Variable exponent Sobolev spaces on metric measure spaces

Petteri Harjulehto[*], Peter Hästö[†] and Mikko Pere

Abstract

In this article we study variable exponent Sobolev spaces on metric measure spaces. We employ two definitions: a Hajłasz type definition, which uses a pointwise maximal inequality, and a Newtonian type definition, which uses an upper gradient. We prove that these spaces are Banach, that Lipschitz functions are dense as well as other basic properties. We also study when these spaces coincide.

Contents

[*]Supported in part by the Academy of Finland.

[†]Supported in part by a Gehring–Finland Postdoctoral Felloship and by the Finnish Academy of Science and Letters.

1 Introduction

The theory of Sobolev spaces was originally developed in domains of $\mathbb{R}^n$ using the notion of distributional derivatives. To generalize this theory to metric spaces alternative ways to define Sobolev spaces were needed. P. Hajłasz showed in [15] that a p-integrable function u, $1 < p < \infty$, belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if there exists a non-negative p-integrable function g such that

$$|u(x) - u(y)| \leqslant |x - y|(g(x) + g(y))$$

for almost every $x, y \in \mathbb{R}^n$. This inequality can be stated also in metric measure spaces if $|x - y|$ is replaced by the distance between the points x and y. Spaces defined using this inequality are often called Hajłasz spaces.

Another way to define Sobolev spaces on metric measure spaces is via the concept of an upper gradient. A non-negative function ρ is said to be an upper gradient of u if

$$|u(x) - u(y)| \leqslant \int_\gamma \rho \, ds \tag{1.1}$$

for every x, y and curve γ connecting x to y. In an open set Ω of $\mathbb{R}^n$ this holds for a Sobolev function u on every curve not belonging to an exceptional family of p-modulus zero [14]. If this holds in a general metric space, we call g a weak upper gradient of u. Existence and p-integrability of a weak upper gradient together with p-integrability of the function lead to another characterization of $W^{1,p}(\Omega)$. These spaces are often called Newtonian spaces. For basic properties of Newtonian spaces see the pioneering work of N. Shanmugalingam [39].

Sobolev spaces on metric measure spaces have been studied very intensively during the last ten years, and the "standard" framework is now

2000 *Mathematics Subject Classification.* Primary 46E35; Secondary 31B15.

Key words and phrases. Variable exponent, metric measure space, Sobolev space, Hajłasz space, Newtonian space.

available in the monograph [25] by J. Heinonen. For instance it is known that if the measure is doubling and the space supports a $(1, p-\varepsilon)$-Poincaré inequality, then the Hajłasz and Newtonian spaces coincide. These advances have made possible the development of potential theory in metric measure spaces, see for example [1, 29].

Variable exponent Lebesgue and Sobolev spaces have attracted a steadily increasing interest over the last couple of years, but most papers have dealt only with the Euclidean case. Variable exponent spaces have been independently discovered by several investigators [12, 31, 38, 40] and are related to differential equations with non-standard coercivity conditions. For some of the latest advances in the Euclidean theory see [9, 21, 30].

In metric measure spaces the variable exponent is very natural. For example, it allows us to study Sobolev spaces with integrability connected to the dimension of the space, which changes with location. Only three papers exist on variable exponent spaces on metric measure spaces, two by T. Futamura, Y. Mizuta and T. Shimomura [13, 33] and one by the authors [24]. All of these papers deal only with variable exponent Lebesgue spaces.

In this article we study Hajłasz and Newtonian spaces with variable exponent in metric measure spaces. In the next section we review some definitions and present the theory of variable exponent Lebesgue spaces on metric measure spaces. In Section 3 we study properties of the variable exponent Hajłasz space. We show that it is a Banach space, that the Poincaré inequality holds and that Lipschitz continuous functions are dense, and study a related Sobolev capacity. In Section 4 we show that the variable exponent Newtonian space is a Banach space. We prove that Lipschitz continuous functions are dense if the measure is doubling and the space supports a Poincaré inequality. In the final section we study when Hajłasz and Newtonian spaces coincide in a metric measure space and also in Euclidean space.

2 Preliminaries

This section contains some material on variable exponent Lebesgue spaces. The results in the first two sections are used throughout this paper. The last two sections are necessary only for specific parts, and can be read later as needed.

2.1 Metric measure spaces

By a *metric measure space* we mean a triple (X, d, μ), where X is a set, d is a metric on X and μ is a non-negative Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$ we denote by $B(x, r)$ the open ball centered at x with radius r. We use the convention that C denotes a constant whose value can change even between different occurrences in a chain of inequalities.

A metric measure space X or a measure μ is said to be *doubling* if there is a constant $C \geqslant 1$ such that

$$\mu(B(x, 2r)) \leqslant C\mu(B(x, r)) \tag{2.1}$$

for every open ball $B(x, r) \subset X$. The constant C in (2.1) is called the *doubling constant* of μ. By the doubling property, if $B(y, R)$ is an open ball in X, $x \in B(y, R)$ and $0 < r \leqslant R < \infty$, then

$$\frac{\mu(B(x, r))}{\mu(B(y, R))} \geqslant C_Q \left(\frac{r}{R}\right)^Q \tag{2.2}$$

for some C_Q and Q depending only on the doubling constant. For example, in $\mathbb{R}^n$ with the Lebesgue measure (2.2) holds with Q equal to the dimension n.

We say that the measure μ is *lower Ahlfors Q–regular* if there exists a constant $C > 0$ such that $\mu(B) \geqslant C \operatorname{diam}(B)^Q$ for every ball $B \subset X$ with $\operatorname{diam} B \leqslant 2 \operatorname{diam} X$. We say that μ is *upper Ahlfors Q–regular* if there exists a constant $C > 0$ such that $\mu(B) \leqslant C \operatorname{diam}(B)^Q$ for every ball $B \subset X$ with $\operatorname{diam} B \leqslant 2 \operatorname{diam} X$. The measure μ is *Ahlfors Q–regular* if it is upper and lower Ahlfors Q–regular, i.e. if $\mu(B) \approx \operatorname{diam}(B)^Q$ for every ball $B \subset X$ with $\operatorname{diam} B \leqslant 2 \operatorname{diam} X$. If X is a bounded doubling metric measure space, so that $\mu(X) < \infty$ and $\operatorname{diam}(X) < \infty$, then it is lower Ahlfors Q-regular.

2.2 Variable exponent Lebesgue spaces

We call a measurable function $p\colon X \to [1, \infty)$ a *variable exponent*. For $A \subset X$ we define $p_A^+ = \operatorname{ess\,sup}_{x \in A} p(x)$ and $p_A^- = \operatorname{ess\,inf}_{x \in A} p(x)$; we further abbreviate $p^+ = p_X^+$ and $p^- = p_X^-$. For a μ–measurable function $u\colon X \to \mathbb{R}$ we define the *modular*

$$\varrho_{p(\cdot)}(u) = \int_X |u(y)|^{p(y)}\, d\mu(y)$$

and the *norm*

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

Sometimes we use the notation $\|u\|_{p(\cdot),X}$ when we want to emphasize in what metric space the norm is taken. The *variable exponent Lebesgue space* $L^{p(\cdot)}(X,d,\mu)$ consists of those μ-measurable functions $u\colon X \to \mathbb{R}$ for which $\|u\|_{p(\cdot)} < \infty$. This is a special case of an Orlicz–Musielak space, cf. [34].

As in the Euclidean setting, we easily see that $\|\cdot\|_{p(\cdot)}$ is a norm. Also, if $\|f\|_{p(\cdot)} \leqslant 1$, then $\varrho_{p(\cdot)}(f) \leqslant \|f\|_{p(\cdot)}$. Moreover, if $p^+ < \infty$, then $\varrho_{p(\cdot)}(f_i) \to 0$ if and only if $\|f_i\|_{p(\cdot)} \to 0$. If $p\colon X \to (1,\infty)$, then Hölder's inequality,

$$\|fg\|_1 \leqslant C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

holds also in variable exponent Lebesgue spaces, where p' is the pointwise Hölder conjugate of p, i.e. $p(x) + p'(x) = p(x)p'(x)$.

Variable exponent Lebesgue spaces on metric measure spaces have been studied in [13, 24, 33]. Some of the basic results are that $L^{p(\cdot)}(X)$ is a Banach space if $p^+ < \infty$ [24, Lemma 3.1] and that continuous functions with compact support are dense in $L^{p(\cdot)}(X)$ provided X is a locally compact doubling space and $p^+ < \infty$ [24, Theorem 3.3]. As in [31, Theorem 2.8] we can prove that

$$\|u\|_{p(\cdot)} \leqslant (1 + \mu(X))\,\|u\|_{q(\cdot)}$$

if $\mu(X) < \infty$ and $p(x) \leqslant q(x)$.

