Pseudo-Prikry sequences
(Joint and ongoing work with Spencer Unger)

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Arctic Set Theory
Kilpisjärvi, Finland
January 2017
I: Historical background
Suppose $\kappa$ is a measurable cardinal and $U$ is a normal measure on $\kappa$. There is a forcing poset, which we denote $\mathbb{P}_U$, such that:

1. $\mathbb{P}_U$ is cardinal-preserving;
2. forcing with $\mathbb{P}_U$ adds an increasing sequence of ordinals, $\langle \gamma_i \mid i < \omega \rangle$, cofinal in $\kappa$;
3. $\langle \gamma_i \mid i < \omega \rangle$ diagonalizes $U$, i.e., for all $X \in U$, for all sufficiently large $i < \omega$, $\gamma_i \in X$.

$\mathbb{P}_U$ is known as Prikry forcing (with respect to $U$).

There is now a large class of variations on Prikry forcing, known collectively as Prikry-type forcings, which add diagonalizing sequences to a large cardinal $\kappa$, to a set of the form $\mathbb{P}_\kappa(\lambda)$, or to a sequence of such objects.
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Outside guessing of clubs

Sequences approximating Prikry sequences appear in abstract settings, as well. In these cases, we may not have a normal measure on the relevant cardinal, so we consider sub-filters of the club filter.
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Theorem (Džamonja-Shelah, [3])

Suppose that:

1. $V$ is an inner model of $W$;
2. $\kappa$ is an inaccessible cardinal in $V$ and a singular cardinal of cofinality $\theta$ in $W$;
3. $(\kappa^+)^W = (\kappa^+)^V$;
4. $\langle C_\alpha \mid \alpha < \kappa^+ \rangle \in V$ is a sequence of clubs in $\kappa$. 
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Then, in $W$, there is a sequence $\langle \gamma_i \mid i < \theta \rangle$ of ordinals such that, for all $\alpha < \kappa^+$ and all sufficiently large $i < \theta$, $\gamma_i \in C_\alpha$. 
Generalized outside guessing of clubs

A similar theorem is proven by Gitik [4], and it is extended by Magidor and Sinapova [5], who also prove the following generalization.

Theorem (Magidor-Sinapova, [5])

Suppose that $n < \omega$

1. $V$ is an inner model of $W$;
2. $\kappa$ is a regular cardinal in $V$ and, for all $m \leq n$, $(\kappa + m)_V$ has countable cofinality in $W$;
3. $(\kappa + n + 1)_W = (\kappa + n + 1)_V$;
4. $\langle D_\alpha | \alpha < \kappa + n + 1 \rangle \in V$ is a sequence of clubs in $P_{\kappa + n + 1}$.

Then, in $W$, there is a sequence $\langle x_i | i < \omega \rangle$ of elements of $(P_{\kappa + n + 1})_V$ such that, for all $\alpha < \kappa + n + 1$ and all sufficiently large $i < \omega$, $x_i \in D_\alpha$. 
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4. \( \langle D_\alpha \mid \alpha < \kappa^{n+1} \rangle \in V \) is a sequence of clubs in \( \mathcal{P}_{\kappa}(\kappa^n) \).

Then, in \( W \), there is a sequence \( \langle x_i \mid i < \omega \rangle \) of elements of \( (\mathcal{P}_{\kappa}(\kappa^n))^V \) such that, for all \( \alpha < \kappa^{n+1} \) and all sufficiently large \( i < \omega \), \( x_i \in D_\alpha \).
## Applications

**Theorem (Cummings-Schimmerling in the context of Prikry forcing, [2])**

Suppose that $V$ is an inner model of $W$, $\kappa$ is inaccessible in $V$ and a singular cardinal of countable cofinality in $W$, and $(\kappa^+)^W = (\kappa^+)^V$.

**Theorem (Brodsky-Rinot, [1])**

Suppose that $\lambda$ is a regular, uncountable cardinal, $2^{\lambda} = \lambda^+$, and $P$ is a $\lambda^+$-c.c. forcing notion of size $\leq \lambda^+$. Suppose moreover that, in $V_P$, $\lambda$ is a singular ordinal and $|\lambda| > \text{cf}(\lambda)$.

Then there is a $\lambda^+$-Souslin tree in $V_P$. 
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Suppose that $\lambda$ is a regular, uncountable cardinal, $2^\lambda = \lambda^+$, and $\mathbb{P}$ is a $\lambda^+$-c.c. forcing notion of size $\leq \lambda^+$. Suppose moreover that, in $V^\mathbb{P}$, $\lambda$ is a singular ordinal and $|\lambda| > \text{cf}(\lambda)$.

