First order inquisitive logics of finite width

Gianluca Grilletti

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ILLC, University of Amsterdam,
Amsterdam, The Netherlands

Who is coming to the party?

Will the people coming to the party bring a present?

If John comes to the party, will someone bring a present?

Who is coming to the party?

Will the people coming to the party bring a present?

If John comes to the party, will someone bring a present?

Every attendee will bring a present.

Is someone coming to the party?

determines

Is someone bringing a present?

Overview

First-order inquisitive logics of finite width

Sano's problems

First-order inquisitive logics of finite width

Syntax: introducing questions

$$\varphi ::= \bot \mid P(x) \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \overline{\exists} x.\varphi$$

where P is a unary predicate.

Shorthands:

$\neg\varphi := \varphi \to \bot$ $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ $\exists x.\varphi := \neg\forall x.\neg\varphi$

A formula is called *classical* if it doesn't contain $\mathbin{\vee\!\!\!\vee}$ and $\overline{\exists}$.

φ, ψ, $\theta\ldots$: generic formulas

α, β, $\gamma\ldots$: classical formulas

p, q, $r\ldots$: atomic formulas

Intuition

Classical formulas represent statements.

$$\forall x.P(x) \quad \equiv \quad \textit{“Every element has property } P\textit{.”}$$

The operator $\mathbin{\rotatebox[origin=c]{0}{$\vee\!\!\!\vee$}}$ introduces *alternative questions*.

$$P(x) \mathbin{\vee\!\!\!\vee} P(y) \quad \equiv \quad \textit{“Does } x \textit{ have property } P \textit{ or does } y \textit{ have property } P\textit{?”}$$

The operator $\overline{\exists}$ introduces *existential questions*.

$$\overline{\exists} x.P(x) \quad \equiv \quad \textit{“What is an element with property } P\textit{?”}$$

Models: representing information

Information model

$$\mathcal{M} = \{ M_w \mid w \in W \}$$

- W is a non-empty set (the *set of worlds*).
- M_w for $w \in W$ are (classical) first order modes over a fixed domain D (the *domain* of $\mathcal{M}$).

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w_0	w_1	w_2

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	w_0	w_1	w_2
a	■	■	■
b	■	■	■

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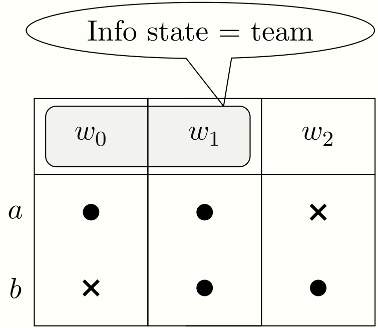
	w_0	w_1	w_2
a	●	●	×
b	×	●	●

A piece of information is encoded by selecting the *set of worlds* where the piece of information is true.



"a has property P"

"If b has property P, so does a"

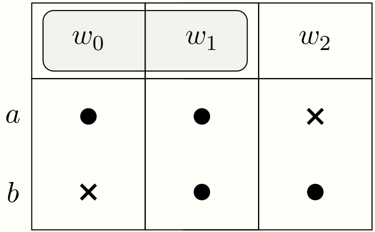
Semantics: support relation

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

s **supports** α iff s entails the information conveyed by α.

$$\{w_0, w_1\} \models P(a) \qquad \{w_0, w_1, w_2\} \not\models P(a)$$

Semantics: support relation



s **supports** α iff s entails the information conveyed by α.

$$\{w_0, w_1\} \vDash P(a) \qquad \{w_0, w_1, w_2\} \nvDash P(a)$$

s **supports** φ iff s solves the issue raised by φ.

$$\{w_0, w_1\} \vDash P(a) \mathbin{\vee\!\!\!\vee} \neg P(a) \qquad \{w_0, w_1, w_2\} \nvDash P(a) \mathbin{\vee\!\!\!\vee} \neg P(a)$$

Semantics: support relation

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

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φ **supports** ψ iff every s that supports φ, supports also ψ.

$$P(a) \vee\!\!\!\vee \neg P(a) \not\models P(b) \vee\!\!\!\vee \neg P(b)$$

$\mathcal{M} \rightsquigarrow$ info structure $\quad s \rightsquigarrow$ info state $\quad g \rightsquigarrow$ assignment

$$\mathcal{M}, s \vDash_g \varphi$$

$$
\begin{array}{lll}
\mathcal{M}, s \vDash_g \bot & \text{iff} & s = \emptyset \\
\mathcal{M}, s \vDash_g P(x) & \text{iff} & \forall w \in s.\ g(x) \in P_w \\
\mathcal{M}, s \vDash_g \varphi \wedge \psi & \text{iff} & \mathcal{M}, s \vDash_g \varphi \text{ and } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \varphi \rightarrow \psi & \text{iff} & \forall t \subseteq s.\ [\ \mathcal{M}, t \vDash_g \varphi \Rightarrow \mathcal{M}, t \vDash_g \psi\ ] \\
\mathcal{M}, s \vDash_g \forall x.\varphi & \text{iff} & \forall d \in D.\ \mathcal{M}, s \vDash_{g[x \mapsto d]} \varphi \\
\mathcal{M}, s \vDash_g \varphi \vee\!\!\!\vee \psi & \text{iff} & \mathcal{M}, s \vDash_g \varphi \text{ or } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \overline{\exists} x.\varphi & \text{iff} & \exists d \in D.\ \mathcal{M}, s \vDash_{g[x \mapsto d]} \varphi
\end{array}
$$