Recall that a Banach space is said to be *uniformly convex* if for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\|u\| = \|v\| = 1$ and $\|u - v\| > \delta$ imply that $\|u + v\| < 2 - \varepsilon$. Recall also that a measure μ is atomless if $\mu(\{x\}) = 0$ for every point $x \in X$. For future reference we record the following simple but useful fact

Proposition 2.3. *If $1 < p^- \leqslant p^+ < \infty$ and μ is atomless, then $L^{p(\cdot)}(X,\mu)$ is uniformly convex.*

Proof. It is clear that the $(x,z) \mapsto z^{p(x)}$ is a uniformly convex modular in the sense of [34] (as also pointed out by Diening, [5]). Therefore it follows from [34, Theorem 11.6] that $L^{p(\cdot)}(X)$ is uniformly convex. □

Some well-known consequences of uniform convexity are summarized in the following corollary.

Corollary 2.4. *If* $1 < p^- \leqslant p^+ < \infty$ *and* μ *is atomless, then* $L^{p(\cdot)}(X,\mu)$ *is reflexive and has the Banach-Saks property, namely, if* $u_i \rightharpoonup u$ *weakly, then* $\frac{1}{i}(u_1 + \ldots + u_i) \to u$ *strongly.*

The following condition has emerged as the right one to guarantee a high degree of regularity for variable exponent spaces in $\mathbb{R}^n$. We say that $p\colon X \to [1,\infty)$ is log-*Hölder continuous* if

$$|p(x) - p(y)| \leqslant \frac{C}{-\log d(x,y)}, \tag{2.5}$$

when $d(x,y) \leqslant 1/2$. This condition has also been called Dini-Lipschitz, weak-Lipschitz and 0-Hölder. Since it is the limiting case of α-Hölder continuity, we think that log-Hölder is the most descriptive term. The following lemma illustrates the geometrical significance of log-Hölder continuous exponents. It corresponds to Lemma 3.2 of [6] on the Euclidean case.

Lemma 2.6 (Lemma 3.6, [24]). *Assume that* $p^+ < \infty$ *and consider two conditions:*

(i) p *is* log-*Hölder continuous;*

(ii) *for all balls* $B \subset X$ *we have* $\mu(B)^{p_B^- - p_B^+} \leqslant C$.

If μ *is lower Ahlfors* Q*-regular, then* (i) *implies* (ii). *If* μ *is upper Ahlfors* Q*-regular, then* (ii) *implies* (i).

2.3 The Hardy–Littlewood maximal operator

Recall that the Hardy–Littlewood maximal operator is defined for a locally integrable function u by

$$\mathcal{M}u(x) = \sup_{r>0} \fint_{B(x,r)} |u(y)|\, d\mu(y).$$

Recall also that the integral to the right denotes the mean value of u in $B(x,r)$.

Assume that $1 < p^- \leqslant p^+ < \infty$, p is log-Hölder continuous and satisfies the decay estimate

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + |x|)}$$

for every $x, y \in \mathbb{R}^n$, $|y| \geqslant |x|$. Then Cruz-Uribe, Fiorenza and Neugebauer proved that the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself [3, Theorem 1.5]. In the local case this result was first derived by Diening [6]. Nekvinda [35] has given another global version of the boundedness result, using a decay condition stated in terms of an integral. Pick and Růžička [37] constructed an example which shows that log-Hölder continuity is in some sense sharp. Nekvinda has given an example which shows that the decay condition is not necessary [36], and Diening [8] has further studied the necessity of decay conditions.

In bounded doubling metric measure spaces the Hardy-Littlewood maximal operator is bounded if p is log-Hölder and satisfies $1 < p^- \leqslant p^+ < \infty$, [24, Theorem 4.3]. However, in this case log-Hölder continuity is not necessary [24, Example 4.5]. A weaker result, derived under fewer assumptions, was given in [13, Theorem 2.3].

2.4 Density of smooth functions in $\mathbb{R}^n$

Variable exponent Sobolev spaces are defined in the obvious way: For $\Omega \subset \mathbb{R}^n$ the *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the subspace of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient exists almost everywhere and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

For the Newtonian space to agree with the classical Sobolev space the density of differentiable functions turns out to be crucial. This question is not as well understood in variable exponent spaces as is the boundedness of the maximal operator, but we do have some results: Samko proved in [38] that smooth function are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ if $p^+ < \infty$ and p is log-Hölder continuous. Diening proved a similar, though slightly weaker, result [6]. Edmunds and Rákosník showed that a certain monotonicity condition on the exponent is also sufficient for the density of smooth functions, see [11]. Hästö [15] gave an example of a variable exponent Sobolev space in which continuous functions are not dense. In this example the continuous exponent has growth just slightly greater than allowed by log-Hölder continuity.

The following simple lemma will be needed later on.

Lemma 2.7. *Suppose that $C^1(\Omega)$ is dense in $W^{1,p(\cdot)}(\Omega)$. Then $C_0^1(\Omega)$ is dense in the set of functions in $W^{1,p(\cdot)}(\Omega)$ with compact support in Ω.*

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$ with compact support in Ω. Let $\phi_i \in C^1(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ be a sequence of functions converging to u in $W^{1,p(\cdot)}(\Omega)$ and let $\Phi \in C_0^1(\Omega)$ be a non-negative cut-off function which equals 1 in the support of u. Then $\Phi\phi_i$ converges to u since

$$\begin{aligned}\|u - \Phi\phi_i\|_{1,p(\cdot)} &= \|\Phi u - \Phi\phi_i\|_{1,p(\cdot)} \\ &\leqslant \max\{\Phi\}\big(\|u-\phi_i\|_{p(\cdot)} + \|\nabla u - \nabla\phi_i\|_{p(\cdot)}\big) \\ &\quad + \max\{|\nabla\Phi|\}\|u-\phi_i\|_{p(\cdot)} \to 0\end{aligned}$$

as $i \to \infty$. □

3 Hajłasz space

In this section we define Sobolev spaces on metric measure spaces with variable exponent by a pointwise maximal inequality. This definition is due to P. Hajłasz, [15]. We show that the variable exponent Hajłasz space is a Banach space, that the $p(\cdot)$-Poincare inequality holds and that Lipschitz continuous functions are dense provided that the exponent is bounded. The proofs follow the ideas used by Hajłasz in [15].

3.1 Basic properties

Throughout this section we restrict our attention to exponents $p : X \to (1,\infty)$ not taking the value 1. We say that a $p(\cdot)$-integrable function u belongs to Hajłasz space $M^{1,p(\cdot)}(X,d,\mu) = M^{1,p(\cdot)}(X)$ if there exists a non-negative $g \in L^{p(\cdot)}(X)$ such that

$$|u(x) - u(y)| \leqslant d(x,y)\,(g(x) + g(y))$$

for μ-almost every $x, y \in X$. The function g is called a *Hajłasz gradient* of u. We equip $M^{1,p(\cdot)}(X)$ with the norm

$$\|u\|_{M^{1,p(\cdot)}(X)} = \|u\|_{p(\cdot)} + \inf \|g\|_{p(\cdot)},$$

where the infimum is taken over all Hajłasz gradients of u.

Proposition 3.1. *Suppose that $1 < p^- \leqslant p^+ < \infty$ and that μ is atomless. Then for every $u \in M^{1,p(\cdot)}(X,\mu)$ there exists a unique Hajłasz gradient of u, denoted by g_u, which minimizes the norm.*

Proof. Let (g_i) be a minimizing sequence of Hajłasz gradients of u, i.e. a sequence such that

$$\lim_{i\to\infty} \|g_i\|_{p(\cdot)} = \inf \|g\|_{p(\cdot)},$$

where the infimum is taken over all Hajłasz gradients g of u. Since $L^{p(\cdot)}(X)$ is reflexive (Corollary 2.4) there is a subsequence, denoted again by (g_i), and a function $g_u \in L^{p(\cdot)}(X)$ such that $g_i \rightharpoonup g_u$ weakly in $L^{p(\cdot)}(X)$. Since $L^{p(\cdot)}(X)$ is a Banach space we obtain by the Mazur lemma [41, Theorem V.1.2, p. 120] that there exist convex combinations $h_k = \sum_{i=k}^{m_k} \alpha_{k,i} g_i$ such that $h_k \to g$ strongly in $L^{p(\cdot)}(X)$, where $\alpha_{k,i} \geq 0$ and $\sum_{i=k}^{m_k} \alpha_{k,i} = 1$.

Since every g_i is a Hajłasz gradient of u, we have

$$\begin{aligned} |u(x) - u(y)| &= \sum_{i=k}^{m_k} \alpha_{k,i} |u(x) - u(y)| \\ &\leqslant \sum_{i=k}^{m_k} \alpha_{k,i}\, d(x,y)\,(g_i(x) + g_i(y)) \\ &= d(x,y)\,(h_k(x) + h_k(y)) \end{aligned} \tag{3.2}$$

for μ-almost all $x, y \in X$, hence h_k is a Hajłasz gradient of u for all k. So g_u is a Hajłasz gradient of u, because a subsequence of (h_k) converges to g_u poinwise μ-almost everywhere in X.

