On the Presburger fragment of logics with multiteam semantics

Richard Wilke

RWTH Aachen University

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Statistical Dependencies

Team semantics is based on sets (of assignments), hence statements such as $x \perp y$ "x is independent of y" are defined on the presence or absence of data, not on their multiplicity.

The table $\mathcal{T}$ lists temperature measurements between 0° and 40° in different countries.

Country	Temperature	Date
	40°	25.07.19
	32°	26.06.19
	40°	07.07.20
	2°	08.02.20
⋮	⋮	⋮

In team semantics, $\mathcal{T} \models \text{Country} \perp \text{Temperature}$.

Multiteams

Approach: extend teams $X = \{s_1 \colon x \mapsto a, s_2 \colon x \mapsto b\}$ to finite multiteams $M = \{\!\{s_1, s_1, s_2\}\!\}$ in which the information of the multiplicity of the assignments is present.

Formally, $X = \{s \colon \mathcal{V} \to A\}$ becomes $M = (X, n \colon X \to \mathbb{N}_{>0})$.

Support $M^{\mathsf{T}} = X$

Union $\{\!\{s_1, s_2\}\!\} \uplus \{\!\{s_1\}\!\} = M$

Subset $\{\!\{s_1\}\!\} \sqsubseteq \{\!\{s_1, s_2\}\!\}$

Probability $\Pr_M(x = a) = 2/3$

Universal extension

For $A = \{a, b\}$

$M[y \mapsto A]$ is

x	y
a	a
a	b
a	a
a	b
b	a
b	b

Skolem extension

$F \colon M \to A$ assigns every $s \in M$ an element of A.

$M[y \mapsto F]$ is

x	y
a	a
a	b
b	b

Statistical dependencies

Dependence $M \models \mathrm{dep}(\bar{x}, y)$ if $M^{\mathsf{T}} \models \mathrm{dep}(\bar{x}, y)$

Independence $M \models \bar{x} \perp\!\!\!\perp \bar{y}$ if $\mathrm{Pr}_M(\bar{x} = \bar{a}) = \mathrm{Pr}_M(\bar{x} = \bar{a} \mid \bar{y} = \bar{b})$
holds for all $\bar{a} \in M(\bar{x})^{\mathsf{T}}$ and $\bar{b} \in M(\bar{y})^{\mathsf{T}}$
In particular $\mathcal{T} \not\models \mathsf{Country} \perp\!\!\!\perp \mathsf{Temperature}$

Inclusion $M \models x \subseteq^{\pm} y$ if $M(x) \subseteq^{\pm} M(y)$
Because M is finite, $x \subseteq^{\pm} y \equiv y \subseteq^{\pm} x$.

First-order multiteam semantics

$\mathrm{FO}^{\mathsf{M}}[\Omega]$ is the closure of atomic multiteam dependencies Ω under first-order connectives.

- $\mathfrak{A} \vDash_M \lambda$ if $\mathfrak{A} \vDash \lambda[s]$ for all $s \in M^{\mathsf{T}}$ if λ is a first-order literal
- $\mathfrak{A} \vDash_M \varphi_1 \wedge \varphi_2$ if $\mathfrak{A} \vDash_M \varphi_i$ for $i = 1, 2$
- $\mathfrak{A} \vDash_M \varphi_1 \vee \varphi_2$ if $\mathfrak{A} \vDash_{R_i} \varphi_i$ for some $R_1 \uplus R_2 = M$
- $\mathfrak{A} \vDash_M \forall x \varphi$ if $\mathfrak{A} \vDash_{M[x \mapsto A]} \varphi$
- $\mathfrak{A} \vDash_M \exists x \varphi$ if $\mathfrak{A} \vDash_{M[x \mapsto F]} \varphi$ for some $F \colon M \to A$

Relationship of logics with multiteam semantics

Under multiteam semantics dependence is downwards closed and inclusion is union closed.

It is also possible to express dependence $\text{dep}(\bar{x}, y)$ / exclusion and inclusion $\bar{x} \subseteq \bar{y}$ in $\text{FO}^M[\perp\!\!\!\perp]$.