$\mathcal{M} \rightsquigarrow$ info structure $\qquad s \rightsquigarrow$ info state $\qquad g \rightsquigarrow$ assignment

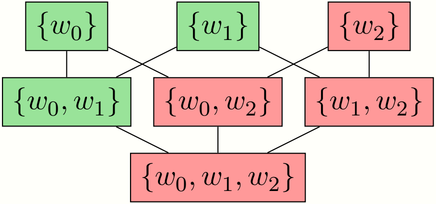
$$\mathcal{M}, s \vDash_g \varphi$$

Inquisitive logic	**Team logic**
$\wedge$	Conjunction $\wedge$
$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$	Boolean disjunction $\sqcup$
$\rightarrow$	Intuitionistic impl. $\rightarrow$
$\overline{\exists}$	$\exists^1$
$\forall$	$\forall^1$

$$\mathcal{M}, s \vDash_g P(x) \quad \text{iff} \quad \forall w \in s.\ g(x) \in P_w$$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

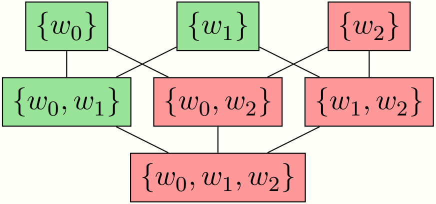
$$\mathcal{M}, s \overset{?}{\vDash} P(a)$$



$$\mathcal{M}, s \models_g P(x) \qquad \text{iff} \qquad \forall w \in s.\ g(x) \in P_w$$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$$\mathcal{M}, s \overset{?}{\models} P(a)$$



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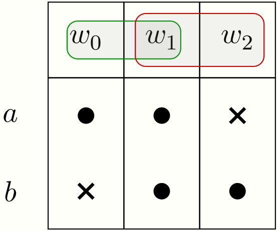

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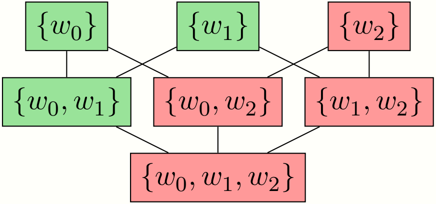
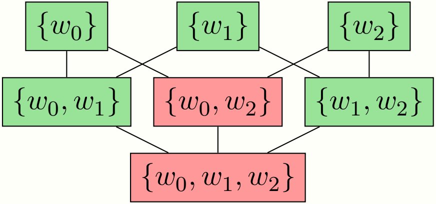


$$\mathcal{M}, s \vDash_g \varphi \vee\!\!\vee \psi \quad \text{iff} \quad \mathcal{M}, s \vDash_g \varphi \quad \text{or} \quad \mathcal{M}, s \vDash_g \psi$$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

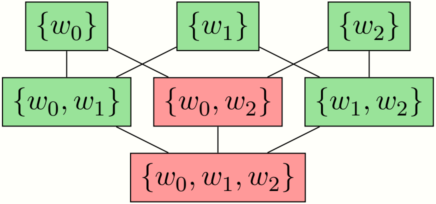
$$\mathcal{M}, s \overset{?}{\vDash} P(a) \vee\!\!\vee P(b)$$



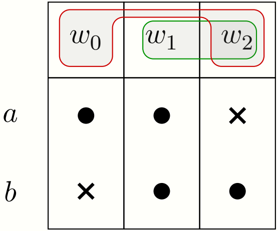
$$\mathcal{M}, s \vDash_g \varphi \mathbin{\mathbb{V}} \psi \quad \text{iff} \quad \mathcal{M}, s \vDash_g \varphi \quad \text{or} \quad \mathcal{M}, s \vDash_g \psi$$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

