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Lebesgue points in variable exponent Sobolev spaces on metric measure spaces

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In this article we prove a version of the Lebesgue point theorem in the variable exponent Newtonian space. The result is proved under the assumptions that the exponent is logarithmically Hölder continuous and that the bounded doubling space supports a $(1,1)$-Poincaré inequality.

1. Introduction. For a metric measure space (X, d, μ) and a bounded measurable function $p\colon X \to [1, \infty)$ the variable exponent Lebesgue spaces $L^{p(\cdot)}(X)$ consists of all measurable functions such that

$$\int_X |u(x)|^{p(x)} d\mu(x) < \infty.$$

For constant p they coincide with classical Lebesgue spaces. These spaces are special cases of Orlicz–Musielak spaces, see [15]. Nowadays the special case has been the focus of quite a lot of research. Such spaces are related to differential equations with non-standard growth and coercivity conditions, arising for instance from modeling certain fluids (e.g, [1, 16]). Many properties of these spaces, like boundedness of maximal and potential operators, density of smooth functions and pointwise properties of Sobolev functions, have been studied in the Euclidean case. For some of the latest advances see for example [2, 4, 10, 13] and the references therein.

Recently the theory has been extended to metric measure spaces [5, 8]. These investigations were concerned with extending results about the

Hardy–Littlewood maximal and Riesz potential operator to this new setting. Subsequently, also variable exponent Sobolev spaces have been introduced in the metric measure spaces in [9], where some basic properties, such as completeness and density of Lipschitz functions were addressed.

We continue along this line of research by studying the capacity version of the Lebesgue point theorem in the case of variable exponent Sobolev functions on metric measure spaces. We combine techniques from [12] (metric measure spaces, fixed exponent) and [6] (Euclidean space, variable exponent). It seems that certain weak type capacity estimates for the maximal operator are the only known way to obtain the desired result. The novelty in our arguments is that our weak type result is somewhat weaker than in previous proofs and the proof of our main theorem is somewhat non-standard using the ideas of [11]. We are also able to slightly improve the main result of [6] by localizing the use of the maximal function.

2. Preliminaries. By a *metric measure space* we mean a triple (X, d, μ), where X is a set, d is a metric on X and μ is a non-negative Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$ we denote by $B(x, r)$ the open ball centered at x with radius r. We use the convention that C denotes a constant whose value can change even between different occurrences in a chain of inequalities.

A metric measure space X or a measure μ is said to be *doubling* if there is a constant $C \geqslant 1$ such that

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)) \tag{1}$$

for every open ball $B(x, r) \subset X$. The constant C in (1) is called the *doubling constant* of μ. By the doubling property, if $y \in X$, $0 < r \leqslant R < \infty$ and $x \in B(y, R)$, then

$$\frac{\mu(B(x, r))}{\mu(B(y, R))} \geqslant C_Q \left(\frac{r}{R}\right)^Q \tag{2}$$

for some C_Q and Q depending only on the doubling constant. For example, in $\mathbb{R}^n$ with the Lebesgue measure (2) holds with Q equal to the dimension n. The constant Q is called the dimension of X.

By a *variable exponent* we mean a bounded measurable function $p \colon X \to [1, \infty)$. For $A \subset X$ we define $p_A^+ = \operatorname{ess\,sup}_{x\in A} p(x)$ and $p_A^- = \operatorname{ess\,inf}_{x\in A} p(x)$; we further abbreviate $p^+ = p_X^+$ and $p^- = p_X^-$. For a

μ–measurable function $u\colon X \to \mathbb{R}$ we define the *modular*

$$\varrho_{p(\cdot)}(u) = \int_X |u(y)|^{p(y)} d\mu(y)$$

and the norm

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(X, d, \mu)$ consists of those μ-measurable functions $u\colon X \to \mathbb{R}$ for which $\|u\|_{p(\cdot)} < \infty$. Some basic properties of variable exponent Lebesgue spaces on metric measure spaces are given in [8], and most of them are straightforward generalizations of the Euclidean case [14].