Let $\varepsilon > 0$. Then there is $k \in \mathbb{N}$ such that $\|g_i\|_{p(\cdot)} < \inf \|g\|_{p(\cdot)} + \varepsilon$ for all $i \geq k$, where the infimum is taken over all Hajłasz gradients of u. With this k we have

$$\begin{aligned} \|h_k\|_{p(\cdot)} &\leqslant \sum_{i=k}^{m_k} \alpha_{k,i} \|g_i\|_{p(\cdot)} \leqslant \sum_{i=k}^{m_k} \alpha_{k,i} \left(\inf \|g\|_{p(\cdot)} + \varepsilon\right) \\ &= \inf \|g\|_{p(\cdot)} + \varepsilon. \end{aligned}$$

This implies that $\|g_u\|_{p(\cdot)} = \inf \|g\|_{p(\cdot)}$, hence g_u is a minimal Hajłasz gradient of u.

Suppose there are two minimal Hajłasz gradients of u, g_1 and g_2 such that $g_1 \neq g_2$, i.e. $\|g_1 - g_2\|_{p(\cdot)} > 0$. Dividing g_1 and g_2 by $m = \|g_1\|_{p(\cdot)} = \|g_2\|_{p(\cdot)} > 0$ we get functions with unit norm. The uniform convexity of the norm implies that $\|g_1/m + g_2/m\|_{p(\cdot)} < 2$. Multiplying this by $m/2$ gives

$$\|\tfrac{1}{2} g_1 + \tfrac{1}{2} g_2\|_{p(\cdot)} < \|g_1\|_{p(\cdot)},$$

which is a contradiction, since $\frac{1}{2}g_1 + \frac{1}{2}g_2$ is also a Hajłasz gradient of u by the argument in (3.2). Therefore g_u is the unique minimal Hajłasz gradient of u. □

Theorem 3.3. *If $p^+ < \infty$, then $M^{1,p(\cdot)}(X)$ is a Banach space.*

Proof. Let (u_i) be a Cauchy sequence in $M^{1,p(\cdot)}(X)$. Since $L^{p(\cdot)}(X)$ is a Banach space, there is a function $u \in L^{p(\cdot)}(X)$ such that $u_i \to u$ in $L^{p(\cdot)}(X)$. We need to prove that $u \in M^{1,p(\cdot)}(X)$ and $u_i \to u$ also in $M^{1,p(\cdot)}(X)$.

We can choose a subsequence, denoted again by (u_i), such that $\|u_{i+1} - u_i\|_{M^{1,p(\cdot)}(X)} < 2^{-i}$ and $u_i \to u$ pointwise μ-almost everywhere in X. Then there exist non-negative functions $g_i \in L^{p(\cdot)}(X)$ such that $\|g_i\|_{p(\cdot)} < 2^{-i}$ and

$$|(u_{i+1} - u_i)(x) - (u_{i+1} - u_i)(y)| \leqslant d(x,y)\,(g_i(x) + g_i(y)) \tag{3.4}$$

for μ-almost all $x, y \in X$. Adding the inequalities (3.4) we get

$$|(u_j - u_i)(x) - (u_j - u_i)(y)| \leqslant d(x,y) \sum_{k=i}^{\infty} (g_k(x) + g_k(y))$$

for $j > i$, and letting $j \to \infty$ yields

$$|(u - u_i)(x) - (u - u_i)(y)| \leqslant d(x,y) \sum_{k=i}^{\infty} (g_k(x) + g_k(y)) \tag{3.5}$$

for μ-almost all $x, y \in X$. Therefore

$$\begin{aligned} |u(x) - u(y)| &\leqslant |(u - u_1)(x) - (u - u_1)(y)| + |u_1(x) - u_1(y)| \\ &\leqslant d(x,y)\Big(g_1(x) + g_1(y) + \sum_{k=1}^{\infty} (g_k(x) + g_k(y))\Big) \end{aligned} \tag{3.6}$$

for μ-almost all $x, y \in X$. Since

$$\Big\| \sum_{k=i}^{\infty} g_k \Big\|_{p(\cdot)} \leqslant \sum_{k=i}^{\infty} 2^{-k} = 2^{-i+1},$$

the inequalities (3.6) and (3.5) imply that $u \in M^{1,p(\cdot)}(X)$ and $u_i \to u$ in $M^{1,p(\cdot)}(X)$, respectively. □

3.2 The Poincaré inequality and density of Lipschitz functions

The Poincaré inequality in variable exponent Sobolev spaces has been studied in [20]. The pointwise Hajłasz equation gives it easily in our case. Recall the following notation for an integrable function u and a finite space X:

$$u_X = \fint_X u(y)\,d\mu(y) = \frac{1}{\mu(X)}\int_X u(y)\,d\mu(y).$$

Theorem 3.7 (The Poincaré inequality). *Let $p^+ < \infty$ and assume that X is bounded and of finite measure. If $u \in M^{1,p(\cdot)}(X)$, then*

$$\|u - u_X\|_{p(\cdot)} \leqslant C\big(p^-, p^+, \mu(X)\big)\operatorname{diam}(X)\|g\|_{p(\cdot)}.$$

Proof. Integrating the inequality $|u(x) - u(y)| \leqslant d(x,y)(g(x) + g(y))$ over y we obtain

$$\begin{aligned}\Big|u(x) - \fint_X u(y)\,d\mu(y)\Big| &\leqslant \fint_X |u(x) - u(y)|\,d\mu(y)\\ &\leqslant \operatorname{diam}(X)\Big(g(x) + \fint_X g(y)\,d\mu(y)\Big).\end{aligned}$$

By Hölder's inequality this yields

$$|u(x) - u_X| \leqslant \operatorname{diam}(X)\Big(g(x) + \frac{C\|1\|_{p'(\cdot)}}{\mu(X)}\|g\|_{p(\cdot)}\Big).$$

Since the previous inequality holds pointwise for every Sobolev function, especially for u/λ and every $\lambda > 0$, we obtain

$$\begin{aligned}\|u - u_X\|_{p(\cdot)} &\leqslant \operatorname{diam}(X)\Big(\|g\|_{p(\cdot)} + \frac{C}{\mu(X)}\|1\|_{p'(\cdot)}\|1\|_{p(\cdot)}\|g\|_{p(\cdot)}\Big)\\ &\leqslant C\operatorname{diam}(X)\left(1 + \max\left\{\mu(X)^{1/p^+ - 1/p^-}, \mu(X)^{1/p^- - 1/p^+}\right\}\right)\|g\|_{p(\cdot)}.\end{aligned}$$

□

Remark 3.8. Notice that, in contrast to the Euclidean case, the constant in the Poincaré inequality depends on the measure of the space X.

Theorem 3.9. *If $p^+ < \infty$, then Lipschitz functions are dense in $M^{1,p(\cdot)}(X)$. Specifically, for every $u \in M^{1,p(\cdot)}(X)$ and every $\varepsilon > 0$ there exists a Lipschitz function $h \in M^{1,p(\cdot)}(X)$ such that*

(i) $\mu\big(\{x \in X : u(x) \neq h(x)\}\big) \leqslant \varepsilon$;

(ii) $\|u - h\|_{M^{1,p(\cdot)}(X)} \leqslant \varepsilon$.

Proof. We fix $u \in M^{1,p(\cdot)}(X)$ and denote by $g \in L^{p(\cdot)}(X)$ a Hajłasz gradient of u. We write

$$E_\lambda = \{x \in X : |u(x)| \leqslant \lambda \text{ and } g(x) \leqslant \lambda\}.$$

The function $u|_{E_\lambda}$ is Lipschitz continuous with constant 2λ. We extend $u|_{E_\lambda}$ by McShane extension, [32], to all of X as a Lipschitz function by defining

$$\bar{u}(x) = \inf_{y \in E_\lambda} \{u|_{E_\lambda}(y) + 2\lambda \operatorname{dist}(x, y)\}.$$

Next we slightly modify this extension by truncating:

$$u_\lambda = (\operatorname{sign} \bar{u}) \min\{|\bar{u}|, \lambda\}.$$

It is clear that u_λ is Lipschitz with constant 2λ, $u|_{E_\lambda} = u_\lambda|_{E_\lambda}$ and $|u_\lambda| \leqslant \lambda$. We have for every $\lambda > 1$ that

$$\begin{aligned}
\mu\big(\{x \in X : u(x) \neq u_\lambda(x)\}\big) &\leqslant \mu(X \setminus E_\lambda) \\
&\leqslant \int_{X \setminus E_\lambda} \left(\frac{|u(x)| + g(x)}{\lambda}\right)^{p(x)} d\mu(x) \\
&\leqslant \frac{2^{p^+}}{\lambda^{p^-}} \left(\int_X |u(x)|^{p(x)} d\mu(x) + \int_X |g(x)|^{p(x)} d\mu(x)\right)
\end{aligned}$$

and hence $\mu\big(\{x \in X : u(x) \neq u_\lambda(x)\}\big) \to 0$ as $\lambda \to \infty$.

