$(*)$, MM^{++} and Δ_1-definability of NS_{ω_1}

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October 30, 2018

Definability in set theory universe

One central notion in logic is Definability. In general, we say an object or a property is definable if there exists a definition describing this object or property in some underlining language.

For set theory, the underlining language is the first order predicate logic. Besides, one benefit of set theory is that any property can be represented as an object or a set.

Definability Hierarchy in first order logic with $\in$ relation

Due to the inductive nature of the first order logic, we use the number of quantifier to indicate the complexity of formulas or definition. For simplicity, we always assume that any formula is written in normal form.

Definition (Levy Hierarchy)

- (Σ_0 or Π_0 formulas) ψ is Σ_0 or Π_0 if any appearances of quantifiers in ψ is bounded (under relation $\in$).
- (Σ_{n+1} formula) ψ is Σ_{n+1} if ψ can be written as $\exists x \varphi$ for some x and Π_n formulas φ.
- (Π_{n+1} formula) ψ is Π_{n+1} if ψ can be written as $\forall x \varphi$ for some x and Σ_n formulas φ.

Now an object O is $\Sigma_n(\Pi_n)$-definable in a structure A if there is a $\Sigma_n(\Pi_n)$ formula ψ such that

$$O = \{a \in A \mid a \text{ satisfy the formulas } \psi \text{ in } A\}$$

Descriptive set theory

Descriptive set theory studies the definable sets in the structure $\mathcal{R}$ or $H(\omega_1)$. It is one of the primary areas of research in set theory.

In recent years, set theorists are increasingly interested in the uncountable analogues 2^κ of the Cantor space (generalised Cantor space) and κ^κ of the Baire space (generalised Baire space) for uncountable cardinals κ. Generalized descriptive set theory studies the definable sets over the structure generalised Cantor space or generalised Baire space.

Canonical models of set theory

Godel's incompleteness theorem and Cohen's forcing technique shows that many mathematical statements are independent from ZFC. In other word, using ZFC as foundation does leave many mathematical statement both unprovable and unrefutable. Nevertheless, form a Platonism point of view, we could still look for the "real" model of set theory or its partial approximations. These are canonical models of set theory.
For a particular canonical model, it is also important to justify why it captures the truth of set theory.

Canonical model	Justification
Godel's L	Σ^1_2 well ordering, no large cardinal, GCH, Rigidity, ...
Core models for large cardinals	large cardinal, Rigidity, absoluteness, ...
strong ideals on small cardinal	large cardinal, small cardinals, combinatorics, ...

MM and its justification

Definition

MM is the statement that if P is a partial order preserving stationary subsets of ω_1, and $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a collection of dense subsets of P, then there is a filter $G \subseteq P$ meeting each D_α with all valuation of $\dot{S}_\alpha$ under G is stationary.

MM has many striking consequence on other field of mathematics. For example,

Theorem (Moore)

MM implies that there is a 5-element base for uncountable linear ordering.

Theorem (Farah)

MM implies that all automorphisms of Calkin algebra are inner.

Definition

$(*)$ is the statement that AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$-generic extension of $L(\mathbb{R})$.

As one of the main feature of $\mathbb{P}_{max}$ theory, $(*)$ captures the maximal Π_2-theory of $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$. Assuming $(*)$ (modulo some large cardinals), for any Π_2 sentence ψ, if in some generic extension $V[G]$,

$$\langle H(\omega_2), \in, NS_{\omega_1} \rangle^{V[G]} \models \psi,$$

then in V, already we have

$$\langle H(\omega_2), \in, NS_{\omega_1} \rangle \models \psi.$$

Definability in canonical models of ZFC

Theorem (Godel)

In Constructible universe L, there is a Σ_2 definable wellordering in structure $\mathbb{R}$. As a consequence, there are Σ_2 definable non-measurable set of reals.

Theorem (Mycielski and Swierczkowski)

Under Axiom of Projective Determinacy, there is no definable non-measurable set of reals in the structure $\mathbb{R}$.

Definability of the nonstationary ideal

The nonstationary ideal on κ (NS_κ) is a typical "pathological" object in generalized descriptive set theory.
From now on, we will talk about definability in the structure is $\langle H(\kappa^+), \in\rangle$ for some regular cardinal κ.

- NS_κ is lightface Σ_1-definable.
- (Friedman-Hyttinen-Kulikov) In L, NS_κ is not boldface Π_1-definable.
- (Mekler-Shelah) $Add(\kappa, \kappa^+)$ forces NS_κ to be not boldface Π_1-definable.
- (Mekler-Shelah) The canary tree order forces NS_{ω_1} to be boldface Π_1-definable.
- (Friedman-Wu-Zdomskyy) If $L \models \kappa$ is regular and not weakly compact, then in some generic extension of L, NS_κ is lightface Π_1-definable.