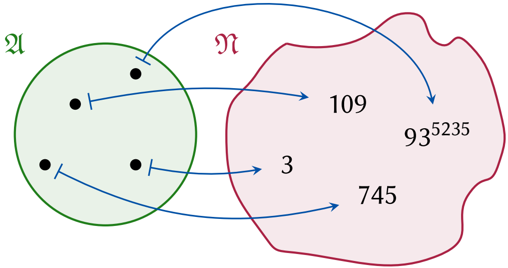
Theorem

Let $\bar{\alpha}$ be any collection of downwards closed atomic formulae and $\bar{\beta}$ be any collection of union closed atomic formulae. There is no formula $\psi \in \text{FO}^M[\bar{\alpha}, \bar{\beta}]$ equivalent to $x \perp\!\!\!\perp y$.

Hence $\text{FO}^M[\subseteq, |\,] \precneqq \text{FO}^M[\perp\!\!\!\perp]$.

Metafinite structures

The basis is a finite structure $\mathfrak{A}$ and an infinite number sort here: $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, \sum)$, where $\sum$ maps multisets of numbers to their sum. Weight functions $w\colon A^k \to \mathbb{N}$ mapping tuples of elements of $\mathfrak{A}$ into $\mathfrak{N}$ connect the finite and infinite part.



Here: $\mathfrak{D} = (\mathfrak{A}, \mathfrak{N}, \{f_M\})$ where $f_M\colon A^k \to \mathbb{N}$ encodes a multiteam M; $f_M(\bar{a})$ is the multiplicity of $s\colon \bar{x} \mapsto \bar{a}$ in M.

Existential second-order logic for multiteams

First-order part $\mathrm{FO}^{\mathrm{mts}}[+,\cdot]$:

$$t ::= 0 \mid f\bar{x} \mid t+t \mid t\cdot t \mid \Sigma_{\bar{x}}\{t : \alpha\}$$
$$\varphi ::= \alpha \mid t = t \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi \mid \forall x \varphi$$

Plus second-order quantification results in $\mathrm{ESO}^{\mathrm{mts}}[+,\cdot]$:

$$\psi(f) = (\exists g_1 \lhd b_1) \dots (\exists g_k \lhd b_k)\varphi$$

where b_i are bounds of the form $|f| \cdot |A|^{\ell}$.

$\mathfrak{D} = (\mathfrak{A}, \mathfrak{N}, \{f_M\}) \vDash (\exists g \lhd b)\varphi$ if $(\mathfrak{A}, \mathfrak{N}, \{f_M, g\}) \vDash \varphi$ for some $g\colon A^k \to \mathbb{N}$ with $|g| = \sum_{\bar{a}} g(\bar{a}) \leq |f| \cdot |A|^{\ell}$.
The Presburger fragment $\mathrm{ESO}^{\mathrm{mts}}[+]$ of $\mathrm{ESO}^{\mathrm{mts}}[+,\cdot]$ uses only addition and summation.

$\mathrm{FO}^{M}[\Subset, \mid] \equiv \mathrm{ESO}^{\mathrm{mts}}[+]$

Theorem

For every $\varphi(\bar{x}) \in \mathrm{FO}^{M}[\Subset, \mid]$ there exists $\psi(f) \in \mathrm{ESO}^{\mathrm{mts}}[+]$, such that

$$\mathfrak{A} \models_M \varphi \text{ if, and only if, } (\mathfrak{A}, \mathfrak{N}, \{f_M\}) \models \psi(f).$$

Theorem

For every $\psi(f) \in \mathrm{ESO}^{\mathrm{mts}}[+]$, such that $\psi(0) \equiv \mathrm{true}$, there exists $\varphi \in \mathrm{FO}^{M}[\Subset, \mid]$ such that

$$\mathfrak{A} \models_M \varphi \text{ if, and only if, } (\mathfrak{A}, \mathfrak{N}, \{f_M\}) \models \psi(f).$$

Forking

Forking, e.g. $x \lessdot_{\geq \lambda} y$, has been studied for team semantics in context of two-sorted structures by Grädel and Hegselmann (2016).
Under multiteam semantics: $\mathfrak{A} \models_M \bar{x} \lessdot_{\trianglelefteq p} \bar{y}$ if
$\Pr_M(\bar{y} = \bar{b} \mid \bar{x} = \bar{a}) \trianglelefteq p$ holds for all tuples $\bar{a}, \bar{b} \in A^*$ for which
$\Pr_M(\bar{y} = \bar{b} \mid \bar{x} = \bar{a}) > 0$ where $\trianglelefteq \in \{\leq, =, \geq\}$ and $p \in [0, 1]$.
Special cases:

- $\bar{x} \lessdot_{\leq 0} \bar{y} \equiv \text{false}$ and $\bar{x} \lessdot_{\geq 0} \bar{y} \equiv \text{true}$.
- $\bar{x} \lessdot_{\geq p} \bar{y} \equiv \text{dep}(\bar{x}, \bar{y})$ for any $p > 1/2$

Closure properties:

1. $\bar{x} \lessdot_{\leq p} \bar{y}$ is union closed
2. $\bar{x} \lessdot_{= p} \bar{y}$ and $\bar{x} \lessdot_{\geq p}$ are neither downwards closed nor union closed

Forking$_{\geq}$

$$M = \begin{array}{cc} x & y \\ \hline a & a \\ a & a \\ a & b \\ a & b \\ b & b \end{array} \text{ satisfies } x \lhd_{\geq 1/2} y \text{, but } \begin{array}{cc} x & y \\ \hline a & a \\ a & a \\ a & b \\ b & b \end{array} \sqsubseteq^{+} M \text{ does not.}$$

Hence $\bar{x} \lhd_{\geq p} \bar{y}$ is not downwards closed.

But $\bar{x} \lhd_{\geq p} \bar{y}$ is downwards closed in the team semantical sense:
If $(X, m) \models \bar{x} \lhd_{\geq p} \bar{y}$ then $(Y, m \upharpoonright Y) \models \bar{x} \lhd_{\geq p} \bar{y}$ for all $Y \subseteq X$.

Fact: $\mathrm{FO}^{M}[\mathrm{dep}] \precneqq \mathrm{FO}^{M}[\lhd_{\geq 1/2}]$.

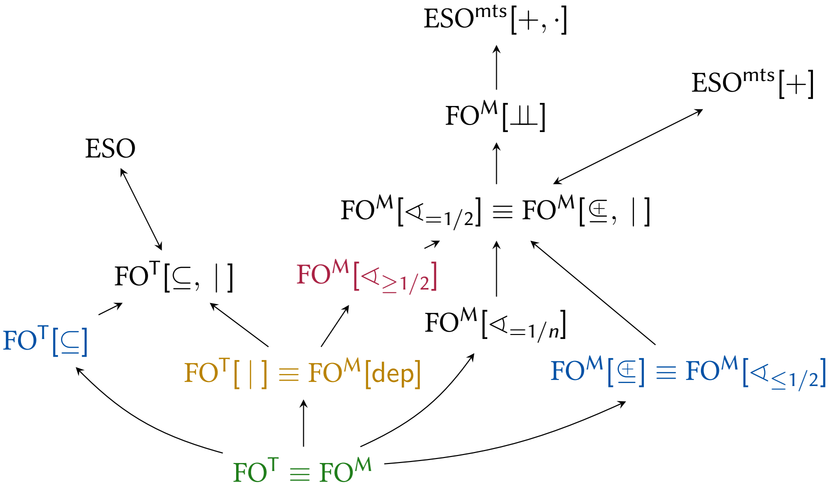
Forking$_{\leq}$ & anonymity

The atom $\bar{x} \lessdot_{\leq p} \bar{y}$ is similar to anonymity $x \Upsilon y$ introduced by Väänänen in team semantics: “given x one cannot determine the value of y”.
In particular, $\mathfrak{A} \models_{(X,n)} x \lessdot_{\leq 1/2} y$ implies $\mathfrak{A} \models_X x \Upsilon y$.

Facts:

- $\mathrm{FO}^M[\lessdot_{\leq 1/2}] \equiv \mathrm{FO}^M[\subseteq]$
- $\mathrm{FO}^M[\lessdot_{\leq p}]$ and $\mathrm{FO}^M[\lessdot_{\geq p}]$ are incomparable
- $\mathrm{FO}^M[\lessdot_{\leq 1/2}, \lessdot_{\geq 1/2}] \equiv \mathrm{FO}^M[\lessdot_{=1/2}] \equiv \mathrm{ESO}^{\mathrm{mts}}[+]$

Overview



flat, (team semantically) downwards closed, union closed