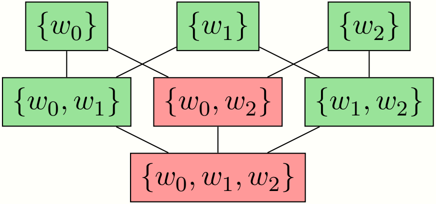
$$\mathcal{M}, s \overset{?}{\vDash} P(a) \mathbin{\mathbb{V}} P(b)$$



$$\mathcal{M}, s \models_g \varphi \Vee \psi \quad \text{iff} \quad \mathcal{M}, s \models_g \varphi \quad \text{or} \quad \mathcal{M}, s \models_g \psi$$



$$\mathcal{M}, s \overset{?}{\models} P(a) \Vee P(b)$$



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a	●	●	×
b	×	●	●

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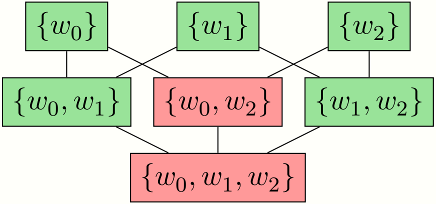

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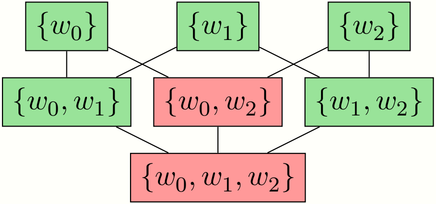

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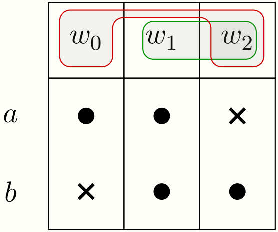

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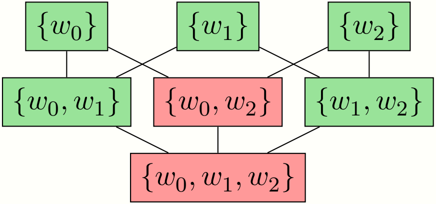

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a	●	●	×
b	×	●	●

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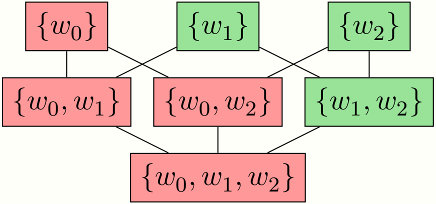

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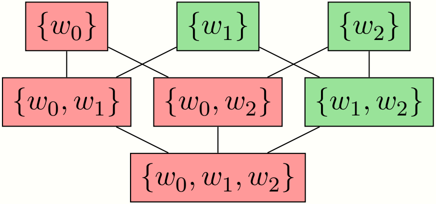

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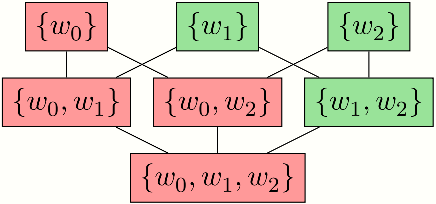

First order inquisitive logic

Definition (Ciardelli, '10): First order inquisitive logic `InqBQ` is the logic of information models.

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Lemma: The semantics of a formula φ is downward closed.

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Lemma: $\vDash$ restricted to *classical formulas* is first order entailment $\vDash^{\texttt{CQC}}$.

$$\Gamma \vDash \alpha \qquad \text{iff} \qquad \Gamma \vDash^{\texttt{CQC}} \alpha$$

Inquisitive logics of finite width

$$\mathbb{M}_n := \{ \mathcal{M} \mid \#W \leq n \}$$

	w_0	w_1
a	●	●
b	×	●

$\in \mathbb{M}_2$

$\notin \mathbb{M}_3$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\in \mathbb{M}_3$

$\notin \mathbb{M}_4$

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a	●	●	×
b	×	●	●

$\in \mathbb{M}_3$

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Definition (Sano, '11): $\mathtt{InqBQ}_n := \mathrm{Log}(\mathbb{M}_n)$.

- $\mathtt{InqBQ}_n \not\supseteq \mathtt{InqBQ}_{n+1}$;
- Axiomatization for $\mathtt{InqBQ}_2$.

$$\color{red}{\mathtt{InqBQ}_2} \; \supsetneq \; \mathtt{InqBQ}_3 \; \supsetneq \; \dots \; \supsetneq \; \bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n$$

Open questions by Sano

[Q1] Whether $\bigcap_{n\in\mathbb{N}} \texttt{InqBQ}_n = \texttt{InqBQ}$.

[Q2] Whether $\texttt{InqBQ}_n$ is axiomatizable for every $n \in \mathbb{N}$.

Open questions by Sano

[Q1] Whether $\bigcap_{n \in \mathbb{N}} \mathtt{InqBQ}_n = \mathtt{InqBQ}$.

[Q2] Whether $\mathtt{InqBQ}_n$ is axiomatizable for every $n \in \mathbb{N}$.

We are going to answer these two questions.

Sano's problems

The classes $\mathbb{M}_n$ are definable

$$\begin{array}{lcl} \mathrm{C}_1 & := & \forall x. \left(P(x) \vee\!\!\vee \neg P(x) \right) \\ \mathrm{C}_n & := & \overline{\exists} x. \bigvee\!\!\!\bigvee_{k=1}^{n-1} \Big(\left(P(x) \to \mathrm{C}_k \right) \wedge \left(\neg P(x) \to \mathrm{C}_{n-k} \right) \Big) \end{array}$$

Lemma (G.): C_n *defines the class* $\mathbb{M}_n$.