We remark that it is unknown how to construct a forcing such that NS_κ is boldface Π_1-definable for regular $\kappa \geq \omega_2$ in the extension (without inner model assumption).

Large cardinal and Δ_1-definability of NS_κ

Theorem (Friedman-Wu, Lücke-Schindler-Schlicht)

If κ is weakly compact or strong limit of ω_1-iterable cardinals, NS_κ is not boldface Π_1 definable.

Theorem (Friedman-Wu, Lücke-Schindler-Schlicht)

NS_{ω_1} is not lightface Π_1 definable, under each of the following assumption:

- *Woodin's $\mathbb{P}_{max}$ axiom $(*)$.*
- *$M_1^\sharp(A)$ exists for all $A \subseteq \omega_1$.*
- *NS_{ω_1} is precipitous and there is a measurable cardinal.*

Large cardinal and Δ_1-definability of NS_κ

Theorem (Friedman-Wu)

In a generic extension of $L[U]$, NS_{ω_1} is both precipitous and lightface Π_1-definable.

Theorem (Hoffelner)

In a generic extension of M_1, NS_{ω_1} is both saturated and lightface Π_1-definable.

$(*)$ and MM^{++}

Definition

$(*)$ is the statement that AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$-generic extension of $L(\mathbb{R})$.

Definition

MM^{++} is the statement that if P is a partial order preserving stationary subsets of ω_1, and $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a collection of dense subsets of P, $\langle \dot{S}_\alpha \mid \alpha < \omega_1 \rangle$ is a collection of name for stationary subset of ω_1, then there is a filter $G \subseteq P$ meeting each D_α with all valuation of $\dot{S}_\alpha$ under G is stationary.

Conjecture (Woodin)

MM^{++} + Large Cardinals implies $(*)$.

Lightface Π_1-definability of NS_{ω_1} can be interpreted in a $\Pi_2^{\langle H(\omega_2), \in NS_{\omega_1}\rangle}$ way.

Proposition

Under certain large cardinal assumption, no forcing can force NS_{ω_1} to be lightface Π_1 definable.

$(*)$, MM^{++} and Δ_1-definability of NS_{ω_1}

Theorem (Larson-Schindler-Wu)

NS_{ω_1} is not boldface Π_1-definable assuming either

- *$(*)$, or*
- *$MM^{++} +$ there is a Woodin cardinal.*

Proof from MM^{++}

Let $Add(\omega_1, 1)$ be the forcing adding a ω_1-Cohen subset. Let g be a generic. In $V[g]$, there are stationary D and $(S_i : i < \omega_2)$ such that

$$\forall S \text{ stationary in } V(\exists i(S_i = S \setminus D)\wedge)).$$

Now it follows that under MM^{++}, there are stationary many uncountable transitive submodel X of $H(\omega_2)$ such that there is stationary $D \subset \omega_1$ with all stationary $S \in X \cap P(\omega_1)$ has the properties that $S \setminus D$ is stationary. We denote this stationary set by S.

Proof from MM^{++}, cont.

Now assume that $\psi(x, A)$ is a Σ_1 definition for stationary set with parameter A. Use stationary tower forcing $P_{<\delta}$ under condition S and targeting Woodin cardinal δ, we have a embedding $j : V \to M \subset V[G]$. Now $H(\omega_2)^V \in j(S)$. By definition of S, M thinks there is stationary $D \subset \omega_1$ with all stationary $S \in H(\omega_2)^V$ has the properties that $S \setminus D$ is stationary. As M is closed under ω_1 sequence, $V[G]$ thinks the same. It follows that $V[G]$ is a stationary preserving extension of V.

Now shooting a club out of D with generic h. It follows that $V[g, h]$ is stationary preserving. Go back to M, D is stationary and hence $\psi(D, A)$. Now $V[g, h] \models \psi(D, A)$ by upward Σ_1-correctness. But D is nonstationary in $V[g, h]$. Hence $V[g, h] \models \exists E(E$ is nonstationary and $\psi(E, A))$. By BMM, this reflect to V. Hence $V \models \exists E(E$ is nonstationary and $\psi(E, A))$. This is a contradiction.

Question

Base on the relationship between $(*)$, MM and MM^{++}, it seems very unlikely that we can reduce MM^{++} to MM in the above theorem. Still we could ask whether Woodin cardinal is necessary.

Question

Does MM^{++} imply NS_{ω_1} is not boldface Π_1-definable?

One other question is the following:

Question

Is NS_{ω_1} Σ_1-complete assuming either

- $(*)$, *or*
- $MM^{++}+$ *certain large cardinal.*