We say that X *supports a* $(1,1)$-*Poincaré inequality* if there exists a constant $C > 0$ such that for all open balls B in X and all pairs of functions u and ρ defined on B, the inequality

$$\fint_B |u - u_B|\, d\mu \leqslant C \operatorname{diam}(B) \fint_B \rho\, d\mu$$

holds whenever ρ is a weak upper gradient of u on B and u is integrable on B. If X is a doubling space that supports a $(1,1)$-Poincaré inequality, $p^+ < \infty$ and the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(X)$ to itself, then Lipschitz continuous functions are dense in $N^{1,p(\cdot)}(X)$ [9, Theorem 4.5].

3. Newtonian spaces. A curve γ in X is a non-constant continuous map $\gamma : I \to X$, where $I = [a, b]$ is a closed interval in $\mathbb{R}$. The image of γ, $\gamma(I)$, is denoted by $|\gamma|$. Let Γ be a family of rectifiable curves. We denote by $F(\Gamma)$ the set of all *admissible functions*, i.e. all Borel measurable functions $\rho\colon X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \geqslant 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to curve length. We define the $p(\cdot)$-*modulus of* Γ by

$$\mathrm{M}_{p(\cdot)}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho(x)^{p(x)} d\mu(x).$$

If $F(\Gamma) = \emptyset$, then we set $\mathrm{M}_{p(\cdot)}(\Gamma) = \infty$. The $p(\cdot)$-modulus is an outer measure in the set of curves. The proof of this is a straightforward generalization of the variable exponent Euclidean case [10].

Let u be a real valued function on X. A non-negative Borel measurable function ρ on X is a (*weak*) *upper gradient of* u if there exists a family Γ of rectifiable curves with $\mathrm{M}_{p(\cdot)}(\Gamma)=0$ such that

$$|u(x)-u(y)| \leqslant \int_\gamma \rho\, ds$$

for every rectifiable curve $\gamma \notin \Gamma$ with endpoints x and y.

The *Newtonian space* $N^{1,p(\cdot)}(X)$ consists of functions in $L^{p(\cdot)}(X)$ with a weak upper gradient in $L^{p(\cdot)}(X)$ equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \inf \|\rho\|_{p(\cdot)}$$

where the infimum is taken over all weak upper gradients of u. If $p^+ < \infty$, then $N^{1,p(\cdot)}(X)$ is a Banach space. For more details on the Newtonian space see [9, Sections 4 and 5].

Lemma 1. *Let $u \in N^{1,p(\cdot)}(X)$. If g_1 is a weak upper gradient of u and $g_1 \leqslant g_2$ μ-almost everywhere, then g_2 is also a weak upper gradient of u.*

Proof. Let Γ be the family of curves γ for which

$$\int_\gamma g_1\, ds > \int_\gamma g_2\, ds.$$

It suffices to show that Γ is exceptional. For every curve $\gamma \in \Gamma$ we clearly have $H^1\big(|\gamma| \cap \big\{x \in X : g_1(x) > g_2(x)\big\}\big) > 0$. Therefore $\infty\chi_{\{g_1(x)>g_2(x)\}}$ is an admissible test function for Γ. On the other hand, this function is zero almost everywhere, so its integral is zero. Thus the modulus of Γ is zero. □

Lemma 2. *Let $u, v \in N^{1,p(\cdot)}(X)$. If g_1 is a weak upper gradient of u and g_2 is a weak upper gradient of v, then $|v|g_1 + |u|g_2$ is a weak upper gradient of uv.*