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\begin{aligned}
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\end{aligned}
$$

Lemma (G.): C_n *defines the class* $\mathbb{M}_n$.

	w_0	w_1
a	●	●
b	×	●

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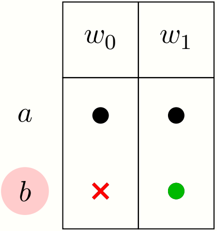

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Lemma (G.): C_n *defines the class* $\mathbb{M}_n$.

	w_0	w_1	w_2
a	●	●	×
b	×	●	●
c	●	●	●

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$$
\begin{aligned}
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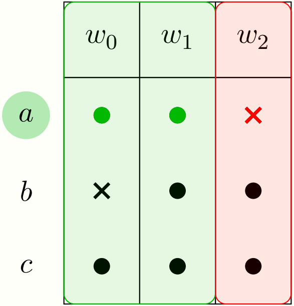

Q1: Are $\bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n$ and $\mathtt{InqBQ}$ the same logic?

Notice: $\bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n$ is the logic of $\mathbb{M}_{<\aleph_0} := \bigcup_{n\in\mathbb{N}} \mathbb{M}_n$.

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Strategy: we find a formula $\theta \to \chi$ such that

1. θ defines a special class of models $\mathbb{C}$;
2. χ is supported by every *finite* model in $\mathbb{C}$;
3. χ is *not* supported by every model in $\mathbb{C}$.

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$$(1+2) \quad \Longrightarrow \quad \theta \to \chi \in \bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n$$
$$(1+3) \quad \Longrightarrow \quad \theta \to \chi \notin \mathtt{InqBQ}$$

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$$(1+3) \quad \Longrightarrow \quad \theta \to \chi \notin \mathtt{InqBQ}$$

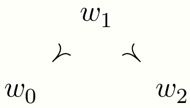
Definable class of models: P-chains

$$P_w := \{\ d \in D \mid \mathcal{M}, \{w\} \vDash P(d)\ \}$$

(the set of dots in the column w).

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$$w \preceq w' \quad \text{iff} \quad P_w \subseteq P_{w'}$$



Definable class of models: P-chains

We call $\mathcal{M}$ a P-*chain* iff $\preceq$ is a total order.

	w_0	w_1	w_2	w_3
a_0	×	●	●	●
a_1	×	×	●	●
a_2	×	×	●	●
a_3	×	×	×	●
a_4	×	×	×	×

$$w_0 \prec w_1 \prec w_2 \prec w_3$$

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	w_0	w_1	w_2	w_3
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a_1	×	×	●	●
a_2	×	×	●	●
a_3	×	×	×	●
a_4	×	×	×	×

$$w_0 \prec w_1 \prec w_2 \prec w_3$$

Claim: The class of P-chains is definable by a formula Pc.

$$\mathrm{Pc} := \forall x, y. \left[\, (P(x) \to P(y)) \Vee (P(y) \to P(x)) \,\right]$$

Question time!

What is a property true for every *finite* total order, but not true for every total order?

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What is a property true for every *finite* total order, but not true for every total order?

Possible answer: having a maximum.

Definable property: $\preceq$ has maximum

	w_0	w_1	w_2	$\cdots$	$w_{\max}$
a_0	×	●	●		●
a_1	×	×	●		●
a_2	×	×	●		●
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
a_n	×	×	×		●
a_{n+1}	×	×	×		×
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Definable property: $\preceq$ has maximum

	w_0	w_1	w_2	$\cdots$	$w_{\max}$
a_0	×	●	●		●
a_1	×	×	●		●
a_2	×	×	●		●
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
a_n	×	×	×		●
a_{n+1}	×	×	×		×
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Definable property: $\preceq$ has maximum

	w_0	w_1	w_2	$\cdots$	$w_{\max}$
a_0	×	●	●		●
a_1	×	×	●		●
a_2	×	×	●		●
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
a_n	×	×	×		●
a_{n+1}	×	×	×		×
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

$$\mathrm{MAX} := \overline{\exists} x.\ (\ P(x) \to \mathrm{C}_1\ )$$

Definable property: $\preceq$ has maximum

	w_0	w_1	w_2	$\cdots$
a_0	×	●	●	
a_1	×	×	●	
a_2	×	×	●	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
a_n	×	×	×	
a_{n+1}	×	×	×	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$\mathrm{MAX} \;:=\; \overline{\exists} x.\ (\ P(x) \to \mathrm{C}_1\ )$$

Definable property: $\preceq$ has maximum

	w_0	w_1	w_2	$\cdots$
a_0	×	●	●	
a_1	×	×	●	
a_2	×	×	●	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
a_n	×	×	×	
a_{n+1}	×	×	×	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$\mathrm{MAX} \;:=\; \overline{\exists} x.\ (\ P(x) \to \mathrm{C}_1\ )$$

Q1: Are $\bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n$ and $\mathtt{InqBQ}$ the same logic?

