On continuous teams

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November 2, 2018
UH-CAS workshop, Helsinki

Starting point

What happens from an algorithmic viewpoint of satisfiability and model checking if we look at infinite teams over a metric space?

Key points of computable analysis

Idea: A real number x is *computable*, if there is an algorithm outputting arbitrarily close approximations to x.

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) such that X is a set and $\delta_X : \{0,1\}^{\mathbb{N}} \to X$ is a surjective partial function.

Here δ_X tells how the elements are coded.

Computable metric spaces

Definition

A *computable metric space* is a separable metric space with a designated dense sequence $(a_n)_{n \in \mathbb{N}}$ such that $\{(m, n, s, t) \in \mathbb{N}^2 \times \mathbb{Q}^2 : s < d(a_n, a_m) < t\}$ is recursively enumerable. A point x is coded by giving a sequence p of indices such thtat $d(a_{p(n)}, x) < 2^{-n}$.

Open and closed sets

For a represented space $\mathbf{X}$

- the open subsets $\mathcal{O}(\mathbf{X})$ are characterized by $x \in U$ being semidecidable (recognizable) in $x \in \mathbf{X}$ and $U \in \mathcal{O}(\mathbf{X})$ (for a computable metric space $\mathbf{X}$, this can be achieved by coding $U \in \mathcal{O}(\mathbf{X})$ as $\langle p, q \rangle$ with $U = \bigcup_{\{n|p(n)\neq 0\}} B(a_{p(n)+1}, 2^{-q(n)}))$
- the closed subsets $\mathcal{A}(\mathbf{X})$ are the formal complements of the open sets, i.e. the codes for $A \in \mathcal{A}(\mathbf{X})$ are just the codes for $(X \setminus A) \in \mathcal{O}(\mathbf{X})$.

Overt and compact spaces

For a represented space $\mathbf{X}$ the overt subsets $\mathcal{V}(\mathbf{X})$ are characterized by $U \cap A \neq \emptyset$ being semidecidable (recognizable) in $U \in \mathcal{O}(\mathbf{X})$ and $A \in \mathcal{V}(\mathbf{X})$. In a computable metric space they can be coded as a list of all basic open balls intersecting them.

All computable metric spaces are computably overt.

A computable metric space $\mathbf{X}$ is *computably compact*, if the set of finite sequences $(n_0, r_0), \ldots, (n_\ell, r_\ell)$ such that $\mathbf{X} \subseteq \bigcup_{i \leq \ell} B(a_{n_i}, 2^{-r_i})$ is recursively enumerable.

Quantification

$\mathbf{X} \models_T \forall x \varphi$ if $\mathbf{X} \models_{\mathbf{X} \times T} \varphi$
$\mathbf{X} \models_T \exists x \varphi$ if there is T' such that $\pi_{-x} T' = T$ and $\mathbf{X} \models_{T'} \varphi$

Viewed as functions

$$\exists : (R, A) \mapsto \{y : \exists x \in A \, (x, y) \in R\}$$
$$\forall : (R, A) \mapsto \{y : \forall x \in A \, (x, y) \in R\}$$

Quantification over overt spaces

Lemma

The following are equivalent for a represented space $\mathbf{X}$*:*

1. $\mathbf{X}$ *is computably overt.*
2. $\exists : \mathcal{O}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{O}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$
3. $\forall : \mathcal{A}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{A}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$

Quantification over compact spaces

Lemma

The following are equivalent for a represented space $\mathbf{X}$*:*

1. $\mathbf{X}$ *is computably compact.*
2. $\forall : \mathcal{O}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{O}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$
3. $\exists : \mathcal{A}(\mathbf{X} \times \mathcal{Y}) \to \mathcal{A}(\mathbf{Y})$ *is computable for all represented spaces* $\mathbf{Y}$

The considered framework

Definition

A compact metric closed (respectively open) structure (over a given signature) consists of a compact separable metric space $\mathbf{X}$ as carrier, a continuous function $f_i : \mathbf{X}^{n_i} \to \mathbf{X}$ for each function symbol of arity n_i in the signature, and a closed (respectively open) subset $R_i \subseteq \mathbf{X}^{n_i}$ for each relation symbol of arity n_i in the signature.

We write ACMS (respectively OCMS) for the represented space of compact metric closed (respectively open) structures, where the carrier is given as $\mathbf{X} \in \mathrm{KPol}$, the functions as $f_i \in \mathcal{C}(\mathbf{X}^{n_i}, \mathbf{X})$ and the relations as $R_i \in \mathcal{A}(\mathbf{X}^{n_i})$ (respectively as $R_i \in \mathcal{O}(\mathbf{X}^{n_i})$).

Assumig teams to be open or closed

Theorem

For positive sentences involving basic closed predicates, $\perp$ and $\subseteq$, the usual team semantics and the teams-are-closed sets semantics agree.

