance of x to the boundary. The pointwise inequality combined with the Hardy–Littlewood maximal theorem implies a "local version near the boundary" of Hardy's inequality (1.1) that does not follow from the earlier results.

Date: June 21, 2005.

2000 *Mathematics Subject Classification.* Primary 46E35; Secondary 26D10.

Key words and phrases. Variable exponent, Sobolev space, Hardy's inequality.

* Supported in part by the Academy of Finland.

† Supported in part by the Finnish Academy of Science and Letters and by a Gehring–Finland Fellowship.

Our aim is to generalize Hardy's inequality to variable exponent Sobolev spaces. Although variable exponent Lebesgue and Sobolev spaces have a long prehistory, the foundations of the theory in $\mathbb{R}^n$ was laid by Kováčik and Rákosník [14] only in the early 1990's. After Diening's breakthrough in the study of the Hardy–Littlewood maximal operator [1], there has been great progress in studying various potential-type operators, for the most recent advances see [2, 3, 4, 9, 12, 13]. Of particular relevance for us here are the papers of Samko [23, 24] and Mashiyev, Çekiç and Ograş [17, 18] dealing with Hardy-type inequalities. There are some differences between these papers and our's. Samko's formulation avoids the use of Sobolev spaces and considers the distance to a single point in the closure of the domain, rather than the distance to the boundary as we do. Mashiyev, Çekiç and Ograş, on the other hand, deal only with the one-dimensional case, but consider also the possibility of different exponents on the left and right side.

The structure of this paper is as follows: first we introduce some notation and present basic results from the theory of variable exponent spaces. Then we show that the local results of Hajłasz' work in the variable exponent case provided that the exponent satisfies a certain continuity condition. In Section 4 we give an example which shows that the continuity assumption on the exponent is optimal. In the last section we study the one-dimensional case – here we give a necessary and a sufficient continuity condition on the exponent for Hardy's inequality to hold.

2. DEFINITIONS

We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geq 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote the open ball with center x and radius r by $B(x, r)$. We denote the complement of $\Omega \subset \mathbb{R}^n$ by Ω^c.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $p\colon \Omega \to [1, \infty)$ be a measurable function (called the *variable exponent* on Ω). We write $p^+_{\Omega'} = \operatorname{ess\,sup}_{x\in\Omega'} p(x)$ and $p^-_{\Omega'} = \operatorname{ess\,inf}_{x\in\Omega'} p(x)$, where $\Omega' \subset \Omega$ and abbreviate $p^+ = p^+_\Omega$ and $p^- = p^-_\Omega$.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)}\,dx < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all measurable functions $u \in L^{p(\cdot)}(\Omega)$ such that the absolute value of the distributional gradient $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ is in $L^{p(\cdot)}(\Omega)$. The norm$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. The basic theory of variable exponent spaces was worked out in [14], which can be consulted for further reference.

We say that the exponent $p\colon \Omega \to [1, \infty)$ is log-*Hölder continuous* if there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}$$

for every $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$. It turns out that this condition is sufficient to guarantee a lot of regularity for variable exponent spaces. For instance, if p is bounded and log-Hölder continuous in $\mathbb{R}^n$, then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, [22]. Under these circumstances, it makes sense to define the space of *zero boundary value Sobolev functions*, $W_0^{1,p(\cdot)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The *Hardy-Littlewood maximal operator* is defined for a locally integrable function u by

$$\mathcal{M}u(x) = \sup_{r>0} \fint_{B(x,r)} |u(y)|\, dy.$$

In the context of maximal operators we extend the function on which the maximal operator is acting by 0 outside its original domain of definition. If Ω is bounded, $1 < p^- \le p^+ < \infty$ and p is log-Hölder continuous in Ω, then $\mathcal{M}$ is bounded from $L^{p(\cdot)}(\Omega)$ to itself [1, Theorem 3.5].

3. Hardy's inequality in $\mathbb{R}^n$

We recall a pointwise version of Hardy's inequality which leads to a norm version of the inequality. We use $\delta(x)$ to denote the distance between the boundary of Ω and $x \in \Omega$.

3.1. Lemma (Proposition 1, [5])**.** *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Suppose that there exists a constant $b > 0$ such that*

$$|B(z,r) \cap \Omega^c| \ge b|B(z,r)|, \tag{3.2}$$

for every $z \in \partial\Omega$ and $r > 0$. Then there exists a constant C depending only on n and b such that the inequality

$$|u(x)| \le C\delta(x)\mathcal{M}|\nabla u|(x)$$

holds for all $u \in C_0^\infty(\Omega)$ and all $x \in \Omega$.

