

PROPERTIES OF CAPACITIES IN VARIABLE EXPONENT SOBOLEV SPACES

PETTERI HARJULEHTO, PETER HÄSTÖ*, AND MIKA KOSKENOJA

ABSTRACT. In this paper we introduce two new capacities in the variable exponent setting: a relative capacity, and a Newtonian capacity based on functions absolutely continuous on curves. We study properties of both capacities and compare them with each other and with the Sobolev capacity. As an application, we consider exceptional sets of variable exponent Sobolev spaces.

1. INTRODUCTION

The notion of capacity offers a standard way to characterize exceptional sets in various function spaces. Depending on the starting point of the study, the capacity of a set can be defined in many appropriate ways. A common property of capacities is that they measure small sets more precisely than the usual Lebesgue measure. The Choquet theory [2] provides a standard approach to capacities. Capacity is a necessary tool in both classical and nonlinear potential theory.

In this paper we continue to develop a basis for studying potential theoretic aspects of variable exponent Sobolev spaces. The authors together with Varonen [6] defined and studied properties of the Sobolev capacity in variable exponent Sobolev space. We continue the earlier study by adopting two new capacities, a relative capacity and a Newtonian capacity, to the variable exponent setting. The nature of the Sobolev capacity is global but both the relative and Newtonian capacities are local since the capacity of a set is taken relative to a surrounding open set. However, it makes sense to compare all three capacities with each other.

Nonlinear theories, such as p -potential theory, are usually equipped with a capacity based on a variational definition. If Ω is an open subset of \mathbb{R}^n and $K \subset \Omega$ is compact, then the relative variational p -capacity of a condenser (K, Ω) is given by

$$\text{cap}_p(K, \Omega) = \inf_u \int_{\Omega} |\nabla u|^p dx,$$

where the infimum is taken over smooth and zero boundary valued functions u in Ω such that $u \geq 1$ in K . The same capacity is obtained if the set of admissible functions u is replaced by the continuous first order Sobolev

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functions with $u \geq 1$ in K . The p -capacity is a Choquet capacity relative to Ω , see [15].

Curve families in variable exponent Sobolev spaces were studied by Harjulehto and Hästö together with Martio in [10] and together with Pere in [11]. Their results establish a solid ground for our studies of the Newtonian capacity based on functions absolutely continuous on curves. Special care is required here since continuous functions are not necessarily dense in $W^{1,p(\cdot)}(\Omega)$. Harjulehto's [4] study of zero boundary value variable exponent Sobolev spaces gives us a reasonable basis for the definition of a relative capacity. This allows us to study removable sets with arguments based on the relative capacity in the variable exponent setting.

Removable sets for Sobolev spaces, that is relatively closed sets $E \subset \Omega$ such that

$$W^{1,p}(\Omega \setminus E) = W^{1,p}(\Omega),$$

have been studied from both analytic and geometric points of view. Analytic studies are usually based on the notion of capacity whereas a geometric approach uses different concepts of small sets. When p is fixed, $1 < p < \infty$, Vodop'janov and Gol'dšteĭn [20] proved that a set $E \subset \Omega$ of Lebesgue measure zero is removable if and only if E is a null set for p -capacity. Kolsrud [17] noted that their result admits a somewhat more general version. Moreover, Hedberg [14] has written a comprehensive study of removable sets in terms of condenser capacities.

The structure of the rest of this paper is as follows. In the next section we review the basic theory of variable exponent spaces. In Section 3 we define the variational relative capacity and study its basic properties. In Section 4 we compare the relative capacity with the Sobolev capacity introduced previously. In Section 5 we introduce a capacity which is based on absolutely continuous functions and prove that a Sobolev function has a certain "best" representative. In Section 6 we use our capacities to characterize removable sets of Sobolev functions.

2. VARIABLE EXPONENT SPACES

Although variable exponent Lebesgue and Sobolev spaces have a long prehistory, the foundations of the theory in \mathbb{R}^n was laid by Kováčik and Rákosník [18] only in the early 1990's. Over the last five years the field has experienced a large increase in activity, with over a hundred research papers published, see [3] for references. In this section we give a brief recap of the most central results in the variable exponent theory.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open set, and let $p : \Omega \rightarrow [1, \infty)$ be a measurable function, called the *variable exponent* on Ω . We write $p_G^+ = \operatorname{ess\,sup}_{x \in G} p(x)$ and $p_G^- = \operatorname{ess\,inf}_{x \in G} p(x)$, where $G \subset \Omega$ and abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)} dx < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1 \}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all measurable functions $u \in L^{p(\cdot)}(\Omega)$ such that the absolute value of the distributional gradient ∇u is in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. For the basic theory of variable exponent spaces see [18].

A capacity for subsets of \mathbb{R}^n was introduced in [6, Section 3]. To define this capacity we denote

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geq 1 \text{ in an open set containing } E\}$$

for $E \subset \mathbb{R}^n$. The *Sobolev $p(\cdot)$ -capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) dx.$$

In case $S_{p(\cdot)}(E) = \emptyset$, we define $C_{p(\cdot)}(E) = \infty$. If $1 < p^- \leq p^+ < \infty$, then the set function $E \mapsto C_{p(\cdot)}(E)$ is an outer measure and a Choquet capacity [6, Corollaries 3.3 and 3.4].

We say that the exponent $p : \Omega \rightarrow [1, \infty)$ is *log-Hölder continuous* if there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|}$$

for every $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$. It turns out that this condition is sufficient to guarantee a lot of regularity for variable exponent spaces. For instance, if p is bounded and log-Hölder continuous, then $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, [19]. Under these circumstances, it makes sense to define the space of zero boundary value Sobolev functions as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. This Banach space is denoted by $H_0^{1,p(\cdot)}(\Omega)$.

Without the log-Hölder assumption for the exponent p , we may still define a zero boundary value Sobolev space using a weaker notion borrowed from the setting of metric measure spaces. This notion coincides with the usual definition if smooth functions are dense.

