Complexity of probabilistic inclusion logic and additive real arithmetics

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Qualitative vs. quantitative dependence

Team logics can reason only about qualitative (relational) dependencies.

What about quantitative (probabilistic) dependencies?

Qualitative:

Functional dependency $X \to Y$

Multivalued dependency $X \twoheadrightarrow Y$

Inclusion dependency $X \subseteq Y$

Quantitative:

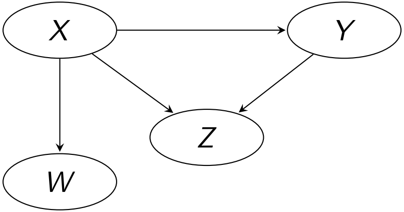
Marginal independence $X \perp\!\!\!\perp Y$

Conditional independence $X \perp\!\!\!\perp Y \mid Z$

Identical distribution of X and Y

Applications of quantitative atoms

- Sample $X_1, \ldots, X_n$ from a population distribution: $X_i \sim X_j$ and $X_i \perp\!\!\!\perp X_j$ for $i \neq j$
- Markov chain $(X_1, X_2, X_3, \ldots)$: $X_1 \ldots X_{i-1} \perp\!\!\!\perp_{X_i} X_{i+1}$ for i
- Bayesian network



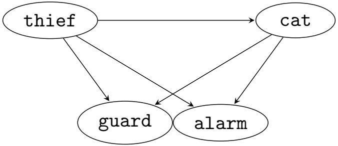
: $\quad Y \perp\!\!\!\perp_X W$ and $Z \perp\!\!\!\perp_{XY} W$

Probabilistic team semantics

Basic concepts:

- Probabilistic team = probability distribution on a finite team (FoIKS 2018)
- Quantitative atoms (e.g., conditional independence $\perp\!\!\!\perp_{\mathrm{c}}$, identical distribution $\approx$)
- $\{\forall, \exists, \wedge, \vee\}$ for complex probability statements

Example 2



guard

thief,cat	T	F
TT	0.8	0.2
TF	0.7	0.3
FT	0	1
FF	0	1

thief

T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

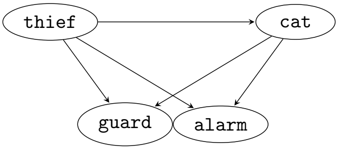
alarm

thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

From the Bayesian network above we obtain that the joint probability distribution for t, c, g, a can be factorized as

$$P(t, c, g, a) = P(t) \cdot P(c \mid t) \cdot P(g \mid t, c) \cdot P(a \mid t, c)$$

Example 2



thief	
T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

guard

thief,cat	T	F
TT	0.8	0.2
TF	0.7	0.3
FT	0	1
FF	0	1

alarm

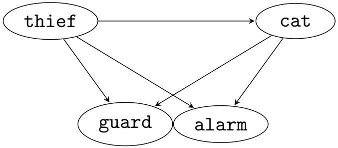
thief,cat	T	F
TT	0.8	0.2
TF	0.7	0.3
FT	0	1
FF	0	1

Given

$$\phi := tca \approx tcg$$

(i.e., conditioned on `thief` and `cat`, `alarm` and `guard` are identically distributed), then the conditional probability tables for `alarm` and `guard` are identical and one of them can be removed.

Example 2



	guard	
thief,cat	T	F
TT	0.45	0.55
TF	0.4	0.6
FT	0.05	0.95
FF	0	1

thief	
T	F
0.1	0.9

	cat	
thief	T	F
T	0.1	0.9
F	0.6	0.4

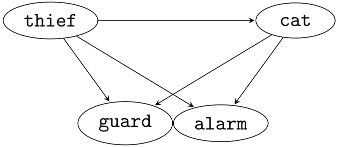
	alarm	
thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

Given

$$\phi := \exists x(tcg \approx tcx \wedge tcga \perp\!\!\!\perp y \wedge x = T \leftrightarrow ay = TT)$$

(i.e., guard is of a factor $P(y = T)$ less sensitive to raise alert than alarm for any given thief and cat), it suffices to store the conditional probability table for alarm and the probability $P(y = T)$.