Next we prove that $u_\lambda \to u$ in $M^{1,p(\cdot)}(X)$. Using $u_\lambda \leqslant \lambda \leqslant |u| + g$ in $X \setminus E_\lambda$ for the second inequality we get

$$\begin{aligned}
\int_X |u(x) - u_\lambda(x)|^{p(x)}\, d\mu(x) &= \int_{X \setminus E_\lambda} |u(x) - u_\lambda(x)|^{p(x)}\, d\mu(x) \\
&\leqslant 2^{p^+} \int_{X \setminus E_\lambda} |u(x)|^{p(x)} + |u_\lambda(x)|^{p(x)}\, d\mu(x) \\
&\leqslant 2^{p^+} \int_{X \setminus E_\lambda} |u(x)|^{p(x)} + (|u(x)| + g(x))^{p(x)}\, d\mu(x) \\
&\leqslant 2^{2p^+ + 1} \int_{X \setminus E_\lambda} |u(x)|^{p(x)} + g(x)^{p(x)}\, d\mu(x).
\end{aligned}$$

Since $u, g \in L^{p(\cdot)}(X)$ and $\mu(X\setminus E_\lambda) \to 0$, this inequality implies that $\varrho_{p(\cdot)}(u - u_\lambda) \to 0$ as $\lambda \to \infty$. Since $p^+ < \infty$, this yields $\|u - u_\lambda\|_{p(\cdot)} \to 0$ as $\lambda \to \infty$.

It is a direct calculation to check that the function $g_\lambda = g\chi_{X\setminus E_\lambda} + 2\lambda\chi_{X\setminus E_\lambda}$ satisfies

$$|(u - u_\lambda)(x) - (u - u_\lambda)(y)| \leqslant d(x, y)(g_\lambda(x) + g_\lambda(y))$$

for almost every $x, y \in X$. We have

$$\int_X |g_\lambda(x)|^{p(x)}\, d\mu(x) \le 2^{p^+} \int_{X\setminus E_\lambda} g(x)^{p(x)} + |2\lambda|^{p(x)}\, d\mu(x)$$

and the same arguments as above implies that $\|g_\lambda\|_{p(\cdot)} \to 0$ as $\lambda \to 0$. This completes the proof. □

Remark 3.10. It is easy to see that the approximating function h constructed in the previous theorem has the additional property of being non-negative whenever u is.

3.3 Sobolev capacity on Hajłasz space

In this section we introduce a Sobolev-type capacity associated to the Hajłasz space. In the fixed exponent case this has been done by Kinnunen and Martio [27, 28]. In fact the proofs are mostly the same and will only be indicated here. The Sobolev capacity has been previously considered by Harjulehto, Hästö, Koskenoja and Varonen [21] in variable exponent Sobolev spaces in the Euclidean case. The proofs in this case do not carry over.

For $u \in M^{1,p(\cdot)}(X)$ we define $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \inf \varrho_{p(\cdot)}(g)$, where the infimum is taken over Hajłasz gradients of u. For $E \subset X$ we denote

$$S_{p(\cdot)}(E) = \{u \in M^{1,p(\cdot)}(X) : u \geqslant 1 \text{ in an open set containing } E\}.$$

The function $u \in S_{p(\cdot)}(E)$ is said to be $p(\cdot)$-*admissible* for the set E. The *Sobolev* $p(\cdot)$-*capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \varrho_{1,p(\cdot)}(u).$$

In case $S_{p(\cdot)}(E) = \emptyset$, we set $C_{p(\cdot)}(E) = \infty$. Notice that the gradient which minimizes the capacity does not necessarily coincide with the (norm-)minimal gradient from Proposition 3.1. However, if $p^+ < \infty$, this has no effect on what the zero sets of the capacity are, and is thus of minor importance.

We can show as in [27, Lemma 2.4] that $M^{1,p(\cdot)}(X)$ is a lattice. The following properties also follow as in [27]:

Theorem 3.11. *Assume that $p^+ < \infty$. The set function $E \mapsto C_{p(\cdot)}(E)$ has the following properties:*

(i) $C_{p(\cdot)}(\emptyset) = 0$.

(ii) *If $E_1 \subset E_2$, then $C_{p(\cdot)}(E_1) \leqslant C_{p(\cdot)}(E_2)$.*

(iii) *If E is a subset of $\mathbb{R}^n$, then*

$$C_{p(\cdot)}(E) = \inf_{\substack{E \subset U \\ U \text{ open}}} C_{p(\cdot)}(U).$$

(iv) *If $K_1 \supset K_2 \supset \dots$ are compact, then*

$$\lim_{i\to\infty} C_{p(\cdot)}(K_i) = C_{p(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i\right).$$

(v) *If $E_i \subset \mathbb{R}^n$ for $i = 1, 2, \dots$, then*

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(E_i).$$

Proof. The first two properties are trivial. Properties (iii) and (iv) follow as in Remark 3.3 and Lemma 3.4 of [27], respectively.

In [21] the assumption $1 < p^-$ was needed for countable subadditivity, Property (v). Let us therefore prove (v) here, following [27]. We assume that $\sum_{i=1}^{\infty} C_{p(\cdot)}(E_i) < \infty$ and fix $\varepsilon > 0$. For every i we choose $u_i \in S_{p(\cdot)}(E_i)$ with Hajłasz gradient g_i such that

$$\varrho_{p(\cdot)}(u_i) + \varrho_{p(\cdot)}(g_i) \leqslant C_{p(\cdot)}(E_i) + 2^{-i}\varepsilon.$$

We define $u = \sup_i u_i$ and $g = \sup_i g_i$. Then

$$\begin{aligned}\int_X |u(x)|^{p(x)} dx &\leqslant \sum_{i=1}^{\infty} \int_{X_i} |u_i(x)|^{p(x)} dx \leqslant \sum_{i=1}^{\infty} \int_X |u_i(x)|^{p(x)} dx \\ &\leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(E_i) + \varepsilon < \infty,\end{aligned}$$

where X_i is the subset of X in which $u_i = u$. Hence we conclude that $u \in L^{p(\cdot)}(X)$. Similarly, $g \in L^{p(\cdot)}(X)$.

Let us next define $v_k = \max_{i \leqslant k} u_i$. By the lattice property, $h_k = \max_{i \leqslant k} g_i$ is a Hajłasz gradient of v_k. Since $v_k \to u$ and $h_k \to g$ pointwise a.e., it follows as in [27, Lemma 2.5] that g is a Hajłasz gradient of u. Since u is greater than or equal to 1 in a neighborhood of $\cup_{i=1}^{\infty} E_i$, it is $p(\cdot)$-admissible. Hence

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \varrho_{1,p(\cdot)}(u) \leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(E_i) + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$. □

Properties (i), (ii), and (v) of the capacity are equivalent by definition to being an outer measure. Thus we have

Corollary 3.12. *If $p^+ < \infty$, then the Sobolev $p(\cdot)$-capacity is an outer measure.*

The following results shows that our capacity is a Choquet capacity. The proof is the same as that of [28, Theorem 4.1], as long as we write $\varrho_{p(\cdot)}(u)$ where they have $\|u\|^p_{L^p(X)}$.

Theorem 3.13. *If $1 < p^- \leqslant p^+ < \infty$ and $E_1 \subset E_2 \subset \ldots$ are subsets of X, then*

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} C_{p(\cdot)}(E_i).$$

The following results describe how to get quasicontinuous representatives of Sobolev functions. The proofs are as in [21].

Lemma 3.14. *Let $p^+ < \infty$. For each Cauchy sequence of functions in $C(X) \cap M^{1,p(\cdot)}(X)$ there is a subsequence which converges pointwise except in a set of zero $p(\cdot)$-capacity. Moreover, the convergence is uniform outside a set of arbitrarily small $p(\cdot)$-capacity.*

Recall that a function u is said to be $p(\cdot)$-*quasicontinuous* if for every $\varepsilon > 0$ there exists an open set $U \subset X$ with $C_{p(\cdot)}(X \setminus U) < \varepsilon$ such that u is continuous in $X \setminus U$.

Theorem 3.15. *Let $p^+ < \infty$. For each $u \in M^{1,p(\cdot)}(X)$ there is a $p(\cdot)$-quasicontinuous function $v \in M^{1,p(\cdot)}(X)$ such that $u = v$ pointwise μ-almost everywhere in X.*

Proof. Notice that Lipschitz functions are dense in Hajłasz space, by Theorem 3.9. Therefore the additional assumption of the density of continuous functions in [21] is not needed here. After this the proof is exactly as in [21, Theorem 5.2]. □

4 Newtonian spaces

In this section we define Sobolev spaces on metric measure spaces using an upper gradient. These so-called Newtonian spaces spaces were first studied, in the fixed exponent case, by N. Shanmugalingam [39], see also [2].