Proof. Let γ be an arc-length parametrization of a rectifiable curve. We define $h\colon [0,\ell(\gamma)] \to (-\infty,\infty)$ by setting $h(s) = (u\circ\gamma)(s)\ (v\circ\gamma)(s)$. The function h is absolutely continuous on γ (not in a certain exceptional family). Hence we obtain

$$h'(s) = (u\circ\gamma)'(s)\ (v\circ\gamma)(s) + (u\circ\gamma)(s)\ (v\circ\gamma)'(s)$$

and, furthermore,

$$|h'(s)| \leqslant g_1(\gamma(s))v(\gamma(s)) + g_2(\gamma(s))u(\gamma(s))$$

for almost every $s \in [0, \ell(\gamma)]$. Since

$$h(s) - h(t) = \int_{\gamma[s,t]} h' ds.$$

Lemma 1 implies that $|v|g_1 + |u|g_2$ is a weak upper gradient of uv. □

Lemma 3. *Suppose that u_i, $i = 1, 2, \ldots$, are measurable functions. Let g_i be an upper gradient of u_i and denote $u = \sup_i u_i$ and $g = \sup_i g_i$. Then g is an upper gradient of u provided that u is finite almost everywhere.*

Proof. Let Γ_i be an exceptional family of curves for u_i and write $\Gamma = \cup_i \Gamma_i$. Then by subadditivity the modulus of Γ is zero. Let $\gamma \notin \Gamma$ be a rectifiable curve with end points x and y. Assume that $u(y) \leqslant u(x) < \infty$. Let $\varepsilon > 0$ and choose i so that $u_i(x) + \varepsilon \geqslant u(x)$. Since $u(y) \geqslant u_i(y)$ we obtain that

$$\begin{aligned}|u(x) - u(y)| &= u(x) - u(y) \leqslant u_i(x) + \varepsilon - u_i(y)\\ &\leqslant \int_\gamma g_i\, ds + \varepsilon \leqslant \int_\gamma g\, ds + \varepsilon.\end{aligned}$$

The claim follows for finite values of u at the end points as $\varepsilon \to 0$.

Using $\infty\chi_{\{u=\infty\}}$ as a test function we find that the modulus of those curves where u is infinity in some subcurve is zero. We denote this family by Γ_∞. Let $\gamma \notin \Gamma \cup \Gamma_\infty$ be a curve from x to y. We may assume that $u(x) = \infty$ and $u(y)$ is finite; otherwise we find a point z from the curve with $u(z) < \infty$ and study subcurves from x to z and z to y. Let u_j be a subsequence with $u_j(x) \to u(x) = \infty$ as $j \to \infty$. For each j we have

$$|u_j(x) - u_j(y)| \leqslant \int_\gamma g_j\, ds \leqslant \int_\gamma g\, ds$$

For large j we have $|u_j(x) - u(y)| \leqslant u_j(x) - u(y) \leqslant u_j(x) - u_j(y)$. Letting $j \to \infty$ we obtain

$$|u(x) - u(y)| \leqslant \int_\gamma g\, ds.$$

Hence the desired inequality holds for those rectifiable curves which do not belong to $\Gamma \cup \Gamma_\infty$. □

Let $E \subset X$. We recall the definition of the capacity in Newtonian space:

$$C_{p(\cdot)}(E) = \inf_u \inf_\rho \int_X |u(x)|^{p(x)} + |\rho(x)|^{p(x)}\, d\mu(x),$$

where the first infimum is taken over all $u \in N^{1,p(\cdot)}(X)$, which are at least 1 in E and the second infimum is taken over upper gradients of u. If the class of test functions is empty we set $C_{p(\cdot)}(E) = \infty$. We need the fact that the capacity is an outer measure provided that $p^+ < \infty$ [9, Section 4.1]. We also need the following lemma which is obtained following standard arguments, see e.g. [7, Lemma 5.1].

Lemma 4. *Let $p^+ < \infty$. For each Cauchy sequence of functions in $N^{1,p(\cdot)}(X)$ there is a subsequence which converges pointwise outside a set of zero $p(\cdot)$-capacity. Moreover, the convergence is uniform outside a set of arbitrary small $p(\cdot)$-capacity.*

4. Maximal operators. In [12] Kinnunen and Latvala introduced the so-called discrete maximal function as a way to overcome the difficulty that the ordinary maximal operator is not bounded in the set of Sobolev functions in metric spaces, see [3]. In this section we study this operator in the variable exponent case.