Example 2



$P(Y = T) = 0.5$

thief

T	F
0.1	0.9

cat

thief	T	F
T	0.1	0.9
F	0.6	0.4

alarm

thief,cat	T	F
TT	0.9	0.1
TF	0.8	0.2
FT	0.1	0.9
FF	0	1

Given

$$\phi := \exists x(tcg \approx tcx \wedge tcga \perp\!\!\!\perp y \wedge x = T \leftrightarrow ay = TT)$$

(i.e., `guard` is of a factor $P(y = T)$ less sensitive to raise alert than `alarm` for any given `thief` and `cat`), it suffices to store the conditional probability table for `alarm` and the probability $P(y = T)$.

Probabilistic inclusion logic $\mathrm{FO}(\approx)$

Syntax: FO (negation normal form) $+$ $\vec{x} \approx \vec{y}$ (only positively)

Semantics: Defined in terms of a finite structure $\mathfrak{A}$ and a probabilistic team $\mathbb{X}$

(1) Team $=$ a set of variable assignments with a shared domain

(2) Probabilistic team $=$ a pair $\mathbb{X} = (X, p)$, where X is a finite team and $p : X \to [0, 1]$ a probability distribution

In general: $\mathrm{FO}(A_1, \ldots, A_n)$ is FO (negation normal form) $+$ $A_1, \ldots, A_n$ (only positively)

Semantics of (probabilistic) dependencies

Let $\mathbb{X} = (X, p)$ be a probabilistic team and $\vec{x}, \vec{a}$ be tuples of variables and values.

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s)$$

The semantics of marginal identity atoms (identical distribution) $\vec{x} \approx \vec{y}$:

$$\mathfrak{A} \models_{\mathbb{X}} \vec{x} \approx \vec{y} \text{ iff } |\mathbb{X}|_{\vec{x}=\vec{a}} = |\mathbb{X}|_{\vec{y}=\vec{a}}, \text{ for each } \vec{a} \in A^k$$

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The semantics of probabilistic conditional independence atoms $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$:

$\mathfrak{A} \models_{\mathbb{X}} \vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$ iff, for all assignments s for $\vec{x}, \vec{y}, \vec{z}$

$$|\mathbb{X}|_{\vec{x}\vec{y}=s(\vec{x}\vec{y})} \cdot |\mathbb{X}|_{\vec{x}\vec{z}=s(\vec{x}\vec{z})} = |\mathbb{X}|_{\vec{x}\vec{y}\vec{z}=s(\vec{x}\vec{y}\vec{z})} \cdot |\mathbb{X}|_{\vec{x}=s(\vec{x})}$$

Semantics of (probabilistic) dependencies

Let $\mathbb{X} = (X, p)$ be a probabilistic team and $\vec{x}, \vec{a}$ be tuples of variables and values.

$$|\mathbb{X}|_{\vec{x}=\vec{a}} := \sum_{\substack{s \in X \\ s(\vec{x})=\vec{a}}} p(s)$$

The semantics of team-based dependencies α:

$$\mathfrak{A} \models_{\mathbb{X}} \alpha \text{ iff } \mathfrak{A} \models_{X^+} \alpha,$$

where X^+ consists of $s \in X$ such that $\mathbb{X}(s) > 0$

Semantics of first-order part I

Definition (FoIKS 2018)

Let $\mathfrak{A}$ be a finite structure and $\mathbb{X} = (X, p)$ a probabilistic team.

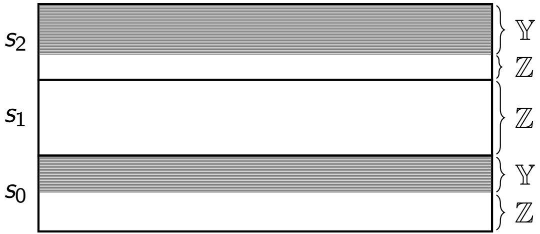
$$
\begin{aligned}
\mathfrak{A} \models_{\mathbb{X}} \ell \quad &\Leftrightarrow \quad \mathfrak{A} \models_{s} \ell \text{ for all } s \in X \text{ such that } p(s) > 0 \\
& \quad \text{(when } \ell \text{ is a first-order literal)} \\
\mathfrak{A} \models_{\mathbb{X}} (\psi \wedge \theta) \quad &\Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{X}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{X}} \theta
\end{aligned}
$$

Semantics of first-order part II

Disjunction via convex combinations:

$$\mathfrak{A} \models_{\mathbb{X}} (\psi \vee \theta) \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{Y}} \psi \text{ and } \mathfrak{A} \models_{\mathbb{Z}} \theta,$$

where $\mathbb{X} = \alpha \cdot \mathbb{Y} + (1 - \alpha) \cdot \mathbb{Z}$, for some $\alpha \in [0, 1]$.



