Enumerating Teams in First-Order Team Logics

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Workshop on Logics of Dependence and Independence
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The logics we use

- $\varphi ::= \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x \varphi \mid \exists x \varphi \mid R(\overline{x}) \mid \neg R(\overline{x}) \mid x = y \mid x \neq y$
- team atoms: $=(\dots), \perp, \subseteq$
- lax semantics
- examples for classes: $\mathrm{FO}(=(\dots)), \#\mathrm{FO}(\subseteq), \mathrm{DelFO}(\perp)$

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Results for decision and counting

- $FO(\perp) = FO(=(\dots)) = \Sigma^1_1 = NP$ [Kontinen, Väänänen, 2009]
- $FO(\subseteq) = GFP^+ = LFP = P$ [Galliani, Hella, 2013]
- $\#FO(\perp) = \#NP$ [Haak et al., 2019]
- $\#FO(\subseteq) \subseteq TotP \subseteq \#P$ [Haak et al., 2019]
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Introduction to enumeration

Problem:	E-SAT
Input:	Propositional forumlua φ
Output:	$\{\theta \mid \theta \models \varphi\}$

Definition

Let C be a decision complexity class. The enumeration class Del$C$ consists of all enumeration problems E, for which there exists a RAM M with oracle $L \in C$ and a polynomial p such that for all inputs x, M enumerates the output set of E with $p(|x|)$ delay and all oracle questions are bounded by $p(|x|)$.

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Hardness

- $E_1 \leq_D E_2$, if there is an oracle-bounded EOM M with oracle E_2 that enumerates E_1 with polynomial delay.
- Exist-E is Σ_k^p-hard $\Rightarrow$ E is DelΣ_k^p-hard [Creignou et al., 2019]
- B is Σ_k^p-hard and B can be decided by a polynomial time EOM with oracle $E \Rightarrow E$ is DelΣ_k^p-hard

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A characterization of DelNP

Problem:	E-SAT$_\varphi^{\mathrm{team}}$
Input:	Structure $\mathcal{A}$
Output:	$\{X \mid \mathcal{A} \models_X \varphi, X \neq \emptyset\}$

Theorem

Let $A \subseteq \{\bot, =(\dots)\}$, *there is* $\varphi \in \mathrm{FO}(A)$, *s. t.* E-SAT$_\varphi^{\mathrm{team}}$ *is* DelNP*-complete and therefore* $[\text{E-SAT}_\varphi^{\mathrm{team}}]^{\leq_{\mathrm{D}}} = \mathrm{DelNP}$.

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Membership

Problem:	$\mathrm{VerifyTeam}_\varphi$
Input:	Structure $\mathcal{A}$, team X
Output:	$\mathcal{A} \models_X \varphi$

Problem:	$\mathrm{ExtendTeam}_\varphi$
Input:	Structure $\mathcal{A}$, team X, set of assignments Y
Output:	$\{X' \mid \mathcal{A} \models_{X'} \varphi, X \subsetneq X', X' \cap Y = \emptyset\} \neq \emptyset$

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DelΣ$_p^2$-complete case?

Problem:	E-MaxSAT$_\varphi^{\mathrm{team}}$
Input:	Structure $\mathcal{A}$
Output:	$\{X \mid \mathcal{A} \models_X \varphi, \forall X'\ X \subsetneq X' \Rightarrow \mathcal{A} \not\models_{X'} \varphi\}$

Problem:	ExtendMaxTeam$_\varphi$
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Inclusion logic cases

Theorem
For any formula $\varphi \in \mathsf{FO}(\subseteq)$ *it holds:* $\text{E-SAT}_{\varphi}^{\text{team}} \in \mathsf{DelP}$.

The same holds for E-MaxSAT, E-MinSAT, E-CMaxSAT.

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Capturing DelNP using inclusion logic

Problem:	E-CMinSAT$_\varphi^{\mathrm{team}}$
Input:	Structure $\mathcal{A}$
Output:	$\{X \mid \mathcal{A} \models_X \varphi, \forall X' \neq \emptyset\ \mathcal{A} \models_{X'} \varphi \Rightarrow \lvert X\rvert \leq \lvert X'\rvert\}$

Problem:	CMinSAT$_\varphi^{\mathrm{team}}$
Input:	Structure $\mathcal{A}, k \in \mathbb{N}$
Output:	$\{X \mid \mathcal{A} \models_X \varphi, \lvert X\rvert \leq k\} \neq \emptyset$

Theorem

There is a formula $\varphi \in \mathrm{FO}(\subseteq)$ *s. t.* CMinSAT$_\varphi^{\mathrm{team}}$ *is* NP*-hard.*