We say that the exponent p is log-Hölder continuous if

$$|p(x) - p(y)| \leqslant \frac{C}{-\log d(x,y)}$$

for $d(x,y) \leq \frac{1}{2}$. The Hardy–Littlewood maximal operator is defined for a locally integrable function u by

$$Mu(x) = \sup_{r>0} \fint_{B(x,r)} |u(y)|\, d\mu(y).$$

In bounded doubling metric measure spaces, the Hardy-Littlewood maximal operator is bounded form $L^{p(\cdot)}(X)$ to itself if $1 < p^- \leqslant p^+ < \infty$ and p is log-Hölder continuous, [8, Theorem 4.3]. A weaker result, derived under fewer assumptions, was given in [5, Theorem 2.3]. In contrast to the Euclidean case, the log-Hölder continuity is by no means necessary – the Hardy-Littlewood maximal operator might be bounded even if the exponent is monotone and discontinuous, see [8, Example 4.5].

Next we recall from [12, Section 3] the definition of the discrete maximal function. Fix $r > 0$ and let $B(x_i, r)$, $i = 1, 2, \ldots$, be a family of balls covering X such that any point $x \in X$ belongs to at most θ balls $B(x_i, 6r)$. Here θ can be chosen to depend only on the doubling constant. Let ϕ_i be a set of functions such that $0 \leqslant \phi \leqslant 1$, $\phi_i = 0$ in the complement of $B(x_i, 3r)$, $\phi_i \geqslant c > 0$ in $B(x_i, r)$, ϕ_i is Lipschitz with a constant c/r and

$$\sum_{i=1}^{\infty} \phi_i = 1$$

on X; the constant c can be chosen depending only on the doubling constant, e.g. as $c = 1/\theta$ (see [12, p. 690] for a construction of such functions). We set

$$u_r(x) = \sum_{i=1}^{\infty} \phi_i(x) \fint_{B(x_i,3r)} |u(y)|\, d\mu(y).$$

Let $(r_j)_{j=1}^{\infty}$ be an enumeration of the positive rationals. For every radius r_j we choose a covering $(B(x_i, r_j))$ as above. We define the discrete maximal function related to the covering $(B(x_i, r_j))$ by

$$M^* u(x) = \sup_j u_{r_j}(x).$$

Note that the defined maximal operator depends on the chosen coverings. However, by [12, Lemma 3.1], the inequalities

$$\frac{1}{c} Mu(x) \leqslant M^* u(x) \leqslant cMu(x) \tag{3}$$

always hold for every $x \in X$ and every locally integrable function u. Here the constant $c \geqslant 1$ depends only on the doubling constant.

The following lemma is a generalization of [12, Lemma 3.3]. For $u \in N^{1,p(\cdot)}(X)$ we denote by ρ_u a weak upper gradient of u.

Lemma 5. *Suppose that X is doubling and supports a $(1,1)$-Poincaré inequality. Assume that $p^+ < \infty$ and $u \in N^{1,p(\cdot)}(X)$. Then $CM\rho_u$ is a weak upper gradient of $M^* u$.*

Proof. Assume without loss of generality that u is non-negative. We first prove the lemma with frozen scale and hence we fix $r > 0$. Denote $B_i = B(x_i, r)$ and $cB_i = B(x_i, cr)$. By Lemmas 1 and 2,

$$\Big(\frac{C}{r}|u_{3B_i} - u| + \rho_u\Big)\chi_{6B_i}$$

is a weak upper gradient of $\phi_i(x)(u_{3B_i} - u(x))$. We have

$$u_r(x) = \sum_{i=1}^{\infty} \phi_i(x) u_{3B_i} = u(x) + \sum_{i=1}^{\infty} \phi_i(x)(u_{3B_i} - u(x)).$$