Using this lemma we can easily adopt the proof of [5, Theorem 1] to prove Hardy's inequality.

3.3. Theorem. *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Let $p\colon \Omega \to [1,\infty)$ be* log-*Hölder continuous in Ω with $1 < p^- \le p^+ < \infty$. Assume that there exists a constant $b > 0$ such that*

$$|B(z,r) \cap \Omega^c| \ge b|B(z,r)|$$

for every $z \in \partial\Omega$ and $r > 0$. Then there exist constants C and a_0 depending only on p, n and b such that the inequality

$$\left\| \frac{u(x)}{\delta(x)^{1-a}} \right\|_{p(x)} \le C \left\| \nabla u(x)\delta(x)^a \right\|_{p(x)} \tag{3.4}$$

holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and all $0 \le a < a_0$.

Proof. We first prove the case $a = 0$. Since smooth functions are dense, it suffices to prove the claim for $u \in C_0^\infty(\Omega)$. It follows from Lemma 3.1 that

$$\left\| \frac{u(x)}{\delta(x)} \right\|_{p(x)} \le C \left\| \mathcal{M}|\nabla u(x)| \right\|_{p(x)},$$

and since the maximal operator is locally bounded [1, Theorem 3.5] the claim follows.

Assume then that $0 < a < 1$. We use the standard bootstrapping procedure (e.g. [25, p. 93]) to deal with this case. We set $v = |u|\delta(x)^a$. Since the Lipschitz constant of $\delta(\cdot)$ equals 1, we obtain $|\nabla v| \le |\nabla u|\delta(x)^a + a|u|\delta(x)^{a-1}$. Applying inequality (3.4) with $a = 0$ to v we obtain

$$\left\| \frac{u(x)}{\delta(x)^{1-a}} \right\|_{p(x)} \le C \left(\big\| |\nabla u(x)|\delta(x)^a \big\|_{p(x)} + a \big\| |u|\delta(x)^{a-1} \big\|_{p(x)} \right).$$

Whenever $Ca < 1$ this yields

$$\left\| \frac{u(x)}{\delta(x)^{1-a}} \right\|_{p(x)} \le \frac{C}{1 - Ca} \big\| |\nabla u(x)|\delta(x)^a \big\|_{p(x)}. \qquad \square$$

If $p^- > n$, then Hardy's inequality holds for every open bounded set, as we will show next. The reason for this has to do with the fact that the $p(\cdot)$-capacity of a point is then positive. In the fixed exponent case the relation between the capacity and the Hardy inequality has been worked out fairly completely [16, 19, 25]. Unfortunately, it turns out to be much harder to even define a relative capacity in variable exponent Sobolev spaces, much less derive the connection (see [8] for more on this issue).

3.5. Theorem. *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Let $p\colon \Omega \to [1, \infty)$ be* log-*Hölder continuous in Ω with $n < p^- \le p^+ < \infty$. Then there exist constants C and $a_0 > 0$ depending only on p and n such that the inequality*

$$\left\| \frac{u(x)}{\delta(x)^{1-a}} \right\|_{p(x)} \le C \left\| \nabla u(x)\delta(x)^a \right\|_{p(x)}$$

holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and all $0 \le a < a_0$.

Proof. We prove only the case $a = 0$ since the other case follows from this as in the proof of Theorem 3.3. Since smooth functions are dense, it suffices to prove the claim for $u \in C_0^\infty(\Omega)$. Hence we obtain by [11, (3.1)] that

$$|u(x)| \le C(n, q)\delta(x) \left(\mathcal{M}(|\nabla u|(x)^q)\right)^{\frac{1}{q}}$$

for every $x \in \Omega$. Here q is a constant with $n < q < p^-$. This yields that

$$\left\| \frac{u(x)}{\delta(x)} \right\|_{p(x)} \le C \left\| \left(\mathcal{M}(|\nabla u|^q)\right)^{\frac{1}{q}} \right\|_{p(\cdot)} = C \big\| \mathcal{M}(|\nabla u|^q) \big\|_{\frac{p(\cdot)}{q}}^{1/q}.$$

