Embedding causal team languages into dependence logic

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In short

Barbero&Sandu (2018,2020) proposed a way of extending the semantics of causal modelling along the lines of team semantics. The languages they considered are easily seen as generalizations of propositional dependence logic.

In this talk we want to clarify the relationship of these languages with *first-order* dependence logic.

Accidental vs. causal dependencies

- In team semantics, one studies *accidental* dependencies, i.e. dependencies in a set of data. They tell us that, by *observing* some property of a set of variables **X**, we can (or cannot) infer something about other variables **Y**.

- In causal inference (Spirtes&al. 1993, Pearl 2000), one studies *causal* dependencies; they tell us how the values of **Y** are affected if we *manipulate* **X**. They are expressed in terms of *interventionist counterfactuals* ("If I fixed the value of **X** to **x**, then Y would take value y") on structures called *causal models*.

Deterministic causal models

A deterministic causal model is a pair $(u, \mathcal{F})$, where:

- $\mathcal{F}$ is a system of "structural equations", e.g.:

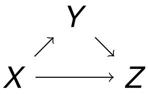
$$\begin{cases} Y := X \\ Z := X + Y \end{cases}$$

  (the variables on the left-hand side are called **endogenous**; the remaining ones **exogenous**)

- u is an assignment to the exogenous variables.

Causal graphs

The structural equations $Y := X, Z := X + Y$ induce a graph:



$PA_Z = \{X, Y\}$ are the *parents* of Z
$PA_Y = \{X\}$ is the unique parent of Y.

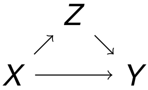
Models with **acyclic** graphs are given causal interpretation. They are called **recursive** models.

If the model is recursive, $(u, \mathcal{F})$ uniquely determine the values of all variables in the model.

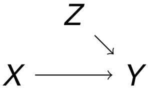
Intervention $do(Z = 3)$

Replace $\begin{cases} Z := X \\ Y := X + Z \end{cases}$ with $\begin{cases} Z := 3 \\ Y := X + Z \end{cases}$

Correspondingly, graph



is turned into



Given e.g. $u(X) = 2$, the counterfactual $Z = 3 \Box\!\!\rightarrow Y = 5$ is true (in the initial model).

Causal team semantics

1) Deterministic causal models describe interventionist counterfactuals in a context of *certainty* (unique assignment, unique system of equations).

2) *Semideterministic* causal models replace the assignment with a probability distribution over the exogenous variables. They model counterfactuals in a context of *quantitative* uncertainty.

$\frac{1}{2}$) But there is an intermediate step: *qualitative uncertainty*. This can be described using a multiplicity of assignments. This step has been proposed in Barbero&Sandu (2018,2020): *causal team semantics*.

(One can also consider uncertainty *over the system of equations*. We won't consider this possibility today.)

Signatures

A **signature** σ is a pair (Dom, Ran) where:

- Dom is a nonempty set of *variables* $(X, Y, Z ...)$
- Ran is a function which assigns to each $V \in Dom$ a nonempty range of possible values v, v', v''...

A **team of signature** σ is a set of assignments s over Dom such that, for all $X \in Dom$, $s(X) \in Ran(X)$.

Causal teams

A **causal team** of signature $\sigma = (Dom, Ran)$ is a triple $T = (T^-, G_T, \mathcal{F}_T)$, where:

1. T^- is a team of signature σ.
2. $G_T = (Dom, E)$ is an irreflexive graph over the set of variables.
3. $\mathcal{F}_T$ is a function $\{(V_i, f_{V_i}) | V_i \in \mathbf{V}\}$ that assigns to each endogenous variable a $|PA_{V_i}|$-ary function $f_{V_i} : Ran(PA_{V_i}) \rightarrow Ran(V_i)$

...such that T^- and $\mathcal{F}_T$ are *compatible*, i.e.:

- if $s \in T^-$, for each variable Y: $s(Y) = f_Y(s(PA_Y))$.

We write $End(T)$ for the set of variables that are endogenous according to G_T (i.e. they have indegree 0).

Interventions on *recursive* causal teams

T as below, with $\mathcal{F}_T(Z)(4,1,2) = 3$ (implicit alphab. order).
Let us evaluate the intervention $do(X = 1)$:

T:

U	X→	Y→	Z
2	1	2	4
3	1	2	4
4	4	1	1

$\rightsquigarrow$

U	X→	Y→	Z
2	**1**	. . .	. . .
3	**1**	. . .	. . .
4	**1**	. . .	. . .