Using the above upper gradient for the terms in the sum and the fact that each point is covered by at most θ balls $6B_i$, we conclude by Lemma 1 that

$$\rho_u + \sum_{i=1}^{\infty} \Big(\frac{C}{r}|u_{3B_i} - u| + \rho_u\Big)\chi_{6B_i} \leqslant (\theta+1)\rho_u + \frac{C}{r}\sum_{i=1}^{\infty} |u_{3B_i} - u|\chi_{6B_i}$$

is a weak upper gradient of u_r. By the Lebesgue points theorem for doubling measures, $\rho_u(x) \leqslant M\rho_u(x)$ for μ-almost every $x \in X$. By the Poincaré inequality, the same arguments as in [12, Lemma 3.3] imply that

$$|u(x) - u_{3B_i}| \leqslant CrM\rho_u(x)$$

for μ-almost every $x \in X$. Thus we have shown that

$$(\theta + 1)\rho_u + \frac{C}{r}\sum_{i=1}^{\infty} |u_{3B_i} - u|\chi_{6B_i} \leqslant CM\rho_u$$

μ-almost everywhere. By Lemma 1, this implies that $CM\rho_u$ is a weak upper gradient of u_r.

Since $M^*u(x) < \infty$ μ-almost everywhere [8, Theorem 5.2], we obtain by Lemma 3 that $CM\rho_u$ is a weak upper gradient of M^*u. □

5. The Lebesgue Theorem. The proof of the main theorem is based on the following lemma.

Lemma 6. *Suppose that X is doubling and supports a $(1, 1)$-Poincaré inequality. Assume that $1 < p^- \leqslant p^+ < \infty$, p is* log*-Hölder continuous and $u \in N^{1,p(\cdot)}(X)$. Then*

$$C_{p(\cdot)}\big(\{x \in B(x_0, r) : Mu(x) > \lambda\}\big) \to 0$$

as $\|u\|_{1,p(\cdot)} \to 0$ for every $x_0 \in X$ and $r, \lambda > 0$.

Proof. Let ϕ be a Lipschitz continuous cut-off function such that $\phi = 1$ in $B(x_0, r)$, $\phi = 0$ in the complement of $B(x_0, 2r)$ and the upper gradient of ϕ is bounded by C depending only on r and the doubling constant. Since $M : L^{p(\cdot)}(B(x_0, 2r)) \to L^{p(\cdot)}(B(x_0, 2r))$ is bounded, we obtain from (3) that

$$\|M^*(u\phi)\|_{p(\cdot)} \leqslant C\|M(u\phi)\|_{p(\cdot)} \leqslant C\|u\phi\|_{p(\cdot)}.$$

Hence $M^*(u\phi) \in L^{p(\cdot)}(B(x_0, 2r))$. By Lemma 5, $M\rho_{u\phi}$ is a weak upper gradient of $M^*(u\phi)$. Since the boundedness of M also implies that $M\rho_{u\phi} \in L^{p(\cdot)}(B(x_0, 2r))$, we obtain that $M^*(u\phi) \in N^{1,p(\cdot)}(B(x_0, 2r))$. By (3) we may use $CM^*(u\phi)/\lambda$ as a test function for the capacity in order to obtain that

$$\begin{aligned}
&C_{p(\cdot)}\big(\{x\in B(x_0,r): Mu(x)>\lambda\}\big)\\
&\quad\leqslant \int_X (C\lambda^{-1}M^*(u\phi)(x))^{p(x)} + (C\lambda^{-1}\rho_{M^*(u\phi)}(x))^{p(x)}\,d\mu(x)\\
&\quad\leqslant C\int_{B(x_0,2r)} (\lambda^{-1}Mu(x))^{p(x)} + (\lambda^{-1}M\rho_{u\phi}(x))^{p(x)}\,d\mu(x)\\
&\quad\leqslant C\max\big\{\|Mu/\lambda\|_{1,p(\cdot)}, \|Mu/\lambda\|_{1,p(\cdot)}^{p^+}\big\}.\\
&\quad\leqslant C\max\big\{\|u/\lambda\|_{1,p(\cdot)}, \|u/\lambda\|_{1,p(\cdot)}^{p^+}\big\},
\end{aligned}$$

where the second-to-last inequality follows by [14, (2.11)], and the norms are in $B(x_0,2r)$ only. □