Recall that a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *$p(\cdot)$ -quasicontinuous* (in \mathbb{R}^n) if for every $\varepsilon > 0$ there exists an open set G with $C_{p(\cdot)}(G) < \varepsilon$ such that $u|_{\mathbb{R}^n \setminus G}$ is continuous. Also, for a subset E of \mathbb{R}^n we say that a claim holds *$p(\cdot)$ -quasieverywhere in E* if it holds everywhere except in a set $N \subset E$ with $C_{p(\cdot)}(N) = 0$. A function u belongs to the space $Q_0^{1,p(\cdot)}(\Omega)$ if there exists a $p(\cdot)$ -quasicontinuous function $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u = \tilde{u}$ almost everywhere in Ω and $\tilde{u} = 0$ $p(\cdot)$ -quasieverywhere in $\mathbb{R}^n \setminus \Omega$. The set $Q_0^{1,p(\cdot)}(\Omega)$ is endowed with the norm

$$\|u\|_{Q_0^{1,p(\cdot)}(\Omega)} = \|\tilde{u}\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}.$$

If $1 < p^- \leq p^+ < \infty$, then $Q_0^{1,p(\cdot)}(\Omega)$ is a Banach space [7, Theorem 3.1], and if, in addition, $C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, then $H_0^{1,p(\cdot)}(\Omega) = Q_0^{1,p(\cdot)}(\Omega)$ [4, Theorem 3.11]. In particular, the last statement holds if $1 < p^- \leq p^+ < \infty$ and p is log-Hölder continuous.

3. RELATIVE $p(\cdot)$ -CAPACITY

In this section we introduce an alternative to the Sobolev $p(\cdot)$ -capacity. This alternative capacity of a set is taken relative to a surrounding open subset of \mathbb{R}^n . Recall that

$$C_0(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : u \text{ is continuous and } \text{spt } u \Subset \Omega\},$$

where $\text{spt } u$ is the support of u , i.e. the smallest set closed in \mathbb{R}^n outside of which u vanishes. Suppose that K is a compact subset of Ω . We denote

$$R_{p(\cdot)}(K, \Omega) = \{u \in W^{1,p(\cdot)}(\Omega) \cap C_0(\Omega) : u \geq 1 \text{ on } K\}$$

and define

$$\text{cap}_{p(\cdot)}^*(K, \Omega) = \inf_{u \in R_{p(\cdot)}(K, \Omega)} \int_{\Omega} |\nabla u(x)|^{p(x)} dx.$$

Further, if $U \subset \Omega$ is open, then

$$\text{cap}_{p(\cdot)}(U, \Omega) = \sup_{\substack{K \subset U \\ \text{compact}}} \text{cap}_{p(\cdot)}^*(K, \Omega),$$

and for an arbitrary set $E \subset \Omega$

$$\text{cap}_{p(\cdot)}(E, \Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot)}(U, \Omega).$$

The number $\text{cap}_{p(\cdot)}(E, \Omega)$ is called the *variational $p(\cdot)$ -capacity of E relative to Ω* . We usually call it simply the relative $p(\cdot)$ -capacity of the pair, or *condenser*, (E, Ω) . The following result follows immediately from the definition:

3.1. Lemma. *The relative $p(\cdot)$ -capacity is an outer capacity, i.e.*

- (i) $\text{cap}_{p(\cdot)}(\emptyset, \Omega) = 0$;
- (ii) If $E_1 \subset E_2 \subset \Omega_1 \subset \Omega_2$, then $\text{cap}_{p(\cdot)}(E_1, \Omega_1) \leq \text{cap}_{p(\cdot)}(E_2, \Omega_2)$; and
- (iii) For an arbitrary set $E \subset \Omega$

$$\text{cap}_{p(\cdot)}(E, \Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot)}(U, \Omega).$$

Although the previous lemma holds for arbitrary measurable exponents, there is one obvious short-coming in our capacity, namely that we do not know whether $\text{cap}_{p(\cdot)}^*(K, \Omega) = \text{cap}_{p(\cdot)}(K, \Omega)$ for compact sets. We next show that this is the case under a very weak assumption, but emphasize, that it is not known whether this assumption is necessary.

3.2. Proposition. *Suppose that $p^+ < \infty$. Then for every compact set $K \subset \Omega$ we have $\text{cap}_{p(\cdot)}^*(K, \Omega) = \text{cap}_{p(\cdot)}(K, \Omega)$.*

Proof. The inequality $\text{cap}_{p(\cdot)}^*(K, \Omega) \leq \text{cap}_{p(\cdot)}(K, \Omega)$ follows directly from the definition. To prove the opposite inequality fix $\varepsilon > 0$ and let $u \in R_0(K, \Omega)$ be such that $\text{cap}_{p(\cdot)}^*(K, \Omega) \leq \varrho_{p(\cdot)}(|\nabla u|) + \varepsilon$. Then $v = (1 + \varepsilon)u$ is greater than one in an open set U which contains K . Thus v is also greater than one in every compact $K' \subset U$, so that

$$\text{cap}_{p(\cdot)}(U, \Omega) \leq \varrho_{p(\cdot)}(|\nabla v|) \leq (1 + \varepsilon)^{p^+} \varrho_{p(\cdot)}(|\nabla u|).$$

This implies that

$$\text{cap}_{p(\cdot)}(K, \Omega) \leq \text{cap}_{p(\cdot)}(U, \Omega) \leq (1 + \varepsilon)^{p^+} \left(\text{cap}_{p(\cdot)}^*(K, \Omega) + \varepsilon \right),$$

from which the conclusion follows as $\varepsilon \rightarrow 0$. \square

We are now ready to study more advanced basic properties:

3.3. Theorem. *Suppose that $p^+ < \infty$. The set function $E \mapsto \text{cap}_{p(\cdot)}(E, \Omega)$ has the following properties:*

(iv) *If K_1 and K_2 are compact subsets of Ω , then*

$$\text{cap}_{p(\cdot)}(K_1 \cup K_2, \Omega) + \text{cap}_{p(\cdot)}(K_1 \cap K_2, \Omega) \leq \text{cap}_{p(\cdot)}(K_1, \Omega) + \text{cap}_{p(\cdot)}(K_2, \Omega).$$

(v) *If $K_1 \supset K_2 \supset \dots$ are compact subsets of Ω , then*

$$\lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(K_i, \Omega) = \text{cap}_{p(\cdot)} \left(\bigcap_{i=1}^{\infty} K_i, \Omega \right).$$

(vi) *If $E_1 \subset E_2 \subset \dots$ are subsets of Ω , then*

$$\lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(E_i, \Omega) = \text{cap}_{p(\cdot)} \left(\bigcup_{i=1}^{\infty} E_i, \Omega \right).$$