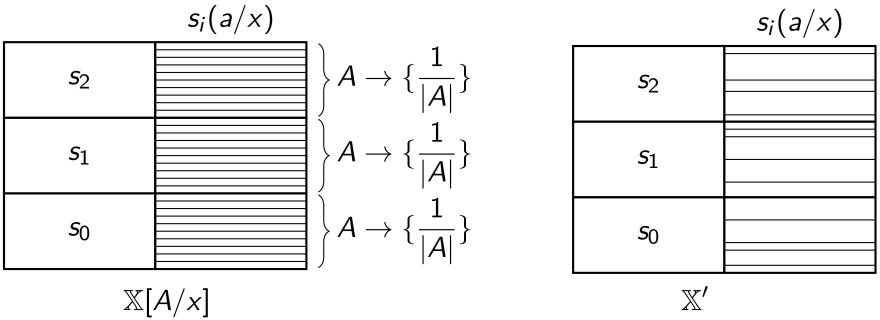
NB. The empty set is considered as a probabilistic team.

Semantics of first-order part III

Quantification introduces a new column:

$$\mathfrak{A} \models_{\mathbb{X}} \forall x \psi \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{X}[A/x]} \psi$$

$$\mathfrak{A} \models_{\mathbb{X}} \exists x \psi \quad \Leftrightarrow \quad \mathfrak{A} \models_{\mathbb{X}'} \psi, \text{ for some } \mathbb{X}' \text{ such that } \mathbb{X}' \upharpoonright \mathrm{Dom}(\mathbb{X}) = \mathbb{X}$$



Formulae vs. sentences

This paper: logical, computational, axiomatic properties of $\mathrm{FO}(\approx)$ and $\mathrm{FO}(\approx, \mathrm{dep}(\cdots))$

Two levels of analysis:

- Sentences $\sim$ finite structures $\sim$ strings of Booleans
- Formulae $\sim$ probabilistic teams $\sim$ strings of reals

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Expressivity over sentences – background

$$\mathrm{FO}(\subseteq) \overset{\mathrm{ord}}{=} \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE}$$

$$\mathrm{FO}(\subseteq, \mathrm{dep}(\cdots)) = \mathsf{NP} = \mathrm{FO}(\perp_{\mathrm{c}})$$

Table: Team semantics

$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \exists[0,1]^{\leq} \subseteq \exists\mathbb{R} \subseteq \mathsf{PSPACE}$$

$$\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) = \exists[0,1]^{\leq}$$

Table: Probabilistic team semantics

Probabilistic independence and existential theory of the reals

- The existential theory of the reals consists of all true sentences of the form

$$\exists x_1 \dots \exists x_n \psi(x_1, \dots x_n)$$

  where ψ is a quantifier-free formula of the real arithmetic

- Gives rise to the Boolean complexity class $\exists\mathbb{R}$:
  the closure of the existential theory of the reals under polynomial-time reductions
- $\exists[0,1]^{\leq}$ defined as $\exists\mathbb{R}$ but in terms of sentences of the form

$$\exists x_1 \dots \exists x_n \Big(\bigwedge_{1 \leq i \leq n} 0 \leq x_i \wedge x_i \leq 1 \wedge \psi \Big),$$

  where ψ does not contain $\neg$ nor $<$.

Theorem (LiCS 2020)

Over finite structures, $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \equiv \exists[0,1]^{\leq}$.

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Expressivity over sentences – new results

$$\begin{array}{ccccccc} \mathrm{FO}(\subseteq) & & \mathrm{FO}(\subseteq, \mathrm{dep}(\cdots)) & & \\ \|_{\mathrm{ord}} & & \| & & \\ \mathsf{P} & \subseteq & \mathsf{NP} & \subseteq & \mathsf{PSPACE} \\ & & \| & & \\ & & \mathrm{FO}(\perp_{\mathrm{c}}) & & \end{array}$$

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\begin{array}{ccccccc}
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\rotatebox{90}{$=$}_{\mathrm{ord}} & & \rotatebox{90}{$=$} & & \\
\mathsf{P} & \subseteq & \mathsf{NP} & \subseteq & \mathsf{PSPACE} \\
 & & \rotatebox{90}{$=$} & & \\
 & & \mathrm{FO}(\perp_{\mathrm{c}}) & &
\end{array}
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\end{array}
$$

Table: Probabilistic team semantics. * New results

Probabilistic inclusion logic over sentences

Lemma

Let $\phi \in \mathrm{FO}(\approx)$ be a sentence. There is a polynomial-time reduction from finite structures $\mathfrak{A}$ to systems of linear inequations $\mathcal{S}$ such that $\mathfrak{A} \models \phi$ if and only if $\mathcal{S}$ has a solution.