$\rightsquigarrow$

U	X→	Y→	Z
2	1	**2**	. . .
3	1	**2**	. . .
4	1	**2**	. . .

$\rightsquigarrow T_{X=1}$:

U	X→	Y→	Z
2	1	2	**4**
3	1	2	**4**
4	1	2	**3**

$\mathcal{F}_{T_{X=x}}$ ($= \mathcal{F}_T$ in this example) is $\mathcal{F}_T$ restricted to $End(T) \setminus \{X\}$.
$G_{T_{X=x}}$($= G_T$) is obtained from G_T by removing all arrows that enter into X.

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Semantics of interventionist counterfactuals

We write $\mathbf{X} = \mathbf{x}$ as an abbreviation for a conjunction $X_1 = x_1 \wedge \cdots \wedge X_n = x_n$.

If the antecedent is consistent, we define

$$T \models \mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \psi \quad \text{iff} \quad T_{\mathbf{X}=\mathbf{x}} \models \psi.$$

If the antecedent is inconsistent, we stipulate $T \models \mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \psi$.

Causal subteams

$S = (S^-, G_S, \mathcal{F}_S)$ is a **causal subteam** of $T = (T^-, G_T, \mathcal{F}_T)$ if $S^- \subseteq T^-$, $G_S = G_T$ and $\mathcal{F}_S = \mathcal{F}_T$.

- $T \models \psi \vee \chi$ iff there are S_1, S_2 causal subteams of T such that $S_1^- \cup S_2^- = T^-$, $S_1 \models \psi$ and $S_2 \models \chi$.

Given a flat formula α, let T^α be the unique causal subteam of T with $(T^\alpha)^- := \{s \in T^- \mid (\{s\}, G_T, \mathcal{F}_T) \models \alpha\}$.

- $T \models \alpha \supset \chi$ iff $T^\alpha \models \chi$.

Other semantical clauses

- $T \models Y = y$ if, for all $s \in T^-$, $s(Y) = y$.
- $T \models Y \neq y$ if, for all $s \in T^-$, $s(Y) \neq y$.
- $T \models \psi \wedge \chi$ if $T \models \psi$ and $T \models \chi$.
- $T \models \psi \sqcup \chi$ iff $T \models \psi$ or $T \models \chi$.
- $T \models =(\mathbf{X}; Y)$ iff for all assignments $s, s' \in T$, $s(\mathbf{X}) = s'(\mathbf{X})$ implies $s(Y) = s'(Y)$.

Summary of the syntax

- $\mathcal{CO}(\sigma)$:

$$\alpha ::= X = x \mid X \neq x \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \supset \alpha \mid \mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \alpha$$

- $\mathcal{COD}(\sigma)$:

$$\varphi ::= X = x \mid X \neq x \mid \; =\!(\mathbf{X}; Y) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \alpha \supset \psi \mid \mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \varphi$$

- $\mathcal{CO}_{\sqcup}(\sigma)$:

$$\varphi ::= X = x \mid X \neq x \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \sqcup \varphi \mid \alpha \supset \psi \mid \mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \varphi$$

$\mathcal{CO}$ has been completely axiomatized in Barbero&Sandu(2020). $\mathcal{COD}, \mathcal{CO}_{\sqcup}$ were given complete natural deduction calculi in Barbero&Yang(2020).

The latter paper also characterizes the expressive power (i.e. definability of classes of causal teams) of the languages.

Statement of the problem

Are the languages $\mathcal{CO}, \mathcal{COD}, \mathcal{CO}_{\sqcup}$ translatable into fragments of first-order logic, or of dependence logic? What do such fragment look like?

Idea of the embedding

1) To each causal team $T = (T^-, G_T, \mathcal{F}_T)$ of signature $\sigma = (Dom, Ran)$ we associate a first-order structure

$$M_T = (M_\sigma, (c^{M_T})_{c \in M_\sigma}, (f_V^{M_T})_{V \in End(G_T)})$$

where:

- $M_\sigma = \bigcup_{V \in Dom} Ran(V)$
- $f_V^{M_T}(\mathbf{c}) = \begin{cases} \mathcal{F}_T(V)(\mathbf{c}) & \text{if } \mathbf{c} \in Ran(PA_V) \\ \text{some } d \in Ran(V) & \text{otherwise.} \end{cases}$.