Lemma (G.): $\bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n \neq \mathtt{InqBQ}$. In particular:

$$\begin{array}{lcl} \mathrm{Pc} \to \mathrm{MAX} & \in & \bigcap_{n\in\mathbb{N}} \mathtt{InqBQ}_n \\ \mathrm{Pc} \to \mathrm{MAX} & \notin & \mathtt{InqBQ} \end{array}$$

Further considerations

Consideration:

`InqBQ` does not have the finite-model property.

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Take-home message:

Find interesting definable classes of models.

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Further considerations

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InqBQ does not have the finite-model property.

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Find interesting definable classes of models.

Open questions:

Are $\texttt{InqBQ}_{\aleph_0}$ and InqBQ the same logic?

For which λ cardinal the logic $\texttt{InqBQ}_{\lambda}$ is axiomatizable?

Q2: Are the logics $\mathtt{InqBQ}_n$ axiomatizable?

Sano axiomatized $\mathtt{InqBQ}_2$ adapting the *canonical model* technique from *intuitionistic logic with constant domain* CD.

- What is CD?
- What is the connection between $\mathtt{InqBQ}$ and CD?

The logic CD

CD is the logic of constant-domain intuitionistic Kripke models.

$$\mathcal{K} := \langle \qquad\qquad\qquad \rangle$$

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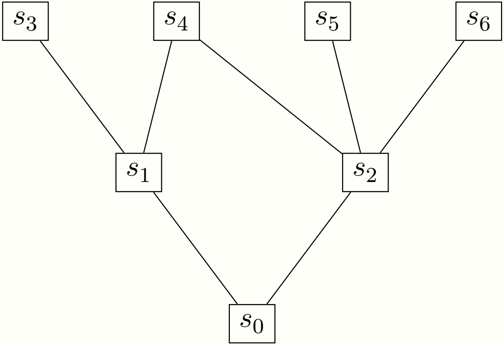
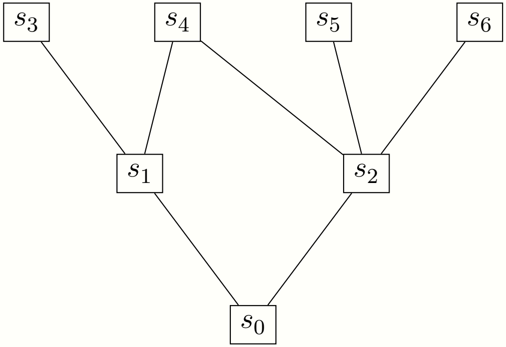
$$\mathcal{K} := \langle\, S,\ \leq \qquad\qquad \rangle$$

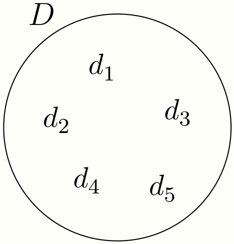


The logic CD

CD is the logic of constant-domain intuitionistic Kripke models.

$$\mathcal{K} := \langle\, S,\ \leq\ ,\ D \qquad \rangle$$

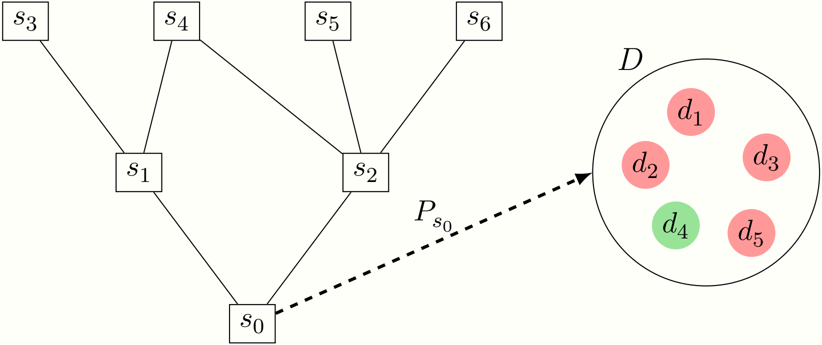




The logic CD

CD is the logic of constant-domain intuitionistic Kripke models.