Corollary

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Summary

- $[\text{E-SAT}^{\text{team}}_{\varphi}]^{\leq_D} = \text{DelFO}(\perp) = \text{DelNP} = [\text{E-CMinSAT}^{\text{team}}_{\psi}]^{\leq_D}$
- $[\text{E-SAT}^{\text{team}}_{\chi}]^{\leq_D} = \text{DelFO}(\subseteq) \subseteq \text{DelP}$

	$\subseteq$	$=(\dots),\perp$
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First-order Team semantics

- For atoms α :
  $\mathcal{A} \models_X \alpha$ iff $\forall s \in X \colon \mathcal{A} \models_s \alpha$
- $\mathcal{A} \models_X \varphi \wedge \psi$ iff $\mathcal{A} \models_X \varphi$ and $\mathcal{A} \models_X \psi$
- $\mathcal{A} \models_X \varphi \vee \psi$ iff there are teams $Y, Z \subseteq X$ such that $Y \cup Z = X$, $\mathcal{A} \models_Y \varphi$ and $\mathcal{A} \models_Z \psi$
- $\mathcal{A} \models_X \forall x \varphi$ iff $\mathcal{A} \models_{X[A/x]} \varphi$
- $\mathcal{A} \models_X \exists x \varphi$ iff there exists a function $F \colon X \to \mathcal{P}(A) \setminus \{\emptyset\}$, such that $\mathcal{A} \models_{X[F/x]} \varphi$

Team semantics new atoms

- New atoms: dependence $=(\dots)$, independence $\perp$, inclusion $\subseteq$.
- $\mathcal{A} \models_X =(\overline{x}, y)$, iff

$$\forall s, s' \in X\colon s(\overline{x}) = s'(\overline{x}) \Rightarrow s(y) = s'(y)$$

- $\mathcal{A} \models_X \overline{x} \perp \overline{y}$ iff

$$\forall s, s' \in X \exists s'' \in X\colon s''(\overline{x}) = s(\overline{x}) \wedge s''(\overline{y}) = s'(\overline{y})$$

- $\mathcal{A} \models_X \overline{x} \subseteq \overline{y}$ iff

$$\forall s \in X \exists s' \in X\colon s(\overline{x}) = s'(\overline{y})$$

Properties of team logics

- For $A \subseteq \{=(\dots), \perp, \subseteq\}$ the Logic FO(A) has the empty team property
- FO($=(\dots)$) is downwards closed, that is for $Y \subseteq X$

$$\mathcal{A} \models_X \varphi \Rightarrow \mathcal{A} \models_Y \varphi$$

- FO($\subseteq$) is union closed, that is

$$\mathcal{A} \models_X \varphi \wedge \mathcal{A} \models_Y \varphi \Rightarrow \mathcal{A} \models_{X \cup Y} \varphi$$

Team logic and second-order logic

- For every σ-formula φ of $\mathrm{FO}(\perp)$, there is an $\sigma(R)$-sentence $\psi(R)$ of Σ_1^1 such that for all σ-structures $\mathcal{A}$ and teams X,

$$\mathcal{A} \models_X \varphi \iff (\mathcal{A}, \mathrm{rel}(X)) \models \psi(R). \tag{1}$$

  Conversely, for every $\sigma(R)$-sentence $\psi(R)$ of Σ_1^1, there is a σ-formula φ of $\mathrm{FO}(\perp)$ such that (1) holds for all σ-structures $\mathcal{A}$ and *non-empty* teams X.
- The same as the above holds for formulae of $\mathrm{FO}(=(\dots))$ as well, except that in both directions for $\mathrm{FO}(=(\dots))$ the relation symbol R is assumed to occur only negatively in the sentence $\psi(R)$.

Team logic and second-order logic cont.

- Let $\varphi(R)$ be a *myopic* σ-formula, that is, $\varphi(R) = \forall \overline{x}(R(\overline{x}) \to \psi(R, \overline{x}))$, where ψ is a first order σ-formula with only positive occurrences of R. Then there exists a σ-formula $\chi \in \mathrm{FO}(\subseteq)$ such that for all σ-structures $\mathcal{A}$ and all teams X:

$$\mathcal{A} \models_X \chi(\overline{x}) \Leftrightarrow \mathcal{A}, \mathrm{rel}(X) \models \varphi(R).$$

Definiton of #FO(A)

For any set $A \subseteq \{=(\dots), \perp, \subseteq\}$, $\#\mathsf{FO}(A)^{\mathsf{team}}$ is the class of all functions $f \colon \{0,1\}^* \to \mathbb{N}$ for which there is a vocabulary σ and an FO(A)-formula $\varphi(\overline{x})$ over σ with a tuple $\overline{x}$ of free first-order variables such that for all σ-structures $\mathcal{A}$,

$$f(\mathrm{enc}_\sigma(\mathcal{A})) = |\{X \in \mathrm{team}(\mathcal{A}, (\overline{x})) : X \neq \emptyset \text{ and } \mathcal{A} \models_X \varphi(\overline{x})\}|.$$