Since the maximal operator is bounded from $L^{\frac{p(\cdot)}{q}}(\Omega)$ to itself [1, Theorem 3.5] we obtain

$$\left\| \frac{u(x)}{\delta(x)} \right\|_{p(x)} \le C \big\| |\nabla u|^q \big\|_{\frac{p(\cdot)}{q}}^{1/q} = C \|\nabla u\|_{p(\cdot)}. \qquad \square$$

4. A counter-example in $\mathbb{R}^n$

In this section we show that the results in the previous section are optimal in so far as the modulus of continuity of the exponent is concerned. In other words we show that for any modulus of continuity which is asymptotically greater that $(-\log(|x-y|))^{-1}$ we find a variable exponent with this modulus of continuity and a bounded domain for which Hardy's inequality does not hold.

We need a fixed exponent variational capacity. Let $R > r > 0$ and denote $B(r) = B(0,r)$. Recall that

$$\operatorname{cap}_q(B(r), B(R)) = \inf_u \int_{\mathbb{R}^n} |\nabla u(y)|^q \, dy,$$

where the infimum is taken over continuous functions in $W^{1,q}(\mathbb{R}^n)$ which equal 1 on $B(r)$ and 0 on $\mathbb{R}^n \setminus B(R)$.

4.1. Lemma. [10, Example 2.12, p. 35] *For fixed $q \neq 1, n$ and $R > r > 0$ we have*

$$\operatorname{cap}_q(B(r), B(R)) = |S^{n-1}| \left| \frac{q-n}{q-1} \right|^{q-1} \left| R^{(q-n)/(q-1)} - r^{(q-n)/(q-1)} \right|^{1-q}.$$

The proof of the following theorem is similar to [6, Example 2.5]. Recall that a modulus of continuity is an increasing function ω with $\omega(0) = 0$.

4.2. Theorem. *Let ω be a modulus of continuity such that $\omega(x) \log \frac{1}{x} \to \infty$ as $x \to 0$. Then there exists a variable exponent p on a bounded set $\Omega \subset \mathbb{R}^n$ such that*

$$|p(x) - p(y)| \leq \omega(|x-y|)$$

and Hardy's inequality does not hold in $W_0^{1,p(\cdot)}(\Omega)$.

Proof. Let $q \in (1,n)$ and (q_i) be a sequence of real numbers in (q,n) tending to q. Let (R_i) and (r_i) be sequences of positive real numbers tending to 0 with $R_i > r_i$ such that $r_i/R_i \to 0$. Consider the set $B(R_i)$ and define the exponent by

$$p(x) = \begin{cases} q, & \text{for } x \in B(R_i) \setminus B(r_i); \\ q + (q_i - q)\frac{r_i - |x|}{r_i/2}, & \text{for } x \in B(r_i) \setminus B(r_i/2); \\ q_i, & \text{for } x \in B(r_i/2). \end{cases}$$

Let $u \in W_0^{1,q}(B(R_i))$ be a continuous function which equals 1 on $B(r_i)$ with

$$\begin{aligned} \|\nabla u\|_{p(\cdot)}^q = \|\nabla u\|_q^q &\leq 2 \operatorname{cap}_q(B(r_i), B(R_i)) \\ &\leq C \left| R_i^{(q-n)/(q-1)} - r_i^{(q-n)/(q-1)} \right|^{1-q} \leq C r_i^{n-q}. \end{aligned}$$

Since $u = 1$ on $B(r_i/2)$ and $\delta(x) \leq R_i$ we find that

$$\|u/\delta\|_{p(\cdot), B(R_i)} \geq \tfrac{1}{R_i} \|u\|_{q_i, B(r_i/2)} \geq C \frac{r_i^{n/q_i}}{R_i}.$$

Combining these two estimates gives

$$\frac{\|u/\delta\|_{p(\cdot)}}{\|\nabla u\|_{p(\cdot)}} \geq C \frac{r_i^{n/q_i - n/q + 1}}{R_i}.$$

Let us choose r_i and q_i so that $r_i^{n/q_i - n/q} \to \infty$ as $i \to \infty$, equivalently such that $(q_i - q) \log \frac{2}{r_i}$ tends to infinity. Then we can choose R_i so that the right hand side in the previous inequality also tends to infinity, for instance

$$R_i = r_i^{1 + \frac{n}{2}(1/q_i - 1/q)}.$$

This means that the domain formed by stringing together the balls $B(R_i)$ does not support a Hardy inequality (by dropping some balls if necessary we may assume that $R_i \le 2^{-i}$ and thus the domain constructed is bounded).