(vii) *If $E_i \subset \Omega$ for $i = 1, 2, \dots$, then*

$$\text{cap}_{p(\cdot)} \left(\bigcup_{i=1}^{\infty} E_i, \Omega \right) \leq \sum_{i=1}^{\infty} \text{cap}_{p(\cdot)}(E_i, \Omega).$$

Proof. To prove (iv), note first that

$$\nabla \max(u, v)(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \geq v\} \\ \nabla v(x) & \text{for a.e. } x \in \{v \geq u\} \end{cases}$$

and similarly for the minimum, so we obtain that $R_{p(\cdot)}(K, \Omega)$ is a lattice. If $u_1 \in R_{p(\cdot)}(K_1, \Omega)$ and $u_2 \in R_{p(\cdot)}(K_2, \Omega)$, then $\max(u_1, u_2) \in R_{p(\cdot)}(E_1 \cup E_2, \Omega)$ and $\min(u_1, u_2) \in R_{p(\cdot)}(E_1 \cap E_2, \Omega)$ and

$$\begin{aligned} & \int_{\Omega} |\nabla \max(u_1, u_2)(x)|^{p(x)} dx + \int_{\Omega} |\nabla \min(u_1, u_2)(x)|^{p(x)} dx \\ &= \int_{\Omega} |\nabla u_1(x)|^{p(x)} dx + \int_{\Omega} |\nabla u_2(x)|^{p(x)} dx. \end{aligned}$$

By Proposition 3.2 this implies the following inequality from which (iv) follows as $\varepsilon \rightarrow 0$:

$$\begin{aligned} & \text{cap}_{p(\cdot)}(K_1 \cup K_2, \Omega) + \text{cap}_{p(\cdot)}(K_1 \cap K_2, \Omega) \\ & \leq \int_{\Omega} |\nabla u_1|^{p(x)} dx + \int_{\Omega} |\nabla u_2|^{p(x)} dx + \varepsilon \\ & \leq \text{cap}_{p(\cdot)}(K_1, \Omega) + \text{cap}_{p(\cdot)}(K_2, \Omega) + \varepsilon. \end{aligned}$$

To prove (v), let $K_1 \supset K_2 \supset \dots$ be compact subsets of Ω . Since $\bigcap_{i=1}^{\infty} K_i \subset K_j$ for each $j = 1, 2, \dots$, property (ii) gives

$$\text{cap}_{p(\cdot)} \left(\bigcap_{i=1}^{\infty} K_i, \Omega \right) \leq \lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(K_i, \Omega).$$

To prove the opposite inequality, choose an open set $U \subset \Omega$ such that $\bigcap_i K_i \subset U$. Because every K_i is compact (so that $\bigcap_i K_i$ is compact, as well), there is a positive integer k such that $K_i \subset U$ for all $i \geq k$. Thus

$$\lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(K_i, \Omega) \leq \text{cap}_{p(\cdot)}(U, \Omega),$$

and by Lemma 3.1 (iii)

$$\lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(K_i, \Omega) \leq \text{cap}_{p(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i, \Omega\right).$$

To prove (vi), we need the following lemma, the proof of which is based on iterated use of (iv). Since the proof is identical to that given in [15, Lemma 2.3], we omit it here.

3.4. Lemma. *Suppose that $E_1, \dots, E_k \subset \Omega$. Then*

$$\begin{aligned} \text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^k E_i, \Omega\right) - \text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^k F_i, \Omega\right) \\ \leq \sum_{i=1}^k \left(\text{cap}_{p(\cdot)}(E_i, \Omega) - \text{cap}_{p(\cdot)}(F_i, \Omega)\right) \end{aligned}$$

whenever $F_i \subset E_i$, $i = 1, 2, \dots, k$, and $\text{cap}_{p(\cdot)}(F_i, \Omega) < \infty$.

We then continue the proof of (vi). Denote $E = \bigcup_{i=1}^{\infty} E_i$. Note first that (ii) implies that

$$\lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(E_i, \Omega) \leq \text{cap}_{p(\cdot)}(E, \Omega).$$

To prove the opposite inequality we may assume that $\lim_{i \rightarrow \infty} \text{cap}_{p(\cdot)}(E_i, \Omega) < \infty$; it follows by (ii) that $\text{cap}_{p(\cdot)}(E_i, \Omega) < \infty$ for each i . Fix $\varepsilon > 0$ and choose open sets U_i such that $E_i \subset U_i \subset \Omega$ and that

$$\text{cap}_{p(\cdot)}(U_i, \Omega) \leq \text{cap}_{p(\cdot)}(E_i, \Omega) + \varepsilon 2^{-i}.$$

Using Lemma 3.4 we derive from this that

$$\text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^k U_i, \Omega\right) - \text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^k E_i, \Omega\right) \leq \sum_{i=1}^k \varepsilon 2^{-i} < \varepsilon.$$

If $K \subset \bigcup_{i=1}^{\infty} U_i$ is compact, then $K \subset \bigcup_{i=1}^k U_i$ for some k , and we have

$$\begin{aligned} \text{cap}_{p(\cdot)}(K, \Omega) &\leq \text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^k U_i, \Omega\right) \leq \text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^k E_i, \Omega\right) + \varepsilon \\ &= \text{cap}_{p(\cdot)}(E_k, \Omega) + \varepsilon, \end{aligned}$$

where we used that $\bigcup_{i=1}^k E_i = E_k$. It follows that

$$\begin{aligned} \text{cap}_{p(\cdot)}(E, \Omega) &\leq \text{cap}_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} U_i, \Omega\right) = \sup_K \text{cap}_{p(\cdot)}(K, \Omega) \\ &\leq \lim_{k \rightarrow \infty} \text{cap}_{p(\cdot)}(E_k, \Omega) + \varepsilon, \end{aligned}$$

where the supremum is taken over all compact sets $K \subset \bigcup_{i=1}^{\infty} U_i$. So (vi) is proved.