Proof.

Sketch. Add a variable $x_{s,\psi}$, for any partial assignment s and any subformula ψ of ϕ. Initialize $\mathcal{S}$ with $x_{\emptyset,\phi} = 1$, and $x_{\psi,s} \geq 0$ for all s and ψ. For each ψ add a set of equations to describe its corresponding team operation. E.g., for disjunction weights of assignments are split to two:

- If ψ is $\theta \vee \theta'$, add $x_{s,\theta} + x_{s,\theta'} = x_{s,\psi}$ for all s.

□

Deciding whether a system of linear inequalities has solutions is in polynomial time

Theorem

Let $\phi \in \mathrm{FO}(\approx)$ be a sentence. The problem of determining whether $\mathfrak{A} \models \phi$ for a given finite structure $\mathfrak{A}$ is in P.

From inclusion to probabilistic inclusion logic

Theorem

Every sentence of $\mathrm{FO}(\subseteq)$ *is equivalent to a sentence of* $\mathrm{FO}(\approx)$.

Proof.

Inclusion atoms definable in terms of *equiextension* atoms $\vec{x}_1 \bowtie \vec{x}_2 := \vec{x}_1 \subseteq \vec{x}_2 \wedge \vec{x}_2 \subseteq \vec{x}_1$ [Galliani, 2012]. However, $\vec{x}_1 \approx \vec{x}_2 \not\equiv \vec{x}_1 \bowtie \vec{x}_2$ as equiextension may hold even if the weights are not in balance.

Proof idea. First balance all positive weights, then apply $\approx$:

$$\forall c \forall \vec{u} \exists v_1 v_2 \forall z'_1 \ldots \forall z'_k \exists z_1 \ldots \exists z_k \big(\bigwedge_{i=1,2} \vec{x}_i = \vec{u} \leftrightarrow v_i = c \wedge \tag{1}$$

$$\bigwedge_{i=1}^{k} z'_i = c \rightarrow z_i = c \wedge (\neg \vec{z} = \vec{c} \vee \vec{u} v_1 \approx \vec{u} v_2)\big),$$

where k is the number of "splits" (from quantification, disjunction) in the underlying sentence.

Probabilistic inclusion logic over sentences cont.

Theorem

$\mathrm{FO}(\approx)$ *corresponds to* P *over finite ordered structures.*

Proof.

1. Over finite structures: $\mathrm{FO}(\subseteq) \subseteq \mathrm{FO}(\approx) \subseteq \mathrm{P}$
2. Over finite ordered structures: $\mathrm{P} \equiv \mathrm{FO}(\subseteq)$ [Galliani and Hella, 2013]

□

Future work:

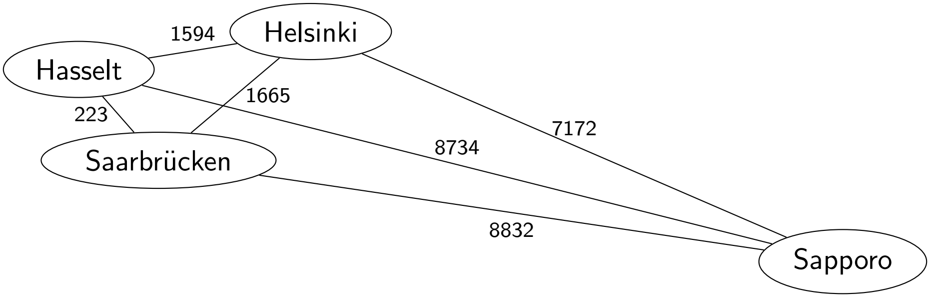
1. $\mathrm{FO}(\subseteq)$ strictly subsumed by $\mathrm{FO}(\approx)$ (over arbitrary finite structures)?
2. Relationship between $\mathrm{FO}(\approx)$ and fixed-point logic/inclusion logic with counting? Cf. [Grädel and Hegselmann, 2016]