$$\mathcal{K} := \langle\, S,\ \leq\ ,\ D,\ P_{\bullet}\,\rangle$$



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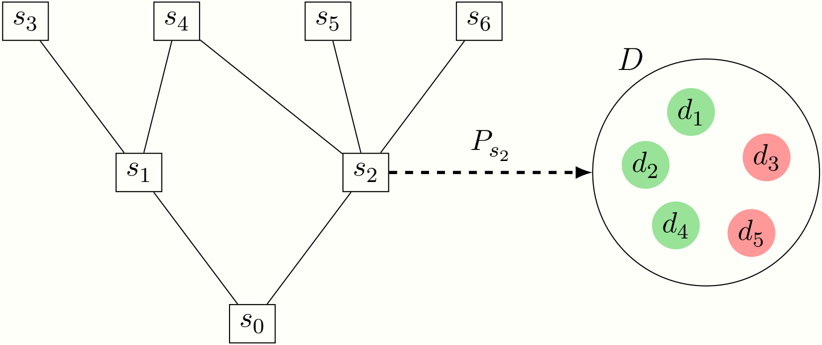

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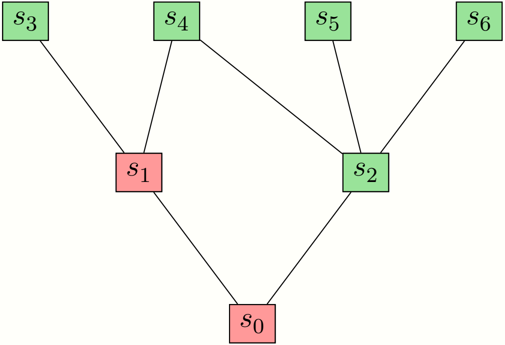
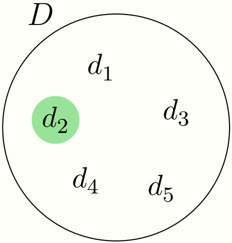
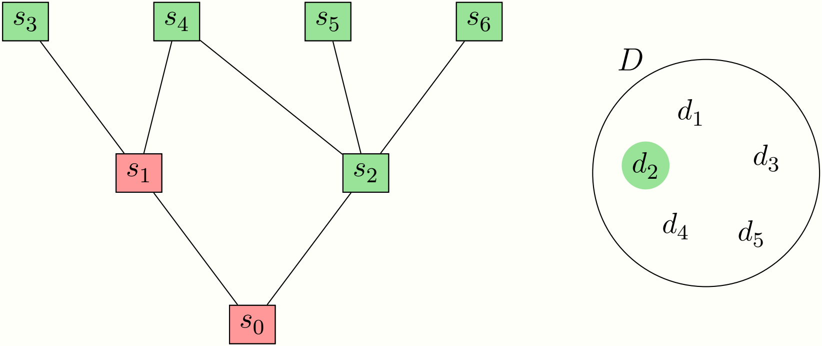




The logic CD

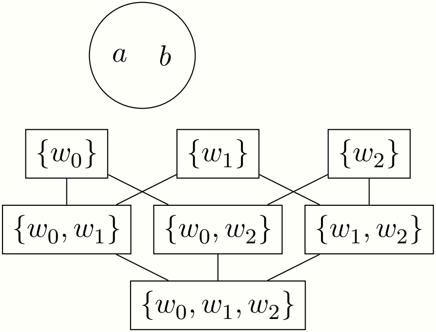
CD is the logic of CD-models.

$$\mathcal{K} := \langle\, S,\ \leq\,,\ D,\ P_{\bullet}\,\rangle$$

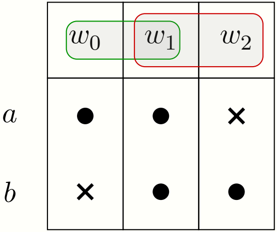


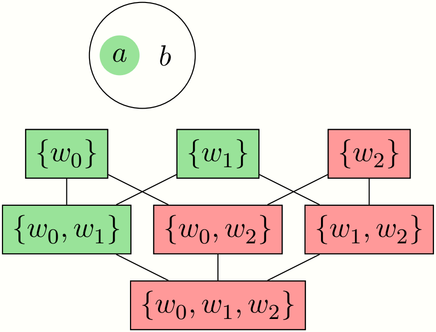
Connections between InqBQ and CD

	w_0	w_1	w_2
a	●	●	×
b	×	●	●



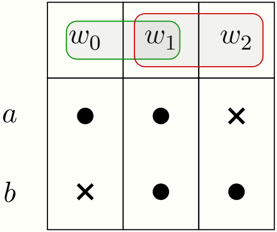
Connections between `InqBQ` and CD

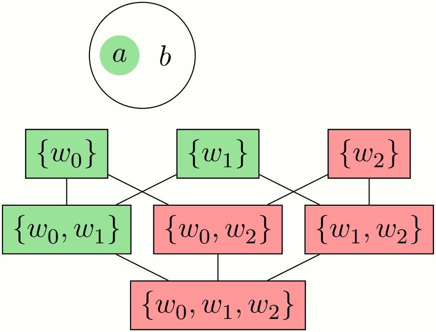




$$\mathcal{M}, s \vDash P(a) \qquad \Longleftrightarrow \qquad \mathcal{K}, s \Vdash P(a)$$