It remains to prove (vii). From (iv) it follows by induction that

$$\text{cap}_{p(\cdot)} \left(\bigcup_{i=1}^k E_i, \Omega \right) \leq \sum_{i=1}^k \text{cap}_{p(\cdot)}(E_i, \Omega)$$

for any finite family of subsets E_1, E_2, \dots, E_k in Ω . Since $\bigcup_{i=1}^k E_i$ increases to $\bigcup_{i=1}^{\infty} E_i$, (vi) implies (vii). This completes the proof of Theorem 3.3. \square

By the definition of outer measure, properties (i), (ii), and (vii) of the relative $p(\cdot)$ -capacity yield:

3.5. Corollary. *If $p^+ < \infty$, then the relative $p(\cdot)$ -capacity is an outer measure.*

A set function which satisfies properties (i), (ii), (v), and (vi) is called a *Choquet capacity*, see [2]. We therefore have the following result:

3.6. Corollary. *If $p^+ < \infty$, then the set function $E \mapsto \text{cap}_{p(\cdot)}(E, \Omega)$, $E \subset \Omega$, is a Choquet capacity. In particular, all Suslin sets $E \subset \Omega$ are capacitable, this is,*

$$\text{cap}_{p(\cdot)}(E, \Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U \text{ open}}} \text{cap}_{p(\cdot)}(U, \Omega) = \sup_{\substack{K \subset E \\ K \text{ compact}}} \text{cap}_{p(\cdot)}(K, \Omega).$$

3.7. Remark. To obtain the preceding basic properties, Choquet capacity and outer measure, of the relative $p(\cdot)$ -capacity, we needed only an upper bound on p , which is assumed also in almost all other results in the variable exponent setting. For the Sobolev $p(\cdot)$ -capacity, in contrast, we needed to assume that $1 < p^- \leq p^+ < \infty$ in order to conclude this, see [6] (but see also [8] for some relaxations on these assumptions).

As has been mentioned, smooth functions are not always dense in the variable exponent Sobolev space. However, when they are, we can use the usual change of test-function set.

3.8. Proposition. *Suppose that $p^+ < \infty$ and that smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$. If $K \subset \Omega$ is compact, then*

$$\text{cap}_{p(\cdot)}(K, \Omega) = \inf_{u \in R_0^\infty(K, \Omega)} \int_{\Omega} |\nabla u(x)|^{p(x)} dx,$$

where $R_0^\infty(K, \Omega) = \{u \in C_0^\infty(\Omega) : u \geq 1 \text{ on } K\}$.

Proof. Clearly $R_0^\infty(K, \Omega) \subset R_{p(\cdot)}(K, \Omega)$ and hence

$$\text{cap}_{p(\cdot)}(K, \Omega) \leq \inf_{u \in R_0^\infty(K, \Omega)} \int_{\Omega} |\nabla u(x)|^{p(x)} dx.$$

Assume then that $u \in R_{p(\cdot)}(K, \Omega)$. Since K is compact, there exists $\Phi \in C^\infty(\Omega)$ such that $\Phi = 0$ in an open set containing K and $\Phi = 1$ in a complement of an open set $U, \bar{U} \subset \Omega$. Let $\phi_i \in C^\infty(\mathbb{R}^n)$ converge to u . Then $\Phi_i = 1 - (1 - \phi_i)\Phi \in C^\infty(\Omega)$ also converge to u . Let $\Psi \in C_0^\infty(\Omega)$ be one in the support of u . Then $\Phi_i \Psi \in R_0^\infty(K, \Omega)$ and $\Phi_i \Psi \rightarrow u$. This completes the proof since u was an arbitrary test function. \square

4. THE RELATIONSHIP BETWEEN THE CAPACITIES

The following two theorems associate the Sobolev $p(\cdot)$ -capacity and relative $p(\cdot)$ -capacity. Specifically, we give sufficient conditions on the exponent p that the sets of capacity zero coincide.

4.1. Lemma. *Assume that $1 < p^- \leq p^+ < \infty$ and fix $R > 0$. If $K \subset \Omega \cap B(0, R)$ is a compact set and satisfies $\text{cap}_{p(\cdot)}(K, \Omega) < t$, then $C_{p(\cdot)}(K) < C(R) \max\{t^{1/p^+}, t\}$.*

Proof. Let $K \subset \Omega \cap B(0, R)$ be a compact set with $\text{cap}_{p(\cdot)}(K, \Omega) < t$, and let $u \in R_{p(\cdot)}(K, \Omega)$ be a function with $\varrho_{L^{p(\cdot)}(\Omega)}(|\nabla u|) < t$. By truncation we may assume that $0 \leq u \leq 1$. Let $\phi: \mathbb{R}^n \rightarrow [0, 1]$ be a $(1/R)$ -Lipschitz function with support in $\overline{B}(0, 2R)$ which equals 1 in $B(0, R)$. Define now

$$v(x) = \begin{cases} \min\{1, 2u(x)\phi(x)\}, & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Note that v equals 1 in an open set containing K . Using that $|\nabla v(x)| \leq 2|\nabla u(x)| + 2|u(x)|/R$ we find that

$$\begin{aligned} \int_{\mathbb{R}^n} |v(x)|^{p(x)} + |\nabla v(x)|^{p(x)} dx \\ \leq \left(3 + \frac{2}{R}\right)^{p^+} \int_{\Omega \cap B(0, 2R)} |v(x)|^{p(x)} + |\nabla u(x)|^{p(x)} dx \end{aligned}$$

Since $0 \leq v \leq 1$, we have

$$\int_{\Omega \cap B(0, 2R)} |v(x)|^{p(x)} dx \leq \int_{\Omega \cap B(0, 2R)} |v(x)| dx.$$

Using the classical Poincaré inequality in $L^1(B(0, 2R))$ and the embedding $L^{p(\cdot)}(B(0, 2R)) \rightarrow L^1(B(0, 2R))$ we obtain that

$$\|v\|_1 \leq CR \|\nabla v\|_1 \leq CR(1 + |B(0, 2R)|) \|\nabla v\|_{L^{p(\cdot)}(\Omega)}$$

Using $v \in S_{p(\cdot)}(K)$ for the first inequality and the above estimates for the second inequality we find that

$$\begin{aligned} C_{p(\cdot)}(K) &\leq \int_{\mathbb{R}^n} |v(x)|^{p(x)} + |\nabla v(x)|^{p(x)} dx \\ &\leq C(R) \|\nabla v\|_{L^{p(\cdot)}(\Omega)} + C(R) \varrho_{p(\cdot)}(|\nabla u(x)|). \end{aligned}$$

It follows trivially that

$$\|\nabla v\|_{L^{p(\cdot)}(\Omega)} \leq C(R) \max\{\varrho_{p(\cdot)}(|\nabla u(x)|)^{1/p^+}, \varrho_{p(\cdot)}(|\nabla u(x)|)^{1/p^-}\},$$

so the claim is proved. \square

4.2. Theorem. *Assume that $1 < p^- \leq p^+ < \infty$. If $E \subset \Omega$ satisfies $\text{cap}_{p(\cdot)}(E, \Omega) = 0$, then $C_{p(\cdot)}(E) = 0$.*

Proof. Let first $K \subset \Omega$ be a compact set with $\text{cap}_{p(\cdot)}(K, \Omega) = 0$. Then by the previous lemma $C_{p(\cdot)}(K) = 0$.