A curve γ in X is a non-constant continuous map $\gamma : I \to X$, where $I = [a, b]$ is a closed interval in $\mathbb{R}$. The image of γ, $\gamma(I)$, is denoted by $|\gamma|$. By Γ_{rect} we denote the family of all rectifiable curves in X.

Let Γ be a family of rectifiable curves. We denote by $F(\Gamma)$ the set of all *admissible functions*, i.e. all Borel measurable functions $\rho\colon X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \geqslant 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to path length. We define the *$p(\cdot)$-modulus of* Γ by

$$\mathrm{M}_{p(\cdot)}(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho(x)^{p(x)}\, d\mu(x).$$

If $F(\Gamma) = \emptyset$, then we set $\mathrm{M}_{p(\cdot)}(\Gamma) = \infty$. The arguments from $\mathbb{R}^n$ imply that the $p(\cdot)$-modulus is an outer measure on the space of all curves of X, for the proof see [23, Lemma 2.1].

A family of curves Γ is said to be *exceptional* if $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$. The following lemma is a generalization of [14, Theorem 3(f)]. The proof is exactly the same as the proof of [23, Lemma 2.2].

Lemma 4.1 (Fuglede's lemma). *Let $(u_i)_{i=1}^\infty$ be a sequence of non-negative Borel functions in $L^{p(\cdot)}(X)$ converging to zero in $L^{p(\cdot)}(X)$. Then there exists* *a subsequence* $(u_{i_k})_{k=1}^{\infty}$ *and an exceptional family* Γ *of rectifiable curves such that for every rectifiable* $\gamma \notin \Gamma$ *we have*

$$\lim_{k\to\infty} \int_\gamma u_{i_k} ds = 0.$$

4.1 Basic properties

Let u be a real valued function on X. A non-negative Borel measurable function ρ on X is a $p(\cdot)$*-weak upper gradient of* u, or weak upper gradient for short, if there exists a family Γ of rectifiable curves with $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$ and

$$|u(x) - u(y)| \leqslant \int_\gamma \rho\, ds$$

for every rectifiable curve $\gamma \notin \Gamma$ with endpoints x and y.

The *Newtonian space* $N^{1,p(\cdot)}(X)$ is the collection of functions in $L^{p(\cdot)}(X)$ with a weak upper gradient in $L^{p(\cdot)}(X)$ equipped with the norm

$$\|u\|_{N^{1,p(\cdot)}(X)} = \|u\|_{p(\cdot)} + \inf \|\rho\|_{p(\cdot)},$$

where the infimum is taken over all weak upper gradients of u. It is easy to see that $N^{1,p(\cdot)}(X)$ is a lattice like classical first order Sobolev space.

Next we show that Newtonian space is a Banach space. For that purpose we introduce a Sobolev type capacity in Newtonian space and study the relation between the capacity and the modulus. Our proof is along the lines of the proofs of [23] and is shorter than the original proof of Shanmugalingam [39].

We define the capacity in Newtonian space by

$$c_{p(\cdot)}(E) = \inf_u \inf_\rho \int_X |u(x)|^{p(x)} + \rho(x)^{p(x)}\, d\mu(x),$$

where the first infimum is taken over all $u \in N^{1,p(\cdot)}(X)$, which are at least 1 in E and the second infimum is taken over weak upper gradients of u. If the class of test functions is empty we set $c_{p(\cdot)}(E) = \infty$. The arguments given in the proof of Theorem 3.11 show that the capacity is an outer measure provided that $p^+ < \infty$.

We denote by Γ_E the family of all rectifiable curves whose image intersects the set E.

Lemma 4.2. *Suppose that $E \subset X$ and $p^+ < \infty$. If $c_{p(\cdot)}(E) = 0$, then* $\mathrm{M}_{p(\cdot)}(\Gamma_E) = 0$.

Proof. For every $i \in \mathbb{N}$ we choose a function $u_i \in N^{1,p(\cdot)}(X)$ with weak upper gradient h_i such that $u_i(x) \geqslant 1$ for every $x \in E$ and

$$\int_X |u_i(x)|^{p(x)} + h_i(x)^{p(x)}\, d\mu(x) \leqslant 2^{-i}.$$

We define

$$v_k = \sum_{i=1}^{k} |u_i|.$$

We find that $\rho_k = \sum_{i=1}^{k} h_i$ is a weak upper gradient of v_k. For every $l > m$ we find that

$$\|v_l - v_m\|_{p(\cdot)} \leqslant \sum_{i=m+1}^{l} \|u_i\|_{p(\cdot)} \leqslant 2^{-m}$$

and

$$\|\rho_l - \rho_m\|_{p(\cdot)} \leqslant \sum_{i=m+1}^{l} \|h_i\|_{p(\cdot)} \leqslant 2^{-m},$$

and therefore the sequences $(v_k)_{k=1}^{\infty}$ and $(\rho_k)_{k=1}^{\infty}$ are Cauchy sequences in the Banach space $L^{p(\cdot)}(X)$. So (ρ_k) converges to a function ρ in $L^{p(\cdot)}(X)$, which we may assume to be Borel. Since the sequence $(v_k(x))$ is non-negative and increasing for every $x \in X$ the limit $v(x) = \lim_{k\to\infty} v_k(x)$ (possibly $+\infty$) exists for every $x \in X$ and $v \in L^{p(\cdot)}(X)$. For $x \in E$ we see that $v_k(x) \geqslant k$ for every k and thus

$$E \subset E_\infty = \big\{x \in X\colon \lim_{k\to\infty} v_k(x) = \infty\big\}.$$

Therefore it suffices to show that $\mathrm{M}_{p(\cdot)}(\Gamma_{E_\infty}) = 0$.

Lemma 4.1 gives a subsequence of (ρ_k), denoted again by (ρ_k), such that there is an exceptional family Γ_1 and

$$\lim_{k\to\infty} \int_\gamma |\rho_k - \rho|\, ds = 0 \tag{4.3}$$

for every rectifiable curve $\gamma \notin \Gamma_1$. Let Γ_2 be the family of all curves γ such that $\int_\gamma v\, ds = \infty$ and Γ_3 the family of curves γ with $\int_\gamma \rho\, ds = \infty$. Since v/i

is admissible for Γ_2 and every $i = 1, 2, \dots$ and since $v \in L^{p(\cdot)}(\mathbb{R}^n)$, we find that

$$\mathrm{M}_{p(\cdot)}(\Gamma_2) \leqslant \int_X \left(\frac{v(x)}{i}\right)^{p(x)} dx \leqslant \frac{\|v\|_{p(\cdot)}}{i}$$

for all $i \geqslant \|v\|_{p(\cdot)}$, by [31, (2.11)]. Therefore $\mathrm{M}_{p(\cdot)}(\Gamma_2) = 0$ and similarly $\mathrm{M}_{p(\cdot)}(\Gamma_3) = 0$. Let $\Gamma_{4,i}$ be the exceptional family of curves from the definition of u_i. By subadditivity we obtain that $\mathrm{M}_{p(\cdot)}(\Gamma_4) = \mathrm{M}_{p(\cdot)}(\bigcup \Gamma_{4,i}) = 0$. This yields that $\mathrm{M}_{p(\cdot)}(\Gamma^*) = 0$, where $\Gamma^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

To complete the proof we show that $\Gamma_{E_\infty} \subset \Gamma^*$. Suppose that $\gamma \notin \Gamma^*$. Since $\gamma \notin \Gamma_2$ there is $y \in |\gamma|$ with $v(y) < \infty$. For any point $x \in |\gamma|$ we find since $\gamma \notin \Gamma_4$ that

$$|v_i(x)| \leqslant |v_i(y)| + |v_i(x) - v_i(y)| \leqslant |v_i(y)| + \int_\gamma \rho_i \, ds.$$

Taking the limit as $i \to \infty$ in this inequality gives, using (4.3) and $\gamma \notin \Gamma_1$ for the inequality, that

$$v(x) = \lim_{i \to \infty} |v_i(x)| \leqslant \lim_{i \to \infty} |v_i(y)| + \int_\gamma \rho \, ds.$$

Since $\gamma \notin \Gamma_3$ and $v(y) < \infty$, the right-hand-side is finite. Hence $v(x) < \infty$ for all $x \in |\gamma|$, which implies that $\gamma \notin \Gamma_{E_\infty}$. Thus $\Gamma_{E_\infty} \subset \Gamma^*$, which completes the proof. □