Connections between InqBQ and CD





$\mathcal{M}, s \vDash P(a)$ $\iff$ $\mathcal{K}, s \Vdash P(a)$

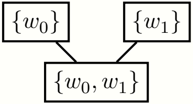
Inquisitive models $\overset{1:1}{\leftrightarrow}$ CD-models s.t.:

- $\langle S, \leq \rangle = \langle \mathcal{P}_0(W), \supseteq \rangle$
- $\neg\neg\alpha \equiv \alpha$ for α classical

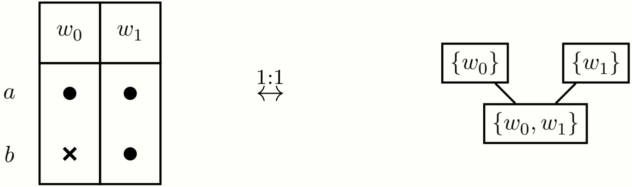
Sano's axiomatization for $\mathtt{InqBQ}_2$

	w_0	w_1
a	●	●
b	×	●

$\overset{1:1}{\leftrightarrow}$

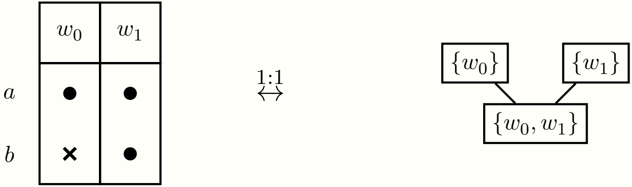


Sano's axiomatization for $\texttt{InqBQ}_2$



Axioms and rules of $\texttt{IQC}$

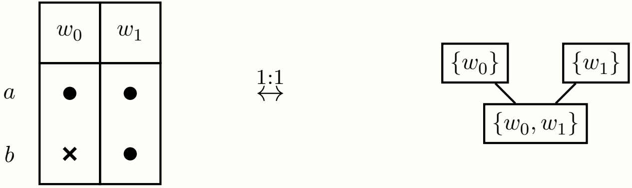
Sano's axiomatization for $\texttt{InqBQ}_2$



Axioms and rules of $\texttt{IQC}$

CD: constant domain

Sano's axiomatization for $\texttt{InqBQ}_2$

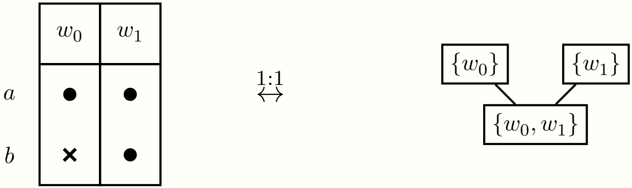


Axioms and rules of $\texttt{IQC}$

CD: constant domain

DNC: $\neg\neg\alpha \to \alpha$ for α classical

Sano's axiomatization for $\texttt{InqBQ}_2$



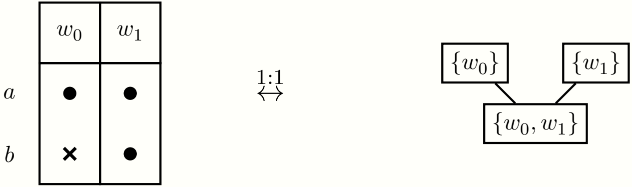
Axioms and rules of $\texttt{IQC}$

CD: constant domain

DNC: $\neg\neg\alpha \to \alpha$ for α classical

H2: height 2

Sano's axiomatization for $\texttt{InqBQ}_2$



Axioms and rules of $\texttt{IQC}$

CD: constant domain

DNC: $\neg\neg\alpha \to \alpha$ for α classical

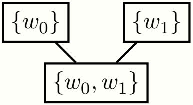
H2: height 2

W2: width 2

Sano's axiomatization for $\texttt{InqBQ}_2$

	w_0	w_1
a	●	●
b	×	●

$\overset{1:1}{\leftrightarrow}$



Axioms and rules of $\texttt{IQC}$

CD: $\forall x.(\varphi \vee\!\!\!\vee \psi) \to \varphi \vee\!\!\!\vee \forall x.\psi$ for x not free in φ

DNC: $\neg\neg\alpha \to \alpha$ for α classical

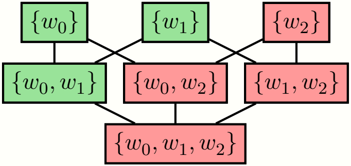
H2: $\varphi \vee\!\!\!\vee (\varphi \to \psi \vee\!\!\!\vee \neg\psi)$

W2: $(\varphi \to \psi) \vee\!\!\!\vee (\psi \to \varphi) \vee\!\!\!\vee ((\varphi \to \neg\psi) \wedge (\psi \to \neg\varphi))$

New axiomatization for $\texttt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