Theorem

For positive sentences involving basic open predicates and $|$, the following all agree:

1. *the semantics allowing arbitrary sets as teams*
2. *the semantics demanding teams to be open sets*
3. *the semantics demanding teams to be closed sets*

Counterexamples

Example

The formula

$$\forall x \forall y \ (x = y) \lor (x \neq y)$$

is valid if arbitrary teams are allowed, but expresses that the space is discrete for closed teams.

Counterexamples

Example

The formula

$$\exists x \forall y \ (x = y) \lor (x \neq y)$$

is valid if arbitrary teams are allowed, but expresses that the space contains some isolated point for closed teams.

Counterexamples

Example

The formula

$$\forall x \exists y \exists z \ \ ((x = y \vee x = z) \wedge y|z)$$

holds over $\mathbf{X} = [0,1]$ if arbitrary teams are allowed, but not if teams have to be closed sets. The reason is that the y and the z values have to be disjoint non-empty sets covering $\mathbf{X}$. For closed teams, this formula expresses that the space is disconnected.

Complexity of formula satisfaction

Theorem

Let ϕ be a positive formula involving basic closed predicates, $\perp$ and $\subseteq$. Then $T \models \phi$ defines a closed predicate in the team (uniformly in ϕ).

Theorem

Let ϕ be a positive formula involving basic open predicates and $|$. Then $T \models \phi$ defines a open predicate in the team (uniformly in ϕ). As a consequence, if $T \models \phi$, then we can effectively find some $n \in \mathbb{N}$ such that any T' with $d(T, T')$ satisfies $T' \models \phi$.

Complexity of model checking

Corollary

It is semidecidable whether a formula $\phi \in \mathcal{L}_+(\perp, \subseteq)$ *does* **not hold** *in a structure* $\mathfrak{S} \in \mathrm{ACMS}$.

Corollary

It is semidecidable whether a formula $\phi \in \mathcal{L}_+(|)$ **holds** *in a structure* $\mathfrak{S} \in \mathrm{OCMS}$.

Open approximation of dependence/independence atoms

- $T \models =^{\varepsilon}_{\delta}(\overline{x}, \overline{y}$ if for any $s, s' \in T$ it holds that if $d(s(\overline{x}), s'(\overline{x})) \leq \delta$ then $d(s(\overline{y}), s'(\overline{y})) < \varepsilon$.
- $T \models \overline{x} \perp^{\delta,\varepsilon}_{\overline{z}} \overline{y}$ if for all $s, s' \in T$, if $d(s(\overline{z}), s'(\overline{z})) \leq \delta$ then there is $s'' \in T$ such that $d(s''(\overline{xz}), s(\overline{xz})) < \varepsilon$ and $d(s''(\overline{zy}), s'(\overline{zy})) < \varepsilon$.
- $T \models \overline{x} \subseteq^{\varepsilon} \overline{y}$ if $(\pi_{\overline{x}} T) \subseteq B(\pi_{\overline{y}} T), \varepsilon)$.
- $T \models \overline{x}|^{\varepsilon}\overline{y}$ if $d(\pi_{\overline{x}} T, \pi_{\overline{y}} T) > \varepsilon$.

Closed approximation of dependence/independence atoms

- $T \models \overline{=^{\varepsilon}_{\delta}}(\overline{x}, \overline{y}$ if for any $s, s' \in T$ it holds that if $d(s(\overline{x}), s'(\overline{x})) < \delta$ then $d(s(\overline{y}), s'(\overline{y})) \leq \varepsilon$.
- $T \models \overline{x}\overline{\perp^{\delta,\varepsilon}}_{\overline{z}}\overline{y}$ if for all $s, s' \in T$, if $d(s(\overline{z}), s'(\overline{z})) < \delta$ then there is $s'' \in T$ such that $d(s''(\overline{xz}), s(\overline{xz})) \leq \varepsilon$ and $d(s''(\overline{zy}), s'(\overline{zy})) \leq \varepsilon$.
- $T \models \overline{x}\overline{\subseteq^{\varepsilon}}\overline{y}$ if $(\pi_{\overline{x}}T) \subseteq \overline{B}(\pi_{\overline{y}}T), \varepsilon)$.
- $T \models \overline{x}\overline{|^{\varepsilon}}\overline{y}$ if $d(\pi_{\overline{x}}T, \pi_{\overline{y}}T) \geq \varepsilon$.

Translations

Many translations between dependence/independence atoms generalise to the approximations, e.g.:

Proposition

1. *If* $\varepsilon > \delta \geq 0$, *then* $=^{\varepsilon}_{\delta}(\overline{x}, \overline{y}) \Rightarrow \overline{y} \perp^{\delta,\varepsilon}_{\overline{x}} \overline{y}$.
2. *For any* $\delta \geq 0, \varepsilon > 0$, $\overline{y} \perp^{\delta,\varepsilon/2}_{\overline{x}} \overline{y} \Rightarrow =^{\varepsilon}_{\delta}(\overline{x}, \overline{y})$.

Next steps

- switching to a logic based on *continuous first order logic*, gives a natural measure of accuracy in the translations
- how does this affect computability?