It is clear that our exponent is continuous. We see that

$$\frac{|p(x)-p(y)|}{\omega(|x-y|)} \le \frac{q_i - q}{\omega(2/r_i)}.$$

By our assumption on ω it is possible to choose r_i and q_i such that $\frac{q_i-q}{\omega(2/r_i)}$ remains bounded but $(q_i-q)\log\frac{2}{r_i}$ tends to infinity. This is the required exponent. □

4.3. Remark. Instead of assuming that $q, q_i \in [1,n)$ we could prove this theorem with $q, q_i \in (n, M)$ (for some finite $M > n$). The proof is essentially the same and is thus omitted.

5. Hardy's inequality in $\mathbb{R}$

In this section we look at Hardy's inequality in one dimension. In this case we are able to prove necessary and sufficient conditions that are much closer to each other than we could in $\mathbb{R}^n$. Specifically, we are able to show that whether or not Hardy's inequality holds depends essentially only on the behavior of the exponent at the boundary. This kind of behaviour was already observed for a Hardy-type operator in [13].

The following statement is proved as part of [1, Lemma 3.3].

5.1. Lemma. *Let q be a bounded variable exponent on an interval $I \subset \mathbb{R}$. Then*

$$\left(\fint_B u(y)\,dy\right)^{q(x)} \le C r^{c(q_B^- - q_B^+)}\left(\fint_B u(y)^{q(y)}dy + 1\right)$$

for positive u with $\varrho_{q(\cdot)}(u) \le 1$, where $B = B(x,r) \subset I$ and the constants depend only on q, n and I.

The proof of the following theorem is similar in spirit to the proof by Diening that the maximal function is locally bounded, [1]. It was pointed out by the referee that a similar theorem had already been derived in [13, Theorem E.II]. That theorem was concerned with slightly different type of Hardy inequalities and allows also negative values for the weight exponent a. Our proof is different from that in [13].

5.2. Theorem. *Let $I = [0, M)$ for $M < \infty$, $p\colon I \to [1,\infty)$ be bounded, $p(0) > 1$ and*

$$\limsup_{x\to 0+} \big(p(x) - p(0)\big)\log\tfrac{1}{x} < \infty \tag{5.3}$$

and $p^-_{(0,x_0)} = p(0)$ for some $x_0 \in (0,1)$. If $a \in [0, 1-\frac{1}{p(0)})$, then Hardy's inequality

$$\left\|\tfrac{u(x)}{x^{1-a}}\right\|_{p(x)} \le C\left\|u'(x)x^a\right\|_{p(x)}$$

holds for every $u \in W^{1,p(\cdot)}(I)$ with $u(0) = 0$.

Proof. Since the case is one-dimensional and p is bounded, smooth functions are now dense in $W^{1,p(\cdot)}(I)$, [7, Theorem 4.1]. Using this it is easy to see that it suffices to consider smooth $u \in W^{1,p(\cdot)}(I)$ with $u(0) = 0$.

The inequality to prove is homogeneous. Thus it suffices to consider the case $\|u'(x)x^a\|_{p(x)} = 1$. We start by considering the norm of u away from x_0. For $x > x_0$ we have

$$|u(x)| - |u(x_0)| \le \int_{x_0}^{x} |u'(y)|dy \le \int_{x_0}^{x} 1 + \big(|u'(y)|\big(\tfrac{y}{x_0}\big)^a\big)^{p(y)} dy \le M + x_0^{-ap^+}.$$

The function $v = u\chi_{(0,x_0)} + u(x_0)\chi_{(x_0,M)}$ has derivative $u'\chi_{(0,x_0)} \le u'$, so this function satisfies Hardy's inequality with exponent p^-. Hence

$$\begin{aligned}|u(x_0)|^{p^-} &\le C\int_{x_0}^{M} (v(x)x^{a-1})^{p^-}\, dx \le C\int_0^M (v'(x)x^a)^{p^-}\, dx\\ &\le C\int_0^M 1 + (u'(x)x^a)^{p(x)}\, dx = C(M+1).\end{aligned}$$

Combining the previous two inequalities, we get a uniform bound for $|u(x)|$ on (x_0, M), and thus we directly see that $\|\frac{u(x)}{x^{1-a}}\|_{L^{p(x)}(x_0,M)}$ has a uniform bound, as well.