Let next $E \subset \Omega$ be a set with $\text{cap}_{p(\cdot)}(E, \Omega) = 0$. Since the relative capacity is an outer capacity, there exists a sequence of open sets $U_i \supset E$ with $\text{cap}_{p(\cdot)}(U_i, \Omega) \rightarrow 0$ as $i \rightarrow \infty$. Let $U = \bigcap_{i=1}^{\infty} U_i$. Then U is a Suslin set and

$\text{cap}_{p(\cdot)}(U, \Omega) = 0$. Hence by the Choquet property $C_{p(\cdot)}(U) = \sup_K C_{p(\cdot)}(K)$ where the supremum is taken over all compact $K \subset U$. For every compact $K \subset U$ we have $\text{cap}_{p(\cdot)}(K, \Omega) = 0$ and hence by the first part of the proof $C_{p(\cdot)}(K) = 0$. Then the Choquet property of $C_{p(\cdot)}$ yields $C_{p(\cdot)}(U) = 0$, and since $E \subset U$, we conclude that $C_{p(\cdot)}(E) = 0$. This completes the proof. \square

To get the converse implication we need to assume that continuous functions are dense in variable exponent Sobolev space. For some sufficient conditions for this, see [13], and for some examples showing that this is not always the case, see [12, 21].

4.3. Theorem. *Let $1 < p^- \leq p^+ < \infty$, and suppose that continuous functions are dense in $W^{1,p(\cdot)}(\Omega)$. If $E \subset \Omega$ satisfies $C_{p(\cdot)}(E) = 0$, then $\text{cap}_{p(\cdot)}(E, \Omega) = 0$.*

Proof. The same argument as in the proof of Theorem 4.2 works in this case, once we establish the property for compact sets in Ω . Let $K \subset \Omega$ be compact and $C_{p(\cdot)}(K) = 0$. By the density of continuous functions, it follows as in [6, Lemma 3.6] that the set of admissible functions in the definition of the Sobolev $p(\cdot)$ -capacity can be replaced by the subset $S_{p(\cdot)}^0(K) = S_{p(\cdot)}(K) \cap C_0(\Omega)$. Therefore we may choose a sequence (u_i) of functions belonging to $S_{p(\cdot)}^0(K)$ such that $\|u_i\|_{W^{1,p(\cdot)}(\mathbb{R}^n)} \rightarrow 0$. However, u_i is also in $R_{p(\cdot)}(K, \Omega)$, so the claim follows. \square

5. CAPACITY AND MODULUS OF A CURVE FAMILY

For a family Γ of rectifiable curves $\gamma: [a, b] \rightarrow \Omega$, we denote by $F(\Gamma)$ the set of all Γ -admissible functions, i.e. all Borel functions $u: \Omega \rightarrow [0, \infty]$ such that

$$\int_{\gamma} u \, ds \geq 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to curve length. We define the $p(\cdot)$ -modulus of Γ by

$$M_{p(\cdot)}(\Gamma) = \inf_{u \in F(\Gamma)} \int_{\Omega} u(x)^{p(x)} \, dx.$$

If $F(\Gamma) = \emptyset$, then we set $M_{p(\cdot)}(\Gamma) = \infty$. The $p(\cdot)$ -modulus is an outer measure on the space of all curves of Ω [10, Lemma 2.1].

Let $u: \Omega \rightarrow \mathbb{R}$ and let Γ be the family of curves γ parametrized by arc-length such that $u \circ \gamma$ is not absolutely continuous on $[0, \ell(\gamma)]$. We say that u is *absolutely continuous on curves*, $u \in ACC_{p(\cdot)}(\Omega)$, if $M_{p(\cdot)}(\Gamma) = 0$. If $1 < p^- \leq p^+ < \infty$ and $C^1(\Omega)$ -functions are dense in $W^{1,p(\cdot)}(\Omega)$, then every function in $W^{1,p(\cdot)}(\Omega)$ has an ACC -representative [10, Theorem 4.6] and [11, Lemma 2.7].

We define a capacity which uses functions absolutely continuous on curves. We first introduce the notation $ACC^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega) \cap ACC_{p(\cdot)}(\Omega)$. Let $E \subset \mathbb{R}^n$. We set

$$NC_{p(\cdot)}(E) = \inf \int_{\Omega} |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \, dx,$$

where the infimum is taken over all $u \in ACC^{1,p(\cdot)}(\Omega)$ which are at least 1 in E . If $p^+ < \infty$, this is a capacity in the Newtonian space and an outer measure, see [11]. The proof of the following lemma is essentially the same as the proof of [9, Lemma 7].

5.1. Lemma. *Let $1 < p^- \leq p^+ < \infty$, and suppose that C^1 -functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Then*

$$C_{p(\cdot)}(E) = NC_{p(\cdot)}(E)$$

for every $E \subset \mathbb{R}^n$.

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be a test function for $C_{p(\cdot)}(E)$. Then $u = 1$ in an open set U containing E . Let \tilde{u} be the ACC -representative of u . Then $\tilde{u} = 1$ almost everywhere in U . Let $F = U \setminus \{\tilde{u} \geq 1\}$. By Γ_F we denote the family of rectifiable curves in \mathbb{R}^n with $|\gamma| \cap F \neq \emptyset$. By Γ_F^+ we denote the subfamily of curves $\gamma \in \Gamma_F$ with positive one-dimensional Hausdorff measure $H^1(|\gamma| \cap E)$. If $\infty \cdot \chi_F$ is not Borel measurable we may replace it by a Borel measurable function h with $h \geq \infty \cdot \chi_F$ in \mathbb{R}^n and $h(x) = \infty \cdot \chi_F(x)$ for almost every $x \in \mathbb{R}^n$. Since every curve in Γ_F^+ has positive H^1 measure, h is admissible, so that

$$M_{p(\cdot)}(\Gamma_F^+) \leq \int_{\Omega} h(x)^{p(x)} dx = 0,$$

and we see that $M_{p(\cdot)}(\Gamma_F^+) = 0$. By absolute continuity, \tilde{u} equals one on every curve in $\Gamma_F \setminus \Gamma_F^+$ not in the exceptional family of \tilde{u} . So we have $M_{p(\cdot)}(\Gamma_F) = 0$. Hence we can redefine \tilde{u} so that it is 1 in E and belongs to $ACC^{1,p(\cdot)}(\mathbb{R}^n)$. This yields that $NC_{p(\cdot)}(E) \leq C_{p(\cdot)}(E)$.

