Kant and Finitism
Crossing Worlds: Mathematical logic, philosophy, art:
For Juliette Kennedy

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The Problems

- Kant in *The Critique of Pure Reason*: How can there be synthetic *a priori* knowledge of geometric propositions “All S are P”?

- BT in “Finitism”: How can there be *finitist* knowledge of propositions like
  - $PA$ is consistent
  - or
  - $\forall x : \mathbb{N}.\phi(x)$

where $\phi(x)$ is quantifier-free number-theoretic formula, i.e. knowledge that does not presuppose infinite totalities. How can we understand as even meaningful the formula $\phi(x)$ which contains function symbols such as $+$ and $\times$ which, as normally understood, are infinite objects?
I will talk about a certain relationship of Kant’s conception of geometric knowledge critical philosophy to finitism in mathematics, as analyzed in my paper “Finitism” (*Journal of Philosophy*, 1981) and which underlies its argument that finitist number theory is precisely Skolem’s primitive recursive arithmetic, *PRA*.
‘Finitism’ here should not be understood in the historical sense, as the foundational stance of the Hilbert school in the 1920’s. There is an extensive literature from members of that school and, more recently, the 1062 page edition of Hilbert’s lectures in logic and proof theory, edited by William Ewald and Wilfried Sieg, to refer to. But there remain questions about how to understand this literature and I won’t talk about that now except to say that, in spite of his well-known claim to affinity with Kant, there seems to be little of Kant in what he actually claims to have derived from him.
Notice that I refer only to Kant’s conception of geometric reasoning. A number of commentators on Kant in relatively recent times have taken the view that Kant was concerned with the theory of whole numbers—that by “number”, i.e. “Zahl” he meant whole number and that by “arithmetic” he meant theory of whole numbers. But this is a misreading. I find in his writings absolutely nothing explicitly about the whole or natural numbers and their theory. He states explicitly that by number he means the real numbers, founded on the ancient theory of proportion.
But mathematics does not merely construct magnitudes (quanta), as in geometry, but also mere magnitude (quantitatem), as in algebra (Buchstabenrechnung), where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it choses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, etc., extraction of roots . . . . (The Critique of Pure Reason)
But to construct a concept means to exhibit a priori the intuition corresponding to it. For the construction of the concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of the concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. (The Critique of Pure Reason, A713/B741)
No image of a triangle would ever be adequate to the concept of it. For it would not attain the generality of the concept, which makes this valid for all triangles, right or acute, etc., but would always be limited to one part of this sphere.

What holds of the triangle of pure intuition holds a priori of all triangles because it holds for the pure intuition.
The additional and crucial ingredient of Kant’s theory involves the idea of

▶ The *construction* of an object *b* of type *T* *from* an object *a* of type *S*, such that (*a*, *b*) falls under a certain concept *R* (Think of Euclid, Book 1, Postulates 1-3), and

▶ The corresponding construction in pure intuition: from the pure object *X* of type *S*, we construct a pure object *f(X)* of type *T* such that (*X*, *f(X)*) is of type *R*.

and it is this latter that confirms *a priori* that from empirical objects *a* of type *S*, *b* = *f(a)* of type *T* can be constructed such that (*a*, *b*) is of type *R*. For Kant, the step from the proposition that (*a*, *b*) is of type *R* to the proposition that *a* falls under the concept *P* is analytic. Thus, the construction in pure intuition of *f(X)* from *X* such that (*X*, *f(X)*) is of type *R* establishes *a priori* the proposition “All *S* are *P*”. 
\[ \psi = \forall x : S \exists y : T \phi(x, y) \]

where \( \phi(x, y) \) contains no quantifiers and the \( x \) and \( y \) may be sequences of variables of different types (sorts). (Read \( x : S \) as “\( x \) of type or ‘sort’ \( S \)”.) So we can replace the axioms \( \psi \) by the corresponding

\[ \psi' = \forall x : S \phi(x, f(x)) \]

where the Skolem functions \( f \) express exactly the construction that Euclid postulated (or should have postulated). In Kant’s terms we have

\[ (X, f(X)) : R \]

.
Basically, Euclid’s theorems in plain geometry are also of this $\Pi_2$ form:

$$\forall x : S' \exists y : T \ \psi(x, g(x))$$

or with Skolem function constants

$$\forall x : S' \ \psi(x, g(x))$$

and again in Kant’s terms

$$(X, g(X)) : R'$$

where the function $g$ is built up from the functions $f$ given in the postulates.

In this way, the synthetic part of geometry for Kant is all in the possibility of the constructions $g$ such that $$(X, g(X)) : R'$$.

The rest is analytic in his sense: “The predicate concept is contained in the subject concept.”
If we turn to this ordinal conception of number, it is reasonable to take the concept $\mathbb{N}$ of a natural number to be simply that of a finite succession or iteration, so that a pure object of type $\mathbb{N}$, i.e. a *generic* natural number, $Y$, is simply an arbitrary finite iteration, which we may represent by

$$0, 1, 2, \ldots, Y$$

iterating the successor operation $x \mapsto x'$ starting with 0.
Now we can ask what constructions $f(X)$ follow specifically from the fact that we are dealing with a sequence $X$ of generic finite iterations. Obviously, we can construct the null iteration, 0, and the successor $Y'$ of the generic number $Y$. In particular, then, each natural number can be constructed.

This differs from the Kantian conception: the pure objects of geometric type $S$ do not live in the same space as the empirical objects of type $S$. 
Moreover, given the generic iteration $Y$ and the construction $X \mapsto f(X)$, where $X$ and $f(X)$ are of some type $S$, we can apply the iteration $Y$ represented by

$$0, 1, 2, \ldots, Y$$

to iterate the operation $X \mapsto g(X)$ starting with $X$:

$$X, g(X), g(g(X)), \ldots, g^Y(X)$$

defining the unique construction $f(X, Y) = g^Y(X)$. Thus, $f(X, Y)$ of type $S$ is uniquely defined by

$$f(X, 0) = X \quad f(X, Y') = g(f(X, Y)).$$
It is along just these lines that the analysis of finitist number theory in question goes. Actually, to found proof by mathematical induction we need to introduce dependent types

$$\Gamma(X) \implies f(X) = g(X)$$

where $\Gamma(X)$ is a conjunction of equations $f_i(X) = g_i(X)$. A proof of

$$f(\bar{n}) = g(\bar{n})$$

where $\bar{n}$ is a sequence of natural numbers is any computation leading to an identity $k = k$ and a proof of an implication is a construction of a proof of the consequent from one of the antecedent — I will spare you the further details. I’ll also spare you further discussion entirely.
THANK YOU!
and
HAPPY BIRTHDAY, JULIETTE!
The constructions built up by the means indicated along with some elementary constructions that are valid independently of the basic types involved, lead precisely to the system of \( PRA \), and the considerations I’ve just rehearsed were the argument that finitist number theory, in the sense that I am considering, is just \( PRA \).

Thus the Kantian pedigree of that analysis of the concept of finitist number theory seems quite distinguished. Moreover, the central role of iteration in this analysis of finitism also may have its historical source in Kant’s discussion of ‘number’ as the schema of magnitude. It would have been nice to have been able to strengthen the pedigree by attributing to Kant a conception of number theory according to which construction by finite iteration is implicit in the concept of an object of type \( \mathbb{N} \). But, alas, as I have argued, he didn’t explicitly consider number theory at all in his account of mathematical demonstration. But there is some reason to believe that he did draw attention to the fundamental role of finite iteration in