Theorem 1. *Suppose that X is a metric space equipped with a doubling measure supporting a $(1,1)$-Poincaré inequality. Assume that $1<p^-\leqslant p^+<\infty$, p is* log*-Hölder continuous and $u\in N^{1,p(\cdot)}(X)$. Then there exists a set $E\subset X$ of zero $p(\cdot)$-capacity such that*

$$u(x)=\lim_{r\to 0} \fint_{B(x,r)} u(y)\,dy$$

and

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y)-u(x)|\,dy = 0$$

for every $x\in X\setminus E$.

Proof. We use ideas from the proof of [11, Lemma 3.1]. Since $X=\cup_{j=1}^{\infty}B(x_0,j)$ for $x_0\in X$ and the capacity is subadditivity [9, Theorem 4.9], it is enough to prove the claim in a given ball $B:=B(x_0,j)$. By [9, Theorem 4.5], Lipschitz functions are dense in $N^{1,p(\cdot)}(B)$ and hence we may choose a sequence (u_i) of continuous functions in $N^{1,p(\cdot)}(B)$ such that $\|u-u_i\|_{1,p(\cdot)}\to 0$ as $i\to\infty$. Using the notation

$$\overline{D}f(x)=\limsup_{r\to 0}\frac{1}{B(x,r)}\int_{B(x,r)} f(y)\,d\mu(y)$$

we find for $x\in B$ that

$$\overline{D}|u-u(x)|(x)\leqslant \overline{D}|u-u_i|(x)+|u(x)-u_i(x)|.$$

Using the subadditivity of the capacity we find, for any $\lambda > 0$, that

$$\begin{aligned} &C_{p(\cdot)}\big(\{x \in B : \overline{D}|u-u(x)|(x) > \lambda\}\big) \\ &\quad \leqslant C_{p(\cdot)}\big(\{x \in B\colon \overline{D}|u-u_i|(x) > \tfrac{\lambda}{2}\}\big) \\ &\qquad + C_{p(\cdot)}\big(\{x \in B\colon |u(x)-u_i(x)| > \tfrac{\lambda}{2}\}\big). \end{aligned}$$

It follows from Lemma 6 that

$$C_{p(\cdot)}\big(\{x \in B\colon \overline{D}|u-u_i|(x) > \tfrac{\lambda}{2}\}\big) \to 0$$

as $i \to \infty$. On the other hand, using $\frac{2}{\lambda}|u-u_i|$ as a test function like in the proof of Lemma 6, we conclude that

$$C_{p(\cdot)}\big(\{x \in B\colon |u(x)-u_i(x)| > \tfrac{\lambda}{2}\}\big) \to 0$$

as $\|u-u_i\|_{1,p(\cdot)} \to 0$. Since the left-hand-side of our previous estimate does not depend on i, this means that

$$C_{p(\cdot)}\big(\{x \in B\colon \overline{D}|u-u(x)|(x) > \lambda\}\big) = 0$$

for any $\lambda > 0$. Denoting by E the set

$$\big\{\, x \in B\colon \overline{D}|u-u(x)|(x) > 0 \,\big\} = \bigcup_{k=1}^{\infty} \big\{\, x \in B\colon \overline{D}|u-u(x)|(x) > k^{-1} \,\big\}$$

we find that $C_{p(\cdot)}(E) = 0$ by using the subadditivity of the capacity. Then the second claim holds and the first claim easily follows. $\square$