So it remains to consider the norm on $(0, x_0)$. Moreover, since $p^+ < \infty$ it suffices to show that $\varrho_{L^{p(x)}(0,x_0)}(u(x)/x^{1-a})$ is bounded by a constant independent of u. We start by observing that $u(x) = \int_0^x u'(y)\, dy$. So we need to show that

$$\int_0^{x_0} \left|\frac{\int_0^x u'(y)\, dy}{x^{1-a}}\right|^{p(x)} dx = \int_0^{x_0} \left|⨍_0^x u'(y)\, dy\right|^{p(x)} x^{ap(x)}dx < C$$

provided that

$$⨍_0^M |u'(x)x^a|^{p(x)}dx = 1.$$

Let us denote $v = u' \in L^{p(\cdot)}(0, x_0)$ and $q(x) = p(x)/p(0)$ for $x \in (0, x_0)$. By Lemma 5.1 we know that

$$\left(⨍_0^x |v(y)|\, dy\right)^{q(x)} \le Cx^{c(q^-_{(0,x)} - q^+_{(0,x)})}\left(⨍_0^x |v(y)|^{q(y)}dy + 1\right).$$

Using the assumption (5.3) it follows as in [1, Lemma 3.2] that $x^{c(q^-_{(0,x)} - q^+_{(0,x)})} < C$. Further we observe that $x^{ap(x)} < Cx^{ap(0)}$, which holds since x is bounded by x_0 and $p(x) \ge p(0)$. Thus we have

(5.4)
$$\int_0^{x_0} \left|⨍_0^x v(y)\, dy\right|^{p(x)} x^{ap(x)}dx \le C\int_0^{x_0} \left(⨍_0^x |v(y)|^{q(y)}dy\right)^{p(0)} x^{ap(0)} + C$$

Expressing a function as the integral of its derivative as we did above, we see that Hardy's inequality for the fixed exponent $p(0) > 1$ gives

$$\begin{aligned}&\int_0^{x_0} \left(⨍_0^x |w(y)|\, dy\right)^{p(0)} x^{ap(0)}dx\\ &\quad\le \left(\frac{p(0)}{p(0)-1-a}\right)^{p(0)} \int_0^{x_0} |w(x)|^{p(0)}x^{ap(0)}dx.\end{aligned}$$

Applying this to the first term of the right hand side of (5.4) (with $w(x) = |v(x)|^{q(x)}$) gives

$$\int_0^{x_0} \left| \fint_0^x v(y)\,dy \right|^{p(x)} x^{ap(x)} dx \le C \int_0^{x_0} |w(x)|^{p(0)} x^{ap(0)} dx + C$$
$$\le C \int_0^{x_0} |v(x)|^{p(x)} x^{ap(x)} dx + C \le 2C.$$

For the second inequality we used that $x^{ap(0)} < Cx^{ap(x)}$, which again follows as in [1, Lemma 3.2]. □

5.5. Remark. Notice that the assumption $p(0) > 1$ is necessary. Also it is worth noting that we did not need the assumption $p^- > 1$, which is commonly associated to variable exponent results if the fixed exponent claim holds in $(1, \infty)$.

Next we adapt the example of Pick and Růžička [21] to show that the condition in the previous theorem is essentially sharp. In the next theorem we denote p_I^+ by p^+.

5.6. Theorem. *Let p be defined in $I = [0, \infty)$ with $p^+ < \infty$. If p is increasing at 0 such that*

$$\limsup_{x \to 0+} \big(p(x) - p(0)\big) \log \tfrac{1}{x} = \infty,$$

then Hardy's inequality

$$\left\| \tfrac{u(x)}{x} \right\|_{p(x)} \le C \left\| u'(x) \right\|_{p(x)}$$

does not hold.

Proof. To show that the Hardy inequality does not hold it suffices to construct a $p(\cdot)$-integrable function u, $u(0) = 0$, with $p(\cdot)$-integrable derivative but with $\varrho_{p(x)}(u(x)/x) = \infty$.