Assume then that $u \in ACC^{1,p(\cdot)}(\mathbb{R}^n)$ is a test function for $NC_{p(\cdot)}(E)$. Since continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, we obtain as in [6, Theorem 5.2] that u has a representative \tilde{u} which is continuous when restricted to the complement of an open set of arbitrarily small $NC_{p(\cdot)}$ -capacity. By [11, Theorem 4.4] we also have that $\tilde{u} \in ACC^{1,p(\cdot)}(\mathbb{R}^n)$. Now it is easy to check that $NC_{p(\cdot)}(\{u \neq \tilde{u}\}) = 0$. Therefore for all $\varepsilon > 0$ there exists an open set F_ε so that \tilde{u} restricted to $\mathbb{R}^n \setminus F_\varepsilon$ is continuous, $\tilde{u} \geq 1$ in $E \setminus F_\varepsilon$ and $NC_{p(\cdot)}(F_\varepsilon) \leq \varepsilon$. Hence there exists a neighborhood U of \bar{E} such that \tilde{u} restricted to $U \setminus F_\varepsilon$ is not less than $1 - \varepsilon$. Let $w_\varepsilon \in ACC^{1,p(\cdot)}(\mathbb{R}^n)$ be such that w_ε restricted to F_ε is one, $0 \leq w_\varepsilon \leq 1$ and

$$\int_{\mathbb{R}^n} |w_\varepsilon(x)|^{p(x)} + |\nabla w_\varepsilon(x)|^{p(x)} dx \leq \varepsilon.$$

Then $\frac{u}{1-\varepsilon} + w_\varepsilon$ is a test function for $C_{p(\cdot)}(E)$ and the inequality $C_{p(\cdot)}(E) \leq NC_{p(\cdot)}(E)$ follows by letting $\varepsilon \rightarrow 0$. Hence $C_{p(\cdot)}(E) = NC_{p(\cdot)}(E)$. \square

Assume that $1 < p^- \leq p^+ < \infty$ and C^1 -functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Then every Sobolev function $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ has a quasicontinuous representative [6, Theorem 5.2]. By [16] the quasicontinuous representative is unique. By [10, Theorem 4.6] we find that if u is quasicontinuous, then $u \in ACC_{p(\cdot)}(\Omega)$. Next we show that in fact the quasicontinuous representative is the only ACC -representative.

5.2. Theorem. *Let $1 < p^- \leq p^+ < \infty$, and suppose that C^1 -functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Assume that $u, v \in ACC^{1,p(\cdot)}(\mathbb{R}^n)$ and $u = v$ almost everywhere in \mathbb{R}^n . Then $u = v$ quasieverywhere, i.e.*

$$C_{p(\cdot)}(\{x \in \mathbb{R}^n : u(x) \neq v(x)\}) = 0.$$

Proof. We define $E = \{x \in \mathbb{R}^n : u(x) \neq v(x)\}$ and denote by Γ_E the family of rectifiable curves γ in \mathbb{R}^n which intersect E . By Γ_E^+ we denote the subfamily of curves with positive one-dimensional Hausdorff measure $H^1(|\gamma| \cap E)$. As in the previous lemma we conclude that Γ_E^+ has modulus zero. The curves γ in $\Gamma_E \setminus \Gamma_E^+$ intersect E only on a set of linear measure zero. Hence the function $u - v$ takes the value 0 H^1 -almost everywhere in $|\gamma|$. This implies that $u - v$ can not be absolutely continuous on γ . Since $u - v \in ACC_{p(\cdot)}(\Omega)$ this implies that $M_{p(\cdot)}(\Gamma_E \setminus \Gamma_E^+) = 0$. The two cases together give that

$$M_{p(\cdot)}(\Gamma_E) = M_{p(\cdot)}(\Gamma_E^+ \cup (\Gamma_E \setminus \Gamma_E^+)) = 0.$$

Hence we find that $1 \cdot \chi_E$ is a test function for $NC_{p(\cdot)}(E)$ and thus, by Lemma 5.1, $C_{p(\cdot)}(E) = 0$. \square

5.3. Corollary. *Let $1 < p^- \leq p^+ < \infty$, and suppose that C^1 -functions are dense in $W^{1,p(\cdot)}(\Omega)$. Assume that $u, v \in ACC^{1,p(\cdot)}(\Omega)$ and $u = v$ almost everywhere in Ω . Then*

$$\text{cap}_{p(\cdot)}(\{x \in \Omega : u(x) \neq v(x)\}, \Omega) = 0.$$

Proof. Let U_i be an increasing sequence of open sets such that $\overline{U_i} \subset \Omega$ and $\bigcup_{i=1}^{\infty} U_i = \Omega$. Let $\phi_i \in C_0^\infty(\Omega)$ with $\phi_i = 1$ in U_i . Now $u\phi_i$ and $v\phi_i$ are in $W^{1,p(\cdot)}(\mathbb{R}^n)$ and thus, by an easy adaptation of Theorem 5.2,

$$C_{p(\cdot)}(\{x \in \mathbb{R}^n : u(x)\phi_i(x) - v(x)\phi_i(x) \neq 0\}) = 0.$$

Since $\text{spt } \phi_i \subset \Omega$ we find by Theorem 4.3 that

$$\text{cap}_{p(\cdot)}(\{x \in \Omega : u(x)\phi_i(x) - v(x)\phi_i(x) \neq 0\}, \Omega) = 0.$$

Hence the claim follows by subadditivity. \square

By $\Gamma(K, \Omega)$ we denote the family of all rectifiable curves in $\overline{\Omega}$ connecting K to $\partial\Omega$. It is well known that relative capacity of K can be characterized by the modulus of $\Gamma(K, \Omega)$. We show that this is true also in our case of variable exponents.