Probabilistic inclusion/dependence logic over sentences

Results via $\mathbb{R}$-structures and Blum-Shub-Smale machines

$\mathbb{R}$-structures [Grädel and Meer, 1995] consist of a finite structure $\mathfrak{A}$ together with an ordered field of reals and a finite set of weight functions from $\mathfrak{A}$ to $\mathbb{R}$

(particular case of metafinite structures [Grädel and Gurevich, 1998])

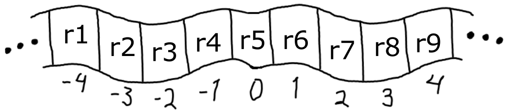


Blum-Shub-Smale machines

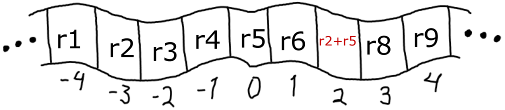
Input: finite string of reals
Output: 0 or 1 (decision problems)

A program is a finite list of instructions:

- Arithmetic instructions:
  $x_i \leftarrow (x_j + x_k)$, $x_i \leftarrow (x_j - x_k)$,
  $x_i \leftarrow (x_j \times x_k)$, $x_i \leftarrow c$.
- Shift left or right.
- Branch on inequality
  if $x_0 \leq 0$ **then** go to α; **else** go to β.



Addition: [2] := [-3]+[0]



Descriptive complexity over the reals

Theorem ([Grädel and Meer, 1995])

$\mathrm{ESO}_{\mathbb{R}}[+, \times, \leq, (r)_{r \in \mathbb{R}}] \equiv \mathsf{NP}_{\mathbb{R}}$

Two-sorted variant of ESO with

1. first-order logic on the finite structure $\mathfrak{A}$
2. existential quantification of functions from $\mathfrak{A}$ to reals
3. constants r for each real
4. complex numerical terms by $\{+, \times\}$
5. (negated) inequality $\leq$ between numerical terms

Too strong for $\mathrm{FO}(\approx, \mathrm{dep}(\cdots))$**:** 1) Lacks $\neg$, $\times$, and real constants 2) Quantification over $[0, 1]$

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Probabilistic inclusion/dependence vs. additive ESO over the reals

We show (adapting techniques from [FoIKS 2018]):

Theorem

$\mathrm{FO}(\approx, \mathrm{dep}(\cdots)) \equiv \mathrm{L}\text{-}\mathrm{ESO}_{[0,1]}[+, \leq, 0, 1]$

- "Loose fragment": no negated atoms $\neg i \leq j$ between two numerical terms
- Existential second-order quantification over functions from $\mathrm{Dom}(\mathfrak{A})$ to $[0, 1]$
- Only constants $0, 1$ allowed

NB. The result holds for formulae of $\mathrm{FO}(\approx, \mathrm{dep}(\cdots))$

Probabilistic inclusion/dependence logic over sentences cont.

Theorem

Over finite structures, $\mathrm{FO}(\approx, \mathrm{dep}(\cdots)) \equiv \mathsf{NP}$.

Proof.

$\boxed{\supseteq}$ Over finite structures: $\mathsf{NP} \subseteq \mathrm{FO}(\mathrm{dep}(\cdots)) \subseteq \mathrm{FO}(\approx, \mathrm{dep}(\cdots))$

$\boxed{\subseteq}$ It is easy to show that over formulae:

$$\mathrm{FO}(\approx, \mathrm{dep}(\cdots)) \subseteq \mathrm{ESO}_{\mathbb{R}}[\leq, +, 0, 1] \subseteq \mathsf{NP}^0_{\mathrm{add}}.$$

$\mathsf{NP}^0_{\mathrm{add}}$ allows guessing a string of reals and then verifying in polynomial time in the **additive** Blum-Shub-Smale model of computation (with machine constants $0, 1$).

It suffices to show that $\mathsf{NP}^0_{\mathrm{add}}$ collapses to NP over Boolean inputs. □

Collapse of additive NP over the reals

Theorem

Over Boolean inputs, $\mathrm{NP}^0_{\mathrm{add}} = \mathrm{NP}$

Proof.