We start by noting that

$$\lim_{x \to 0+} \Big(\frac{1}{x}\Big)^{\frac{1}{p(x/2)} - \frac{1}{p(0)}} = \exp\Big(-\lim_{x \to 0+} \frac{p(x/2) - p(0)}{p(0)p(x/2)} \log \tfrac{1}{x} \Big) = 0.$$

We can choose $b_k \in (0, 1)$ so small that

$$\Big(\frac{1}{b_k}\Big)^{\frac{1}{p(b_k/2)} - \frac{1}{p(0)}} \le 2^{-\frac{k}{p(0)}},$$

and $0 < b_{k+1} < \frac{b_k}{2}$. By throwing away a finite number of terms and relabeling we assume that p is increasing on $(0, b_1)$. Having fixed such sequence (b_k), we denote $a_k = \frac{b_k}{2}$, $c_k = (a_k + b_k)/2$ and $\lambda_k = \big(\frac{1}{b_k}\big)^{\frac{1}{p(a_k)}}$. Notice that $\lambda_k \le (2^{-k} b_k)^{-1/p(0)}$ because of the way the b_k's were chosen. We define

$$u'(x) = \sum_{k=1}^{\infty} \lambda_k \chi_{(a_k, b_k)}(x).$$

(If we require that $u \in L^{p(\cdot)}(I)$, then we have to add a negative part to the derivative at values of x larger than b_1. However it is clear that such a modification will not affect anything in what follows.) Then it is easy to see that

$$u(x) = \int_0^\infty \sum_{k=1}^\infty \lambda_k \chi_{(a_k,b_k)}(x)\,dx \geq \tfrac{1}{2} \sum_{k=k(x)}^\infty \lambda_k b_k,$$

where $k(x)$ is the smallest k such that $b_k \leq x$. From this we conclude that $u(x) \geq \frac{1}{4}\lambda_k b_k$ for $x \in (c_k, b_k)$.

Let us check that $\varrho_{p(x)}(u(x)/x) = \infty$ but $\varrho_{p(\cdot)}(u') < \infty$, so that Hardy's inequality does not hold. As for the first, we have

$$\begin{aligned}\int_0^\infty \Big(\frac{|u(x)|}{x}\Big)^{p(x)} dx &\geq \sum_{k=1}^\infty \int_{c_k}^{b_k} (\tfrac{1}{4}\lambda_k)^{p(x)} dx \geq \frac{1}{2^{p^+}} \sum_{k=1}^\infty \int_{c_k}^{b_k} b_k^{-p(x)/p(a_k)} dx \\ &\geq \frac{1}{2^{p^+}} \sum_{k=1}^\infty \int_{c_k}^{b_k} b_k^{-1} dx = \infty,\end{aligned}$$

where we used that $p(x)/p(a_k) \geq 1$ for $x \in (a_k, b_k)$ and that $b_k < 1$. To estimate the modular of the derivative we find, again keeping in mind that p is increasing, that

$$\begin{aligned}\int_0^\infty u'(x)^{p(x)} dx &= \sum_{k=1}^\infty \int_{a_k}^{b_k} \lambda_k^{p(x)} dx \leq \sum_{k=1}^\infty \int_{a_k}^{b_k} (2^k b_k)^{-p(x)/p(0)} dx \\ &\leq \sum_{k=1}^\infty 2^{-k+1} \int_{a_k}^{b_k} (2b_k)^{-p(b_k)/p(0)} dx \\ &= \sum_{k=1}^\infty 2^{-k} (2b_k)^{1-p(b_k)/p(0)}.\end{aligned}$$

So we will be done once we show that $(2b_k)^{1-p(b_k)/p(0)}$ is bounded. This obviously follows if we show that $x^{1-p(x/2)/p(0)}$ is bounded for all $x \in (0, 2b_1)$. Since

$$x^{1-p(x/2)/p(0)} = \big(x^{1/p(x/2)-1/p(0)}\big)^{p(x/2)} \leq \max\Big\{1, \big(x^{1/p(x/2)-1/p(0)}\big)^{p^+}\Big\},$$

it suffices to show that $x^{1/p(x/2)-1/p(0)}$ is bounded. Since $x^{1/p(x/2)-1/p(0)} \to 0$ as $x \to 0$, we can choose x_0 such that $x^{1/p(x/2)-1/p(0)} \leq 1$ for $x < x_0$. For $x > x_0$ we have $x^{1/p(x/2)-1/p(0)} \leq x_0^{1/p^+-1/p(0)}$. □

Acknowledgements.

We thank the referees for some useful comments and Bilal Çekiç for pointing out an error in a proof of Theorem 5.2 in an earlier version of the manuscript.

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