5.4. Theorem. *Let K be a compact set in a bounded open set $\Omega \subset \mathbb{R}^n$. Let $1 < p^- \leq p^+ < \infty$, and suppose that C^1 -functions are dense in $W^{1,p(\cdot)}(\Omega)$. Then*

$$M_{p(\cdot)}(\Gamma(K, \Omega)) \leq \text{cap}_{p(\cdot)}(K, \Omega) \leq n^{p^+/2} M_{p(\cdot)}(\Gamma(K, \Omega)).$$

Proof. Let $u \in R_{p(\cdot)}(K, \Omega)$. By definition we have $\text{spt } u \subset \Omega$. Let v be a Borel regularization of ∇u in Ω , i.e., v is Borel and $v = |\nabla u|$ almost everywhere. Let ϕ_i be a sequence of $C^1(\Omega)$ -functions converging to u . Since u is continuous, $\phi_i(x) \rightarrow u(x)$ for every $x \in \Omega$. Since v is Borel and $\nabla \phi$

is continuous, we see that $v - |\nabla\phi_i|$ is a Borel function. Thus we can use Fuglede's lemma [10, Lemma 2.2] to conclude that

$$\int_{\gamma[x,y]} |v| ds \geq |u(x) - u(y)| \geq 1$$

for $p(\cdot)$ -almost every curve in $\Gamma(K, \Omega)$ with endpoints x and y . We have thus shown that v is admissible for calculating $M_{p(\cdot)}(\Gamma(K, \Omega))$ and hence we have

$$\text{cap}_{p(\cdot)}(K, \Omega) \geq M_{p(\cdot)}(\Gamma(K, \Omega)).$$

To prove the opposite inequality assume that ρ is a non-negative Borel function such that

$$\int_{\gamma} \rho ds \geq 1$$

for every $\gamma \in \Gamma(K, \Omega)$. We may assume that $M_{p(\cdot)}(\Gamma(K, \Omega))$ is finite (otherwise there is nothing to prove) and that $\rho \in L^{p(\cdot)}(\Omega)$ with $\rho = 0$ in $\mathbb{R}^n \setminus \Omega$. We define

$$u(x) = \min \left\{ 1, \inf_{\gamma} \int_{\gamma} \rho ds \right\}$$

where the infimum is taken over all rectifiable curves γ connecting x to $\partial\Omega$. Now $u|_K = 1$ and ρ is an upper gradient of u , that is

$$|u(x) - u(y)| \leq \int_{\gamma} \rho ds$$

for every curve γ joining x to y in Ω . Since $0 \leq u(x) \leq 1$ and Ω is bounded, we have $u \in L^{p(\cdot)}(\Omega)$. These facts together imply that u is absolutely continuous on $p(\cdot)$ -almost every curve and that $|\nabla u(x)| \leq \sqrt{n}\rho(x)$ for almost every $x \in \Omega$; for more details about this, see [11]. This yields that u is a test function for $NC_{p(\cdot)}(E)$ and hence by Lemma 5.1 we obtain

$$\text{cap}_{p(\cdot)}(K, \Omega) \leq n^{p^+/2} M_{p(\cdot)}(\Gamma(K, \Omega)). \quad \square$$

6. EXCEPTIONAL SETS IN VARIABLE EXPONENT SOBOLEV SPACES

Let $E \subset \Omega$ be a relatively closed set of measure zero. By $W^{1,p(\cdot)}(\Omega \setminus E) = W^{1,p(\cdot)}(\Omega)$ we mean that the zero extension of every $u \in W^{1,p(\cdot)}(\Omega \setminus E)$ belongs to $W^{1,p(\cdot)}(\Omega)$. In fact, since $|E| = 0$, we could extend u from $\Omega \setminus E$ to Ω in an arbitrary way. The essential question is whether the function has a gradient in the larger space. Since $u \in W^{1,p(\cdot)}(\Omega \setminus E)$, we know that

$$\int_{\Omega} \phi \nabla u = \int_{\Omega} u \nabla \phi$$

for every $\phi \in C_0^\infty(\Omega \setminus E)$. In order for u to have a gradient also in Ω we need this equation to hold also for $\phi \in C_0^\infty(\Omega)$.

Let $1 < p^- \leq p^+ < \infty$. It is known that the zero boundary value Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega \setminus E)$ coincide if and only if E is of Sobolev $p(\cdot)$ -capacity zero, see [7, Theorem 3.9]. Therefore it follows from Theorems 4.2 and 4.3:

6.1. Corollary. *Let $1 < p^- \leq p^+ < \infty$, and suppose that continuous functions are dense in $W^{1,p(\cdot)}(\Omega)$. Suppose that $E \subset \Omega$ is a relatively closed set of measure zero. Then $Q_0^{1,p(\cdot)}(\Omega \setminus E) = Q_0^{1,p(\cdot)}(\Omega)$ if and only if $\text{cap}_{p(\cdot)}(E, \Omega) = 0$.*

We consider the problem of removability in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ without the zero boundary value assumption. The proof given in terms of the Sobolev $p(\cdot)$ -capacity follows the proof in the fixed-exponent case given in [15, Chapter 2].

6.2. Theorem. *Let $1 < p^- \leq p^+ < \infty$ in \mathbb{R}^n . Suppose that $E \subset \Omega$ is a relatively closed set. If E is of Sobolev $p(\cdot)$ -capacity zero, then*

$$W^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega \setminus E).$$

Proof. Let $C_{p(\cdot)}(E) = 0$ and let $u \in W^{1,p(\cdot)}(\Omega \setminus E)$. Assume first that u is bounded. Choose a sequence (v_j) of functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$, $0 \leq v_j \leq 1$, such that $v_j = 1$ in an open neighborhood U_j of E , $j = 1, 2, \dots$, and $v_j \rightarrow 0$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Since u and $1 - v_j$ are bounded functions we find that $u_j = (1 - v_j)u \in W^{1,p(\cdot)}(\Omega \setminus E)$. Moreover $u_j = 0$ in a neighborhood of E , so it is clear that u_j can be extended by 0 to E and $u_j \in W^{1,p(\cdot)}(\Omega)$. We easily calculate that

$$\begin{aligned} \varrho_{p(\cdot)}(|\nabla(u_i - u_j)|) &= \int_{\Omega \setminus E} |\nabla((v_j - v_i)u)|^{p(x)} dx \\ &\leq \int_{\Omega \setminus E} (|v_j| + |v_i|)^{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega \setminus E} (|\nabla v_j| + |\nabla v_i|)^{p(x)} |u|^{p(x)} dx. \end{aligned}$$

