Meaning and interpretation in mathematics

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Interpretation in non-Euclidean geometry
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Poincaré, “Les géométries non euclidiennes” (1891)

Let us construct a kind of dictionary by making a double series of terms written in two columns, and corresponding each to each, just as in ordinary dictionaries the words in two languages which have the same meaning correspond to one another. Let us now take Lobachevsky’s theorems and translate them by the aid of this dictionary, as we would translate a German text with the aid of a German-French dictionary. We shall then obtain the theorems of ordinary geometry.
Mutual interpretability

$T$ is interpretable in $T^*$ if there is a way of translating the primitives of $T$ into formulas of $T^*$ such that the induced map $\varphi \mapsto \varphi^*$ is such that

$$\text{if } T \text{ proves } \varphi, \text{ then } T^* \text{ proves } \varphi^*.$$ 

One then says that two theories are mutually interpretable if each interprets the other.

The interpretation serves as a dictionary for translating statements in one theory into statements of another, in such a way that provability is preserved.
Desargues’ theorem

If two triangles lying in the same plane are such that the lines connecting their corresponding vertices intersect at a point, then the intersections of their corresponding sides are collinear.
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Division rings and fields

A **division ring** is a set with two operations $+$, $\times$ such that

- $+$ is associative, commutative, has an identity $0$, and every element of the ring has a $+$-inverse;

- $\times$ is associative, is distributive over $+$, has an identity $1$, and every nonzero element of the ring has a $\times$-inverse.

A **field** is a ring with commutative $\times$.

Examples of division rings/fields: the rational numbers, the real numbers

Example of a division ring that is not a field: the quaternions
In the *Grundlagen der Geometrie* (1899), David Hilbert showed:

- Desargues’ Theorem holds in a projective plane iff that plane can be coordinatized by a division ring.
- Pappus’ Theorem holds in a projective plane iff that plane can be coordinatized by a field.

Hilbert showed, given a projective plane, how to construct its “algebra of segments”, and that if this plane satisfies Desargues’ / Pappus’ theorem, multiplication in this algebra is associative / commutative.

He also showed how to recover the relevant geometries from the relevant algebraic structures.
Paying attention to the proofs of these theorems, one sees quickly that:

- The division ring axioms are *mutually interpretable* (with parameters) with the axioms for Desarguesian projective planes.

- The field axioms are *mutually interpretable* (with parameters) with the axioms for Pappian projective planes.
Interpretation and meaning

Thus one theory can be interpretable in a quite different theory.

One theory may be geometrical, while the other is algebraic, for example.

An interpretation provides a dictionary for translating statements of a theory $T$ into statements of another theory $T^*$, in such a way that if a statement in the language of $T$ is provable in $T$, then its translation into the language of $T^*$ is provable in $T^*$.

This type of translation preserves provability. But ordinarily we look to dictionaries to preserve meaning.

Do interpretations preserve meaning?
Crispin Wright has argued that they do.

He notes that second-order Peano arithmetic is interpretable in second-order logic plus Hume’s Principle (Fregean arithmetic).

Thus one can express successor and zero and other such arithmetic concepts in Fregean arithmetic, in such a way that provability is preserved.

Crispin Wright, “Is Hume’s Principle Analytic?”, 1999
Well, I imagine it will be granted that to define the distinctively arithmetical concepts is so to define a range of expressions that the use thereby laid down for those expressions is indistinguishable from that of expressions which do indeed express those concepts. The interpretability of Peano arithmetic within Fregean arithmetic ensures that has already been accomplished as far as all pure arithmetical uses are concerned.
Wright’s thesis

Wright asserts that the *meanings* of expressions are *determined* by their *uses*, following Wittgenstein.

He also holds that the mathematical *uses* of such *expressions* are determined by their *inferential role* in mathematics.

Since interpretations preserve inferential role, they thus preserve meaning.

**Wright’s thesis.** If a theory \( T \) can be interpreted in another theory \( T^* \), then the translation in \( T^* \) of an expression \( \varphi \) of \( T \) has the same meaning as \( \varphi \) in \( T \).

Thus by Hilbert’s results, Wright’s thesis implies that statements concerning Desarguesian projective planes have the same meaning as their translations concerning division rings.
Roughly, a solution to a problem, or a proof of a theorem, is pure if it draws only on what is “close” or “intrinsic” to that problem or theorem.

Other common language: avoids what is “extrinsic”, “extraneous”, “distant”, “remote”, “alien” or “foreign” to the problem or theorem.

Such judgments of purity are usually made in the context of asserting purity as an ideal of proof: that pure proofs/solutions are valuable.
Newton on purity in geometry

Newton, *Universal Arithmetick*, 1707

Equations are Expressions of Arithmetical Computation, and properly have no Place in Geometry, except as far as Quantities truly Geometrical (that is, Lines, Surfaces, Solids, and Propositions) may be said to be some equal to others. Multiplications, Divisions, and such sort of Computations, are newly received into Geometry, and that unwarily, and contrary to the first Design of this Science.... Therefore these two Sciences ought not to be confounded. The Antients did so industriously distinguish them from one another, that they never introduced Arithmetical Terms into Geometry. And the moderns, by confounding both, have lost the Simplicity in which all the Elegancy of Geometry consists.
Let’s concentrate on one type of purity that is arguably at the heart of mathematical practice: *topical* purity.

**Hilbert, “Lectures on Euclidean Geometry”, 1898–1899**

Therefore we are for the first time in a position to put into practice a *critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve the *purity of method* [*die Reinheit der Methode*], i.e. to prove theorems if possible using means that are suggested by [*nahe gelegt*] the content [*Inhalt*] of the theorem.

A proof is “topically pure” if it draws only on what belongs to the *content* of the theorem it is proving, i.e. on what must be grasped and accepted in order to comprehend that theorem.
Wright’s thesis and the autonomy of mathematical domains

Wright’s thesis implies that statements concerning Desarguesian projective planes have the same meaning as their translations concerning division rings.

Thus purely geometric talk of projective planes and purely algebraic talk of division rings has the same meaning.

This goes against five hundred years of thinking in mathematics, where algebraic thinking and geometric thinking have been thought to be distinct, engendering different types of understanding.
Wright’s thesis and topical purity

If the semantic boundary between algebra and geometry is dissolved, then topical purity for algebra and geometry is also dissolved.

A topically pure proof of a theorem draws only on what belongs to the content of the theorem.

Wright’s thesis implies that what belongs to the content of a statement concerning Desarguesian projective planes is the same as what belongs to the content of its translation concerning division rings.

Thus a topically pure proof of a purely geometric theorem could draw as much on algebraic concepts as it does on geometric concepts.

This obliterates the traditional understanding of what purity in practice seeks to avoid.
Wright’s thesis versus understanding practice

Topical purity has been and remains today important to mathematical practice, where the boundaries between different domains remain settled.

Dissolving the semantic boundary between algebra and geometry would dissolve topical purity as a genuine constraint of mathematical practice.

This dissolving would thus impair our ability to understand mathematical practice.

That is too high a price to pay for a controversial semantic view like Wright’s thesis, whatever its other virtues.

Thus we reject Wright’s thesis.