Theorem 4.4. *If $p^+ < \infty$, then $N^{1,p(\cdot)}(X)$ is a Banach space.*

Proof. Let u_i be a Cauchy sequence in $N^{1,p(\cdot)}(X)$. Passing to a subsequence if necessary we assume that

$$\|u_{i+1} - u_i\|_{N^{1,p(\cdot)}(X)} < 2^{-2i}.$$

Let ρ_i' be a weak upper gradient of $2^i|u_{i+1} - u_i|$ such that

$$2^i \|u_{i+1} - u_i\|_{p(\cdot)} + \|\rho_i'\|_{p(\cdot)} < 2^{-i}.$$

Let $E_i = \{x \in X : |u_{i+1}(x) - u_i(x)| > 2^{-i}\}$. Then $2^i|u_{i+1} - u_i| \in N^{1,p(\cdot)}(X)$ and $2^i|u_{i+1} - u_i| \geqslant 1$ in E_i. Hence we obtain, using $2^i|u_{i+1} - u_i|$ as a test function for the capacity,

$$\begin{aligned} c_{p(\cdot)}(E_i) &\leqslant \varrho_{p(\cdot)}(2^i|u_{i+1} - u_i|) + \varrho_{p(\cdot)}(\rho_i') \\ &\leqslant 2^i \|u_{i+1} - u_i\|_{p(\cdot)} + \|\rho_i'\|_{p(\cdot)} \leqslant 2^{-i}. \end{aligned}$$

Let $F_j = \bigcup_{i=j}^{\infty} E_i$ and $F = \cap_{j\in\mathbb{N}} F_j$. Then we get $c_{p(\cdot)}(F_j) \leqslant \sum_{i=j}^{\infty} c_{p(\cdot)}(E_i) \leqslant 2^{-j+1}$ and $c_{p(\cdot)}(F) = 0$. For $x \in X \setminus F$ the sequence $u_i(x)$ is a Cauchy sequence in $\mathbb{R}$ and we set $u(x) = \lim_{i\to\infty} u_i(x)$.

Next we show that u has a weak upper gradient. Let g_1 be a weak $p(\cdot)$-integrable upper gradient of u_1 and let g_{i+1} that a weak upper gradient of $u_{i+1} - u_i$ with $\|g_i\|_{p(\cdot)} \leq 2^{-2i}$ for $i = 1, 2, \ldots$. Define $\rho_i = g_1 + \ldots + g_i$ and note that ρ_i is a weak upper gradient of u_i. Since (ρ_i) is a Cauchy sequence it converges to a function ρ in $L^{p(\cdot)}(X)$. Passing to a subsequence, if necessary, we obtain by Lemma 4.1 that

$$\lim_{i\to\infty} \int_\gamma \rho_i \, ds = \int_\gamma \rho \, ds$$

except for γ in a family of curves Γ of zero $p(\cdot)$-modulus. By Lemma 4.2, the modulus of Γ_F is zero. Hence we obtain for every rectifiable $\gamma \notin \Gamma \cup \Gamma_F$ joining $x, y \in X$ that

$$|u(x) - u(y)| = \lim_{i\to\infty} |u_i(x) - u_i(y)| \leqslant \lim_{i\to\infty} \int_\gamma \rho_i \, ds = \int_\gamma \rho \, ds.$$

The first equality follows since $x, y \notin F$, as they lie on a curve not in Γ_F. Thus we have shown that ρ is a weak upper gradient of u.

Now we have only to prove that $u_i \to u$ in $L^{p(\cdot)}(X)$. For every $k \in \mathbb{N}$ we have

$$\|u - u_k\|_{p(\cdot)} \leq \sum_{i=k}^{\infty} \|u_{i+1} - u_i\|_{p(\cdot)} \leqslant \sum_{i=k}^{\infty} 2^{-2i} \leqslant 4^{1-k}.$$

This completes the proof of Theorem 4.4 □

4.2 Density of Lipschitz continuous functions

Next we study when Lipschitz functions are dense in the Newtonian space. For this result we need to assume that the Hardy-Littlewood maximal operator is locally bounded from $L^{p(\cdot)}(X) \to L^{p(\cdot)}(X)$. For a summary of what is known about this, see Section 2.3.

We say that X *supports a* $(1,1)$*-Poincaré inequality* if there exists a constant $C > 0$ such that for all open balls B in X and all pairs of functions u and ρ defined on B the inequality

$$\fint_B |u - u_B| \, d\mu \leqslant C \operatorname{diam}(B) \fint_B \rho \, d\mu$$

holds whenever ρ is an upper gradient of u on B and u in integrable on B the inequality. The proof of the next theorem follows that of [39, Theorem 4.1].

Theorem 4.5. *Let X be a doubling space that supports a $(1,1)$-Poincaré inequality. Assume that $p^+ < \infty$ and the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(X) \to L^{p(\cdot)}(X)$. Then Lipschitz continuous functions are dense in $N^{1,p(\cdot)}(X)$.*

Proof. It is easy to see by a truncation argument that bounded functions are dense in $N^{1,p(\cdot)}(X)$ (e.g. [39, Lemma 4.3]). Hence it suffices to consider the case of bounded $u \in N^{1,p(\cdot)}(X)$, say $|u(x)| < u_0$. Let $\rho \in L^{p(\cdot)}(\mathbb{R}^n)$ be a weak upper gradient of u. We set

$$E_\lambda = \{x \in X : \mathcal{M}\rho(x) > \lambda\}.$$

Note that E_λ is open since $\mathcal{M}$ is lower semi-continuous. If $x \in X \setminus E_\lambda$, then for all $r > 0$ and for balls $B = B(x,r)$ we have

$$\fint_B |u - u_B|\, d\mu \leqslant cr \fint_B \rho\, d\mu \leqslant cr\mathcal{M}\rho(x) \leqslant cr\lambda.$$

Hence for $s \in [\frac{r}{2}, r]$ and $x \in X \setminus E_\lambda$ the doubling property implies that

$$\begin{aligned} |u_{B(x,s)} - u_{B(x,r)}| &\leqslant \fint_{B(x,s)} |u - u_{B(x,r)}|\, d\mu \\ &\leqslant \frac{\mu(B(x,r))}{\mu(B(x,s))} \fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \\ &\leqslant cr\lambda. \end{aligned}$$

Using this inequality for $r = r_0, r_0/2, r_0/4, \ldots$ we find that

$$\begin{aligned} |u_{B(x,s)} - u_{B(x,r)}| &\leqslant |u_{B(x,s)} - u_{B(x,2^{-i}r)}| + \sum_{j=1}^{i} |u_{B(x,2^{-j}r)} - u_{B(x,2^{1-j}r)}| \\ &\leqslant c(2^{-i}r + 2^{1-i}r + \ldots + r)\lambda \leqslant cr\lambda, \end{aligned}$$

where i is the largest integer for which $s \leqslant 2^{-i}r$. Hence any sequence $(u_{B(x,r_i)})$, $r_i \to 0$, is a Cauchy sequence in $\mathbb{R}$. Therefore on $X \setminus E_\lambda$ we can define

$$u_\lambda(x) = \lim_{r \to 0} u_{B(x,r)}.$$

Since μ-almost every point is a Lebesgue point for every function in $L^1_{loc}(X)$ we note that $u(x) = u_\lambda(x)$ for μ-almost every $x \in X \setminus E_\lambda$. For $x, y \in X$ we define a chain of balls $(B_i)_{i\in\mathbb{Z}\setminus\{0\}}$ by setting

$$B_1 = B(x, d(x,y)) \quad \text{and} \quad B_{-1} = B(y, d(x,y))$$

and inductively

$$B_i = \tfrac{1}{2}B_{i-1} \text{ (for } i > 1) \quad \text{and} \quad B_{-i} = \tfrac{1}{2}B_{-i+1} \text{ (for } i < -1).$$

We calculate by the doubling property and the $(1,1)$-Poincaré inequality that

$$\begin{aligned}
\mu(B_1)|u_{B_1} - u_{B_{-1}}| &\leqslant \mu(B_1)|u_{B_1} - u_{2B_1}| + \mu(B_1)|u_{2B_1} - u_{B_{-1}}| \\
&\leqslant \int_{B_1} |u - u_{2B_1}|\, d\mu + \frac{\mu(B_1)}{\mu(B_{-1})}\int_{B_{-1}} |u - u_{2B_1}|\, d\mu \\
&\leqslant \int_{2B_1} |u - u_{2B_1}|\, d\mu + \frac{\mu(2B_{-1})}{\mu(B_{-1})}\int_{2B_1} |u - u_{2B_1}|\, d\mu \\
&\leqslant C \operatorname{diam}(B_1) \int_{2B_1} \rho\, d\mu.
\end{aligned}$$

Dividing by $\mu(B_1)$ and using the doubling property we get

$$|u_{B_1} - u_{B_{-1}}| \leqslant c\, d(x,y)\mathcal{M}\rho(x) \leqslant c\, d(x,y)\lambda.$$

If $x, y \in X \setminus E_\lambda$, then they are Lebesgue points also of u_λ and hence

$$\begin{aligned}
|u_\lambda(x) - u_\lambda(y)| &\leqslant \sum_{i=1}^{\infty} |u_{B_{i+1}} - u_{B_i}| + |u_{B_1} - u_{B_{-1}}| + \sum_{i=1}^{\infty} |u_{B_{-i-1}} - u_{B_{-i}}| \\
&\leqslant c\lambda d(x,y).
\end{aligned}$$

Hence u_λ is $c\lambda$-Lipschitz in $X \setminus E_\lambda$. We extend u_λ as a Lipschitz function to all of X by McShane extension, [32], by setting

$$u_\lambda(x) = \inf_{y\in X\setminus E_\lambda} \{u_\lambda(y) + c\lambda d(x,y)\}.$$

We may assume that the extension is bounded by u_0. This can be done by truncation.