Since u , p and all the functions v_j are bounded, we easily see that $C|\nabla u|^{p(x)}$ and $C \max_i \varrho_{p(\cdot)}(|\nabla v_i|)$ are majorants of the integrands which do not depend on i and j . Since $v_i(x) \rightarrow 0$ for almost every $x \in \mathbb{R}^n$, this implies, by Lebesgue's dominated convergence theorem, that $\varrho_{p(\cdot)}(|\nabla(u_i - u_j)|) \rightarrow 0$ as $i, j \rightarrow \infty$. The same holds for $\varrho_{p(\cdot)}(u_i - u_j)$, so (u_i) is a Cauchy sequence. Since $W^{1,p(\cdot)}(\Omega)$ is a Banach space, we see that the limit u of (u_i) lies in this space too.

The same Banach space argument also allows us to get the general case from the case of bounded functions proven above. \square

6.3. Remark. Removability of a relatively closed set $E \subset \Omega$ with measure zero can be characterized also in terms of relative $p(\cdot)$ -capacity, Newtonian $p(\cdot)$ -capacity, and the modulus of $\Gamma(E, \Omega)$. If any of the above mentioned quantities for E is equal to zero, then $C_{p(\cdot)}(E) = 0$ under some additional conditions on p , E and Ω , see Theorem 4.3, Lemma 5.1 and Theorem 5.4, and it follows from the previous theorem that $W^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega \setminus E)$. In fact, with respect to the relative $p(\cdot)$ -capacity of $E \subset \Omega$, the assumption $p^- > 1$ is not needed to prove that $W^{1,p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega \setminus E)$. This can be shown similarly to the proof of Theorem 6.2 if assumed that $E \subset \Omega$ is compact and $\text{cap}_{p(\cdot)}(E, \Omega) = 0$.

6.4. Corollary. *Let $1 < p^- \leq p^+ < \infty$, and suppose that continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Then $W^{1,p(\cdot)}(\Omega) = Q_0^{1,p(\cdot)}(\Omega)$ if and only if $\mathbb{R}^n \setminus \Omega$ has zero Sobolev $p(\cdot)$ -capacity.*

Proof. Suppose first that $C_{p(\cdot)}(\mathbb{R}^n \setminus \Omega) = 0$. Note that since continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, each Sobolev function from $W^{1,p(\cdot)}(\mathbb{R}^n)$ has a quasicontinuous representative [6, Theorem 5.2], and thus $W^{1,p(\cdot)}(\mathbb{R}^n) = Q_0^{1,p(\cdot)}(\mathbb{R}^n)$. Theorem 6.2 and Corollary 6.1 now yield

$$\begin{aligned} W^{1,p(\cdot)}(\Omega) &= W^{1,p(\cdot)}(\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega)) = W^{1,p(\cdot)}(\mathbb{R}^n) \\ &= Q_0^{1,p(\cdot)}(\mathbb{R}^n) = Q_0^{1,p(\cdot)}(\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega)) = Q_0^{1,p(\cdot)}(\Omega). \end{aligned}$$

Suppose then that $W^{1,p(\cdot)}(\Omega) = Q_0^{1,p(\cdot)}(\Omega)$. Let $F = (\mathbb{R}^n \setminus \Omega) \cap B(0, r)$ for a fixed $r > 0$. Since F is bounded, we can choose $u \in C_0^\infty(\mathbb{R}^n)$ such that $u|_F = 1$ and $u|_\Omega \in W^{1,p(\cdot)}(\Omega) = Q_0^{1,p(\cdot)}(\Omega)$. Thus there exists a quasicontinuous function $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $v = u$ almost everywhere in Ω and $v = 0$ quasieverywhere in $\mathbb{R}^n \setminus \Omega$. Since u is continuous and v is quasicontinuous, we find that $u(x) = v(x)$ for quasievery $x \in \Omega$. Let $\varepsilon > 0$. Choose $E \subset \mathbb{R}^n$ such that $v|_{\mathbb{R}^n \setminus E}$ is continuous, $\{x \in \Omega : v(x) \neq u(x)\} \subset E$ and $C_{p(\cdot)}(E) < \varepsilon$. Let $x_i \in \Omega \setminus E$ be points converging to $x \in F \setminus E$. Then $v(x_i) = u(x_i) \rightarrow u(x) = 1$. By the continuity of $v|_{\mathbb{R}^n \setminus E}$ this yields $v(x) = 1$. On the other hand, $v(x) = 0$ for quasievery $x \in F$. This gives $C_{p(\cdot)}(F \setminus E) = 0$, and hence we have

$$C_{p(\cdot)}(F) \leq C_{p(\cdot)}(F \setminus E) + C_{p(\cdot)}(F \cap E) < \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ we obtain $C_{p(\cdot)}(F) = 0$. The claim follows since

$$C_{p(\cdot)}(\mathbb{R}^n \setminus \Omega) \leq \sum_{i=1}^{\infty} C_{p(\cdot)}((\mathbb{R}^n \setminus \Omega) \cap B(0, i)) = 0. \quad \square$$

We omit the proof of the following corollary which is similar to the proof of [15, Lemma 2.46].

6.5. Corollary. *Let $1 < p^- \leq p^+ < \infty$. If $E \subset \mathbb{R}^n$ is a set of Sobolev $p(\cdot)$ -capacity zero, then $\mathbb{R}^n \setminus E$ is connected.*

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P. HARJULEHTO AND M. KOSKENOJA
 DEPARTMENT OF MATHEMATICS AND STATISTICS,
 P.O. BOX 68 (GUSTAF HÄLLSTRÖMIN KATU 2B),
 FIN-00014 UNIVERSITY OF HELSINKI, FINLAND

P. HÄSTÖ
 DEPARTMENT OF MATHEMATICAL SCIENCES,
 NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY,
 N-7491 TRONDHEIM, NORWAY

E-mail addresses:

`petteri.harjulehto@helsinki.fi`
`peter.hasto@helsinki.fi`
`mika.koskenoja@helsinki.fi`