6. The Euclidean case. In this section we prove that in the Euclidean setting our new capacity is equivalent to the variable exponent Sobolev capacity studied in [7]. Suppose that E is an arbitrary subset of $\mathbb{R}^n$. The *Sobolev* $p(\cdot)$*-capacity* of E is defined by

$$Cap_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\, dx,$$

where $S_{p(\cdot)}(E)$ is the set of all $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ which are at least one in an open set containing E. If $S_{p(\cdot)}(E) = \emptyset$, we set $Cap_{p(\cdot)}(E) = \infty$. If $1 < p^- \leqslant p^+ < \infty$, then the Sobolev $p(\cdot)$-capacity is an outer measure and Choquet capacity [7, Corollaries 3.3 and 3.4].

Lemma 7. *Assume that* $1 < p^- \leqslant p^+ < \infty$ *and* p *is* log*-Hölder continuous. Then for every* $E \subset \mathbb{R}^n$ *we have*

$$C_{p(\cdot)}(E) \leqslant Cap_{p(\cdot)}(E) \leqslant (\sqrt{n})^{p^+} C_{p(\cdot)}(E).$$

In particular, $C_{p(\cdot)}(E) = 0$ *if and only if* $Cap_{p(\cdot)}(E) = 0$.

Proof. Since $p^+ < \infty$ and p is log-Hölder continuous, C_0^∞-functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ [17]. This yields that $N^{1,p(\cdot)}(\mathbb{R}^n)$ and $W^{1,p(\cdot)}(\mathbb{R}^n)$ coincide and the norms are comparable [9, Theorem 5.3]. More precisely, we have $|\nabla u(x)| \leq \sqrt{n}\rho_u(x)$ for almost every $x \in \mathbb{R}^n$.

If $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is a test function for $Cap_{p(\cdot)}(E)$ then it is also a test function for $C_{p(\cdot)}(E)$ and thus the first inequality holds, since $|\nabla u|$ is a weak upper gradient of u.

Assume then that $u \in N^{1,p(\cdot)}(\mathbb{R}^n)$ and u is a test function for $C_{p(\cdot)}(E)$. Since continuous functions are dense in $N^{1,p(\cdot)}(\mathbb{R}^n)$ we obtain by Lemma 4 that u has a representative which is continuous outside a set of arbitrarily small $C_{p(\cdot)}$-capacity. Therefore for all $\varepsilon > 0$ there exists a set F_ε so that u restricted to $\mathbb{R}^n \setminus F_\varepsilon$ is continuous and $C_{p(\cdot)}(F_\varepsilon) \leq \varepsilon$. Hence there exists a neighborhood U of E such that u restricted to $U \setminus F_\varepsilon$ is not less that $1-\varepsilon$. Let $w_\varepsilon \in N^{1,p(\cdot)}(\mathbb{R}^n)$ be such that w_ε restricted to F_ε is one, $0 \leqslant w_\varepsilon \leqslant 1$ and

$$\int_{\mathbb{R}^n} |w_\varepsilon(x)|^{p(x)} + |\nabla w_\varepsilon(x)|^{p(x)} dx \leqslant \varepsilon.$$

Then $1/(1-\varepsilon)u + w_\varepsilon$ is a test function for $Cap_{p(\cdot)}(E)$ and the second claim follows by letting $\varepsilon \to 0$. □

By Theorem 1 and Lemma 7 we obtain the following theorem, which slightly improves [6, Theorem 4.7] by dropping out assumption of p near infinity.

Theorem 2. *Assume that $1 < p^- \leqslant p^+ < \infty$, p is* log*-Hölder continuous and $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is $p(\cdot)$-quasicontinuous. Then there exists a set $E \subset X$ of zero $p(\cdot)$-capacity such that for every $x \in \mathbb{R}^n \setminus E$ we have*

$$u(x) = \lim_{r \to 0} \fint_{B(x,r)} u(y)\, dy$$

and

$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - u(x)|\, dy = 0.$$

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