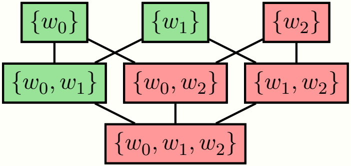
$\overset{1:1}{\leftrightarrow}$



New axiomatization for $\texttt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\overset{1:1}{\leftrightarrow}$

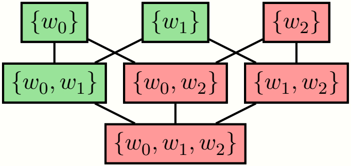


Axioms and rules of $\texttt{IQC}$

New axiomatization for $\texttt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\overset{1:1}{\leftrightarrow}$



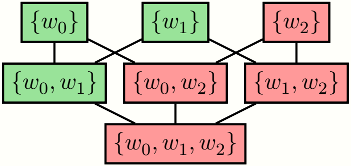
Axioms and rules of IQC

CD: constant domain

New axiomatization for $\texttt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\overset{1:1}{\leftrightarrow}$



Axioms and rules of $\texttt{IQC}$

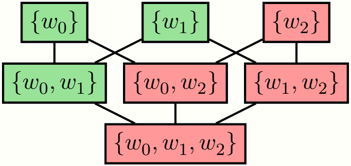
CD: constant domain

DNC: $\neg\neg\alpha \to \alpha$ for α classical

New axiomatization for $\texttt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\overset{1:1}{\leftrightarrow}$



Axioms and rules of $\texttt{IQC}$

CD: constant domain

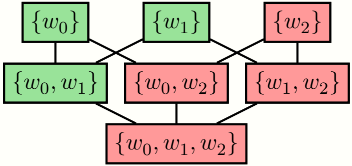
DNC: $\neg\neg\alpha \to \alpha$ for α classical

C_n: at most n endpoints

New axiomatization for $\mathtt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\overset{1:1}{\leftrightarrow}$



Axioms and rules of $\mathtt{IQC}$

CD: constant domain

DNC: $\neg\neg\alpha \rightarrow \alpha$ for α classical

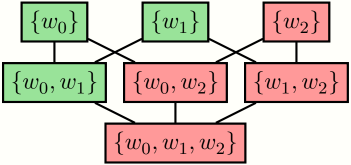
C_n: at most n endpoints

KP and UP: structure of $\mathcal{P}_0$

New axiomatization for $\texttt{InqBQ}_n$

	w_0	w_1	w_2
a	●	●	×
b	×	●	●

$\overset{1:1}{\leftrightarrow}$



Axioms and rules of $\texttt{IQC}$

CD: $\forall x.(\varphi \vvee \psi) \to \varphi \vvee \forall x.\psi$ for x not free in φ

DNC: $\neg\neg\alpha \to \alpha$ for α classical

C_n: $\overline{\exists} x.\ \mathbb{V}_{k=1}^{n-1} \big(\big(P(x) \to \mathrm{C}_k \big) \wedge \big(\neg P(x) \to \mathrm{C}_{n-k} \big) \big)$

KP: $(\neg\varphi \to \psi \vvee \chi) \to (\neg\varphi \to \psi) \vvee (\neg\varphi \to \chi)$

UP: $(\neg\varphi \to \overline{\exists} x.\psi) \to \overline{\exists} x.(\neg\varphi \to \psi)$ for x not free in φ

Q2: Are the logics $\texttt{InqBQ}_n$ axiomatizable?

Theorem (G.): The logics $\texttt{InqBQ}_n$ are axiomatizable.

Axioms and rules of $\texttt{IQC}$
CD: $\forall x.(\varphi \vee\!\!\!\vee \psi) \to \varphi \vee\!\!\!\vee \forall x.\psi$ for x not free in φ
DNC: $\neg\neg\alpha \to \alpha$ for α classical
C_n: $\overline{\exists} x.\ \bigvee\!\!\!\bigvee_{k=1}^{n-1} ((P(x) \to \mathrm{C}_k) \wedge (\neg P(x) \to \mathrm{C}_{n-k}))$
KP: $(\neg\varphi \to \psi \vee\!\!\!\vee \chi) \to (\neg\varphi \to \psi) \vee\!\!\!\vee (\neg\varphi \to \chi)$
UP: $(\neg\varphi \to \overline{\exists} x.\psi) \to \overline{\exists} x.(\neg\varphi \to \psi)$ for x not free in φ

Further considerations

Take-home message:

Different semantics, same logic.

Further considerations

Take-home message:
Different semantics, same logic.

Open questions:
Can we adapt the methodology to `InqBQ` and $\mathtt{InqBQ}_{<\aleph_0}$?

Lemma (G.): There are

- a faithful translation of *Medvedev's logic* `ML` in $\mathtt{InqBQ}_{<\aleph_0}$;
- a faithful translation of *Skvortsov's logic* $\mathtt{ML}_1$ in `InqBQ`.

Thank you for the attention!