Then we easily see that $u_\lambda \to u$ in $L^{p(\cdot)}(X)$:

$$\begin{aligned}\int_X |u(x) - u_\lambda(x)|^{p(x)}\, d\mu(x) &= \int_{E_\lambda} |u(x) - u_\lambda(x)|^{p(x)}\, d\mu(x) \\ &\leqslant 2^{p^+} \int_{E_\lambda} |u(x)|^{p(x)} + |u_\lambda(x)|^{p(x)}\, d\mu(x) \\ &\leqslant 2^{p^+ + 1} \mu(E_\lambda) \max\{u_0, u_0^{p^+}\}.\end{aligned}$$

Since $p^+ < \infty$, the norm goes to zero when the modular goes to zero, so we have shown convergence in $L^{p(\cdot)}(X)$.

Non-zero values of $u - u_\lambda$ are obtained only at points in E_λ and on a set L whose measure is zero. Since E_λ is open and $u - u_\lambda$ is zero μ-almost everywhere in the complement of E_λ, we may assume by [39, Lemma 4.3] that the upper gradient of $u - u_\lambda$ is zero in $X \setminus E_\lambda$. Notice that $\lambda \chi_{E_\lambda} \in L^{p(\cdot)}(X)$ by the definition of λ since $\mathcal{M}u$ is in $L^{p(\cdot)}(X)$. Therefore we find that the function $(c\lambda + \rho)\chi_{E_\lambda}$ is a weak upper gradient of $u - u_\lambda$. Hence $u - u_\lambda$ is in $N^{1,p(\cdot)}(X)$ and therefore so is u_λ. We obtain

$$\begin{aligned}\int_X |(c\lambda + \rho(x))\chi_{E_\lambda}(x)|^{p(x)}\, d\mu(x) &= \int_{E_\lambda} |c\lambda + \rho(x)|^{p(x)}\, d\mu(x) \\ &\leqslant C \int_{E_\lambda} |\rho(x)|^{p(x)} + |\lambda|^{p(x)}\, d\mu(x) \\ &\leqslant C \int_{E_\lambda} |u(x)|^{p(x)} + [\mathcal{M}\rho(x)]^{p(x)}\, d\mu(x).\end{aligned}$$

Since both ρ and $\mathcal{M}\rho$ belong to $L^{p(\cdot)}(X)$, the right hand side converges to zero as $\lambda \to \infty$. Hence the sequence u_λ converges to u in $N^{1,p(\cdot)}(X)$. □

Remark 4.6. It is easy to see that the approximating functions constructed in the previous theorem has the additional property of being non-negative whenever the function itself is.

5 Equivalence of function spaces

In this section we study when Hajłasz, Newtonian and classical Sobolev spaces agree. We will see that, roughly speaking, Hajłasz space agrees with Sobolev space if the maximal operator is bounded, whereas the Newtonian space agrees with Sobolev space if differentiable functions are dense. This reflects the fact that the boundedness of the maximal function is somehow built into the definition of Hajłasz space, as the density of differentiable functions is into Newtonian space. For what is known about when these conditions hold, see Sections 2.3 and 2.4.

Proposition 5.1. *We have* $M^{1,p(\cdot)}(\mathbb{R}^n) \subset W^{1,p(\cdot)}(\mathbb{R}^n)$. *If the maximal operator is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to itself, then* $M^{1,p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n)$.

Proof. Fix first $u \in M^{1,p(\cdot)}(\mathbb{R}^n)$ and let $g \in L^{p(\cdot)}(\mathbb{R}^n)$ be a nonnegative function such that

$$|u(x) - u(y)| \leqslant |x - y|(g(x) + g(y))$$

for almost every $x, y \in X$. We know that $g \in L^1(B)$ in every ball B and hence by [16, Proposition 1] (or [25, Remark 5.13]) ∇u exists and satisfies $|\nabla u| \leqslant C(n)\, g$ almost everywhere. Thus we obtain that $|\nabla u| \in L^{p(\cdot)}(\mathbb{R}^n)$, and so $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$, since $u \in L^{p(\cdot)}(\mathbb{R}^n)$ by definition.

To prove the second claim we fix $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and assume that $\mathcal{M}$ is bounded. We find as in [15, Chapter 2] that

$$|u(x) - u(y)| \leqslant |x - y| \left(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)\right).$$

for almost every $x, y \in \mathbb{R}^n$. Since the maximal operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, we find that $\mathcal{M}|\nabla u|$ is a Hajłasz gradient of u in $L^{p(\cdot)}(\mathbb{R}^n)$, and so $u \in M^{1,p(\cdot)}(\mathbb{R}^n)$. □

We give two alternative characterizations of the Hajłasz space. For this purpose we introduce a fractional sharp maximal operator. For every locally integrable function u we define

$$u^{\#}(x) = \sup_{r>0} \frac{1}{r} \fint_{B(x,r)} |u(x) - u_{B(x,r)}|\, d\mu(x).$$

In the variable exponent setting the sharp maximal operator has been studied by Diening and Růžička [9, 10]. The following theorem is a generalization of [17, Theorem 3.4], and the proof given in that paper also works in our case.

Theorem 5.2. *If the Hardy-Littlewood maximal operator is bounded from* $L^{p(\cdot)}(X)$ *to itself, then the following three statements are equivalent:*

(i) $u \in M^{1,p(\cdot)}(X)$;

(ii) $u \in L^{p(\cdot)}(X)$ *and there exists a non-negative* $g \in L^{p(\cdot)}(X)$ *such that*

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leqslant cr \fint_{B(x,r)} g \, d\mu$$

holds for every $x \in X$ *and* $r > 0$;

(iii) $u \in L^{p(\cdot)}(X)$ *and* $u^{\#} \in L^{p(\cdot)}(X)$.

Moreover, we have that

$$\begin{aligned} \|u\|_{M^{1,p(\cdot)}(X)} &\approx \|u\|_{p(\cdot)} + \inf\{\|g\|_{p(\cdot)} : g \text{ as in (ii)}\} \\ &\approx \|u\|_{p(\cdot)} + \|u^{\#}\|_{p(\cdot)}. \end{aligned}$$

Theorem 5.3. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set. We have* $N^{1,p(\cdot)}(\Omega) \subset W^{1,p(\cdot)}(\Omega)$. *If* $1 < p^- \leqslant p^+ < \infty$ *and* $C^1(\Omega)$ *is dense in* $W^{1,p(\cdot)}(\Omega)$, *then* $N^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega)$.

Proof. Let $u \in N^{1,p(\cdot)}(\mathbb{R}^n)$. Then u is absolutely continuous on every curve except a family of zero $p(\cdot)$-modulus, and hence u has classical derivatives almost everywhere. Denote by ρ a weak upper gradient of u.

Let $Q = (-r,r)^n$ for $r > 0$. We have that

$$\begin{aligned} \int_{(-r,r)^{n-1}} \int_{(-r,r)} \rho(t,y) \, dt \, dH^{n-1}(y) &\leqslant \int_Q \rho(x) \, dx \\ &\leq (1 + |Q|) \int_Q \rho(x)^{p(x)} \, dx < \infty, \end{aligned}$$

which means that $\int_{(-r,r)} \rho(t,x) \, dt < \infty$ except in a set E_k with $m_{n-1}(E_k) = 0$. For $y \notin E_k$ H^1-almost every point in a line segment from $(-r,y)$ to (r,y) is a Lebesgue point. Thus for almost every point in Q we have

$$\lim_{t \to 0} \fint_{[y,y+te_1]} \rho \, ds = \rho(y),$$

where $[z,w]$ denotes the line segment joining z and w. On the other hand we have

$$\frac{1}{|t|} |u(y) - u(y+te_1)| \leqslant \fint_{[y,y+te_1]} \rho \, ds,$$

Since u is differentiable almost everywhere, this implies that $|\partial_1 u(x)| \leqslant \rho(x)$ almost everywhere in Q. Letting $r \to \infty$ and using the subadditivity we derive the same claim in all of $\mathbb{R}^n$. Finally the same argument applies in directions $e_2, \ldots, e_n$ as well, so we get $|\nabla u(x)| \leqslant \sqrt{n}\rho(x)$ for almost every x. Therefore $|\nabla u| \in L^{p(\cdot)}(\mathbb{R}^n)$ and so $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$.