Sketch. $\boxed{\supseteq}$ trivial. $\boxed{\subseteq}$ Suppose $L \subseteq \{0,1\}^* \cap \mathrm{NP}^0_{\mathrm{add}}$ is decided non-deterministically by a BSS machine M whose running is bounded by some polynomial p. Let $x \in \{0,1\}^n$ be an input. First, guess the outcome of each comparison of the BSS computation; the outcome is a Boolean string z of length $p(n)$. During a computation the value of each coordinate x_i is a linear function on the constants 0 and 1, the input x, and the real guess y of length $p(n)$. Thus it is possible to construct in polynomial time a system:

$$\sum_{j=1}^{p(n)} a_{ij} y_j \leq 0 \quad (1 \leq i \leq m), \quad \sum_{j=1}^{p(n)} b_{ij} y_j < 0 \quad (1 \leq i \leq l), \quad a_{ij}, b_{ij} \in \mathbb{Z} \qquad (2)$$

such that y is a (real-valued) solution iff M accepts (x, y) wrt. z. □

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Expressivity over formulae

$$\mathrm{FO}(\subseteq) \quad \subsetneq \quad \mathrm{FO}(\subseteq, \mathrm{dep}(\cdots)) \equiv \mathrm{FO}(\perp_{\mathrm{c}})$$

Table: Team semantics

$$\mathrm{FO}(\approx) \quad \subsetneq \quad \mathrm{FO}(\approx, \mathrm{dep}(\cdots)) \quad \subsetneq^{*} \quad \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$$

Table: Probabilistic team semantics. * New results

Probabilistic inclusion/dependence vs. independence

Both $\approx$ and $\mathrm{dep}(\cdots)$ expressible in $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ (JELIA 2019)

Example

Define $\phi(x) = \exists c \exists y \forall z \theta$ where θ is defined as

$$\mathrm{dep}(c) \wedge x \perp\!\!\!\perp y \wedge x \approx y \wedge ((x = c \wedge y = c) \leftrightarrow z = c). \tag{3}$$

Suppose $\{0,1\} \models_{\mathbb{X}} \phi$. Then

1. $c \in \{0,1\}$ is a constant;
2. z is uniformly distributed, so $z = c$ holds for weights $1/2$;
3. $x = c \wedge y = c$ holds for weight $1/2$;
4. x and y are independent and identically distributed, so $x = c$ holds for weight $1/\sqrt{2}$.

NB. Irrational weights not definable in $\mathrm{FO}(\approx, \mathrm{dep}(\cdots))$.

Theorem

Over formulae, $\mathrm{FO}(\approx, \mathrm{dep}(\cdots)) \subsetneq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$

Axioms for quantitative dependence

- Marginal independence ✓ [Geiger et al., 1991]
- Conditional independence ✗ [Studený, 1992]
- Marginal identity ?

Axioms for quantitative dependence

- Marginal independence ✓ [Geiger et al., 1991]
- Conditional independence ✗ [Studený, 1992]
- Marginal identity ✓

Theorem

The following axiomatization is sound and complete:

1. *reflexivity:* $x_1 \ldots x_n \approx x_1 \ldots x_n$;
2. *symmetry: if* $x_1 \ldots x_n \approx y_1 \ldots y_n$, *then* $y_1 \ldots y_n \approx x_1 \ldots x_n$;
3. *projection and permutation: if* $x_1 \ldots x_n \approx y_1 \ldots y_n$, *then* $x_{i_1} \ldots x_{i_k} \approx y_{i_1} \ldots y_{i_k}$, *where* $i_1, \ldots, i_k$ *is a sequence of distinct integers from* $\{1, \ldots, n\}$.
4. *transitivity: if* $x_1 \ldots x_n \approx y_1 \ldots y_n$ *and* $y_1 \ldots y_n \approx z_1 \ldots z_n$, *then* $x_1 \ldots x_n \approx z_1 \ldots z_n$.

Conclusion

- We studied quantitative variants of inclusion and inclusion/dependence logics
- Qualitative and quantitative variants in many ways **analogous**:
  1. Inclusion logic captures P (over ordered models)
  2. Inclusion/dependence logic captures NP
  3. Marginal identity and independence has axioms, conditional independence has not
- Where the **analogy breaks down**:
  1. Relationship between inclusion/dependence logic, independence logic, and NP
     - Qualitative: Both logics capture NP properties of teams
     - Quantitative: additive vs. multiplicative properties of prob.teams. For sentences, former captures NP, latter $\exists[0,1]^{\leq}$

Thanks!

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