2) For every graph G we define a translation $tr(\cdot, G)$ such that:

$$T \models \varphi \iff M_T, T^- \models tr(\varphi, G).$$

The translation (recursive case), 1

- $tr(\eta, G) := \eta$ if η is $X = x$, $X \neq x$ or $=(\mathbf{X}; Y)$.
- $tr(\psi \circ \chi, G) := tr(\psi, G) \circ tr(\chi, G)$ for $\circ = \wedge, \vee$ or $\sqcup$
- $tr(\theta \supset \chi, G) := tr(\theta^d, G) \vee tr(\chi, G)$.

...where α^d is the *dual* of the flat formula α, defined as usual – with additional clauses:

- $(\beta \supset \gamma)^d := \beta \wedge \gamma^d$

- $(\mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \gamma)^d := \mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \gamma^d$.

The translation (recursive case), 2

- For any graph G, we write:

$$Eq(G) := \bigwedge_{V \in End(G)} V = f_V(PA_V).$$

- Write $\mathbf{D}^G_{\mathbf{X}}$ for the set of strict descendants of $\mathbf{X}$ in graph G.

- $tr(\mathbf{X} = \mathbf{x} \,\Box\!\!\rightarrow \psi, G) := \exists \mathbf{X} \exists \mathbf{D}^G_{\mathbf{X}} (\mathbf{X} = \mathbf{x} \wedge Eq(G_{\mathbf{X}=\mathbf{x}}) \wedge tr(\psi, G_{\mathbf{X}=\mathbf{x}}))$

Embedding results (recursive case)

Theorem

Let $T = (T^-, G, \mathcal{F})$ be a recursive causal team. Then:

$$T \models \varphi \iff M_T, T^- \models tr(\varphi, G).$$

Corollary

a) $\mathcal{CO}$ embeds into the existential fragment of FO.

b, c) $\mathcal{COD}$ and $\mathcal{CO}_{\sqcup}$ embed into the existential fragment of dependence logic.

For the $\mathcal{CO}_{\sqcup}$ case, occurrences of $\sqcup$ are removed by the equivalence:

$$\psi \sqcup \chi \equiv \exists P \exists Q (=(P) \land =(Q) \land (P = Q \to \psi) \land (P \neq Q \to \chi))$$

On satisfiability (1)

The embeddings have an interesting consequence for the satisfiability problem of the causal languages over each *finite* domain of variables.

Corollary

Let Dom be a finite set of causal variables, and let φ be any $\mathcal{COD}$ or $\mathcal{CO}_{\sqcup}$ formula over the variables of Dom. Then it is decidable whether φ is satisfiable in some causal team over Dom.

It is proved by reducing the satisfiability of such formulas to the satisfiability of sentences of existential first-order logic.

On satisfiability (2): proof sketch

φ satisfied by a nonempty causal team $T \Rightarrow$ (by embedding and downward closure) $\forall s \in T^-$: $M_T, \{s\} \models Eq(G_T) \wedge tr(\varphi, G_T)$.

"Conversely": if $M, \{s\} \models Eq(G_T) \wedge tr(\varphi, G_T)$ (s over *Dom*) then the causal team obtained from T by replacing T^- with $\{s\}$ satisfies φ.
By the definition of $Eq(G_T)$, $\{s\}$ is compatible with G_T.

Thus, there is a causal team with graph G_T that satisfies $\varphi \iff \exists \mathbf{Z}(Eq(G_T) \wedge tr(\varphi, G_T))$ (sentence of existential $\mathcal{DL}$) satisfiable.

Thus, φ is satisfiable by some causal team of domain *Dom* $\iff \bigvee_{G_T} \exists \mathbf{Z}(Eq(G_T) \wedge tr(\varphi, G_T))$ (for G_T ranging over the acyclic graphs with nodes in *Dom*) is satisfiable.

But a sentence of existential $\mathcal{DL}$ is equivalent to its first-order *flattening*, which is obtained by the mechanical procedure of removing all dependence atoms. So its satisfiability reduces to satisfiability in existential first-order logic, which is decidable (Bernays&Schonfinkel 1928).

Future related work

- Work in progress: translations for the general (non-recursive) case. It seems that in this case we obtain embeddings into the Bernays-Schönfinkel ($\exists^*\forall^*$) fragment of dependence logic, and similar decidability results.
- It is natural to add *might-counterfactals* $\mathbf{X} = \mathbf{x} \Diamond\!\!\rightarrow \psi$, asserting that after the intervention *do*($\mathbf{X} = \mathbf{x}$), some nonempty causal subteam satisfies ψ. Find translations into e.g. dependence logic with possibility operator?
- Find sharper characterizations of the target fragments of dependence logic, and complexity bounds.

Thank you!