To prove the inclusion $W^{1,p(\cdot)}(\Omega) \subset N^{1,p(\cdot)}(\Omega)$ we need to show that every function in $W^{1,p(\cdot)}(\Omega)$ has a weak upper gradient. But it follows as in [23, Theorem 4.2] that the absolute value of the distributional gradient is an upper gradient in $\mathbb{R}^n$. From this it follows that the same claim holds in $\Omega \subset \mathbb{R}^n$ using the argument of [23, Theorem 4.6], taking into account Lemma 2.7. □

Finally we prove relations between Hajłasz and Newtonian spaces. The following theorem is a generalization of [39, Lemma 4.8].

Theorem 5.4. *Let $p^+ < \infty$. Then $M^{1,p(\cdot)}(X) \subset N^{1,p(\cdot)}(X)$ and $\|u\|_{N^{1,p(\cdot)}} \leqslant C\|u\|_{M^{1,p(\cdot)}}$. If X supports a $(1,1)$-Poincaré inequality and if the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(X)$ to itself, then $M^{1,p(\cdot)}(X) = N^{1,p(\cdot)}(X)$ and the norms are comparable.*

Proof. Let $u \in M^{1,p(\cdot)}(X)$. If u is continuous, we find as in [39, Lemma 4.7] that $4g$ is an upper gradient for u, where g is the Hajłasz gradient of u. Since continuous functions are dense in $M^{1,p(\cdot)}(X)$ by Theorem 3.9, we can approximate $u \in M^{1,p(\cdot)}(X)$ with continuous functions u_i. Since $u_i \to u$, $g_i \to g_u$, and $N^{1,p(\cdot)}(X)$ is a Banach space by Theorem 4.4, we find that $4g$ is an upper gradient of u.

Next let $u \in N^{1,p(\cdot)}(X)$, and let ρ be a weak upper gradient of u. Since by the definition $\rho \in L^{p(\cdot)}(X)$, ρ is non-negative and X supports $(1,1)$-Poincaré inequality, the inclusion $M^{1,p(\cdot)}(X) \supset N^{1,p(\cdot)}(X)$ follows by Theorem 5.2. This completes the proof. □

Acknowledgement

The authors wish to thank Ilkka Holopainen and Olli Martio for comments on this manuscript.

References

[1] A. Björn, J. Björn and N. Shanmugalingam: The Dirichlet problem for p-harmonic functions on metric spaces, *J. Reine Angew. Math.* **556** (2003), 173–203.

[2] J. Cheeger: Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal* **9** (1999), 428–517.

[3] D. Cruz-Uribe, A. Fiorenze and C. J. Neugebauer: The maximal operator on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 223–238.

[4] D. Cruz-Uribe, A. Fiorenze and C. J. Neugebauer: Correction to "The maximal function on variable L^p spaces", *Ann. Acad. Sci. Fenn. Math.* **29** (2004), 247–249.

[5] L. Diening: *Theoretical and Numerical Results for Electrorheological Fluids* Ph.D. thesis, University of Freiburg, Germany, 2002.

[6] L. Diening: Maximal operator on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.*, to appear.

[7] L. Diening: Riesz Potential and Sobolev Embeddings of generalized Lebesgue and Sobolev Spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Math. Nachr.* **263** (2004), no. 1, 31–43.

[8] L. Diening: Maximal Function on Musielak Orlicz Spaces and Generalized Lebesgue Spaces, preprint (2003).

[9] L. Diening and M. Růžička: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, *J. Reine Angew. Math.* **563** (2003), 197–220.

[10] L. Diening and M. Růžička: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, preprint (2003).

[11] D. E. Edmunds and J. Rákosník: Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* **437** (1992), 229–236.

[12] X. Fan and D. Zhao: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424–446.

[13] T. Futamura, Y. Mizuta and T. Shimomura: Sobolev embeddings for variable exponent Riesz potentials on metric spaces, preprint (2004).

[14] B. Fuglede: Extremal length and functional completion, *Acta Math.* **98** (1957), 171–218.

[15] P. Hajłasz: Sobolev spaces on arbitrary metric spaces, *Potential Anal.* **5** (1996), 403–415.

[16] P. Hajłasz: Geometric approach to Sobolev spaces and badly degenerated elliptic equations, in *Nonlinear Analysis and Applications (Warzawa, 1994)*, N. Kenmochi et all. (eds.) GAGUTO Internat. Ser. Math. Sci. Appl. 7, Gakkotosho, Tokyo, 1996, 141–168.

[17] P. Hajłasz and J. Kinnunen: Hölder quasicontinuity of Sobolev functions on metric spaces, *Rev. Mat. Iberoamericana* **14** (1998), no. 3, 601–622.

[18] P. Hajłasz and P. Koskela: Sobolev met Poincaré, *Mem. Amer. Math. Soc.* **145** (2000), no. 688.

[19] P. Hajłasz and O. Martio: Traces of Sobolev functions on fractal type sets and characterization of extension domains, *J. Funct. Analysis* **143** (1997), no. 1, 221–246.

[20] P. Harjulehto and P. Hästö: A capacity approach to the Poincaré inequality and Sobolev imbedding in variable exponent Sobolev space, *Rev. Mat. Complut.* **17** (2004), no. 1., to appear.

[21] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: Sobolev capacity on the space $W^{1,p(\cdot)}(\mathbb{R}^n)$, *J. Funct. Spaces Appl.* **1** (2003), no. 1, 17–33.

[22] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen: The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, preprint (2003).

[Available at `http://www.math.helsinki.fi/analysis/varsobgroup/`.]

[23] P. Harjulehto, P. Hästö and O. Martio: Fuglede's theorem in variable exponent Sobolev space, *Collect. Math.* **55**, (2004), no. 3, to appear.

[24] P. Harjulehto, P. Hästö and M. Pere: Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator, *Real Anal. Exchange* **30**, no. 1, to appear.

[25] J. Heinonen: *Lectures on analysis on metric spaces*, Springer-Verlag, New York, 2001.

[26] J. Heinonen, T. Kilpeläinen and O. Martio: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1993.

[27] J. Kinnunen and O. Martio: The Sobolev capacity on metric spaces, *Ann. Acad. Sci. Fenn. Math.* **21** (1996), 367–382.

[28] J. Kinnunen and O. Martio: Choquet property for the Sobolev capacity in metric spaces, *Proceedings on Analysis and geometry*, Novosibirsk, Sobolev institute press 2000, 285–290.

[29] J. Kinnunen and O. Martio: Potential theory of quasiminimizers, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), no. 2, 459–490.

[30] V. Kokilasvili and S. Samko: Maximal and fractional operators in weighted $L^{p(x)}$ spaces, *Rev. Mat. Iberoamericana* **20** (2004), no. 2, to appear.

[31] O. Kováčik and J. Rákosník: On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41(116)** (1991), 592–618.

[32] E. J. McShane: Extension of range of functions, *Bull. Am. Math. Soc.* **40** (1934), 837–842.

[33] Y. Mizuta and T. Shimomura: Continuity of Sobolev functions of variable exponent on metric spaces, preprint (2004).

[34] J. Museliak: *Orlicz Spaces and Modular Spaces*, Springer-Verlag, Berlin, 1985.

[35] A. Nekvinda: Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$, *Math. Inequal. Appl.*, to appear.

[36] A. Nekvinda: A note on maximal operator on l^{p_n} and $L^{p(x)}(\mathbb{R}^n)$, preprint (2004).

[37] L. Pick and M. Růžička: An example of a space $L^{p(x)}$ on which the Hardy–Littlewood maximal operator is not bounded, *Expo. Math.* **19** (2001), 369–371.

[38] S. Samko: Denseness of $C_0^\infty(\mathbb{R}^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(\mathbb{R}^n)$, pp. 333–342 in *Direct and inverse problems of mathematical physics (Newark, DE, 1997)*, Int. Soc. Anal. Appl. Comput. **5**, Kluwer Acad. Publ., Dordrecht, 2000.

[39] N. Shanmugalingam: Newtonian spaces: An extension of Sobolev spaces to metric measure space, *Rev. Mat. Iberoamericana* **16** (2000), no. 2, 243–279.

[40] I. I. Sharapudinov: On the topology of the space $L^{p(t)}([0;1])$, *Math. Notes* **26** (1979), no. 3–4, 796–806. [translation of *Mat. Zametki* **26** (1978), no. 4, 613–632.]

[41] K. Yosida: *Functional Analysis*, Springer-Verlag, 1980.

Department of Mathematics and Statistics
P.O. Box 4
FIN-00014 University of Helsinki
Finland

Research group home-page:
http://www.math.helsinki.fi/analysis/varsobgroup/

E-mail addresses:
petteri.harjulehto@helsinki.fi
peter.hasto@helsinki.fi
mikko.pere@helsinki.fi