Then there is a $\lambda^+$-Souslin tree in $V^\mathbb{P}$.
II: Fat trees and pseudo-Prikry sequences
Fat trees

Definition

Suppose \( \kappa \) is a regular, uncountable cardinal, \( n < \omega \), and, for all \( m \leq n \), \( \lambda_m \geq \kappa \) is a regular cardinal. Then

\[
T \subseteq \bigcup_{k \leq n+1} \prod_{m<k} \kappa_m
\]

is a fat tree of type \((\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)\) if:
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1. for all $\sigma \in T$ and $\ell < \text{lh}(\sigma)$, we have $\sigma \upharpoonright \ell \in T$;
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1. for all $\sigma \in T$ and $\ell < \lh(\sigma)$, we have $\sigma \upharpoonright \ell \in T$;
2. for all $\sigma \in T$ such that $k := \lh(\sigma) \leq n$, $\text{succ}_T(\sigma) := \{ \alpha \mid \sigma \frown \langle \alpha \rangle \in T \}$ is $(< \kappa)$-club in $\kappa_k$. 

Lemma

If $C$ is a club in $P_{\kappa}(\kappa + n)$, then there is a fat tree of type $(\kappa, \langle \kappa + n, \kappa + n - 1, \ldots, \kappa \rangle)$ such that, for every maximal $\sigma \in T$, there is $x \in C$ such that, for all $m \leq n$, $\sup(x \cap \kappa + m) = \sigma(n - m)$. 

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1. for all \( \sigma \in T \) and \( \ell < \text{lh}(\sigma) \), we have \( \sigma \upharpoonright \ell \in T \);
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Lemma
If \( C \) is a club in \( \mathcal{P}_\kappa(\kappa^{+n}) \), then there is a fat tree of type \( (\kappa, \langle \kappa^{+n}, \kappa^{+n-1}, \ldots, \kappa \rangle) \) such that, for every maximal \( \sigma \in T \), there is \( x \in C \) such that, for all \( m \leq n \), \( \sup(x \cap \kappa^{+m}) = \sigma(n - m) \).
Outside guessing of fat trees

Theorem

Suppose that:

1. $V$ is an inner model of $W$;
2. in $V$, $\kappa < \lambda$ are cardinals, with $\kappa$ regular;
3. in $W$, $\theta < \theta^+ < |\kappa|$, $\theta$ is a regular cardinal, and there is a $\subseteq$-increasing sequence $\langle x_i \mid i < \theta \rangle$ from $(\mathcal{P}_\kappa(\lambda))^V$ such that $\bigcup_{i<\theta} x_i = \lambda$;
4. $(\lambda^+)^V$ remains a cardinal in $W$;
5. $n < \omega$ and, in $V$, $\langle \lambda_i \mid i \leq n \rangle$ is a sequence of regular cardinals from $[\kappa, \lambda]$ and $\langle T(\alpha) \mid \alpha < \lambda^+ \rangle$ is a sequence of fat trees of type $\langle \kappa, \langle \lambda_0, \ldots, \lambda_n \rangle \rangle$. 
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Then, in $W$, there is a sequence $\langle \sigma_i | i < \theta \rangle$ such that, for all $\alpha < \lambda^+$ and all sufficiently large $i < \theta$, $\sigma_i$ is a maximal element of $T(\alpha)$. 
Proof sketch \((n = 0)\)

Our sequence of fat trees is just a sequence \(\langle C_\alpha \mid \alpha < \lambda^+ \rangle\) of clubs in \(\lambda_0\).
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Work first in \(V\). Fix a sequence \(\langle e_\beta \mid \beta < \lambda^+ \rangle\) such that \(e_\beta : \beta \to \lambda\) is an injection.
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3. for all \(\alpha, \beta < \lambda^+\), there is \(\gamma < \lambda^+\) such that \(f^{C_\alpha}_\beta < f_\gamma\).
Move now to $W$, where we have $\langle x_i \mid i < \theta \rangle$. Define a sequence $\vec{g} = \langle g_\beta \mid \beta < \lambda^+ \rangle$ from $\theta$ to $\lambda_0$ by letting $g_\beta(i) = f_\beta(x_i)$. Note that:

1. $\vec{g}$ is $\ast$-increasing;
2. for all $\gamma \in S_{\lambda^+} > \theta$, there is a club $D_\gamma$ in $\gamma$ such that, for all $\beta \in D_\gamma$, $g_\beta < g_\gamma$;
3. $\theta^+ < \lambda^+ + 3$.

Therefore, $\vec{g}$ has an exact upper bound, i.e. a $\ast$-upper bound $h$ such that, for every $h' < \ast h$, there is $\beta < \lambda^+ + 3$ such that $h' < \ast g_\beta$.

Moreover, we may assume $\text{cf}(h(i)) > \theta$ for all $i < \theta$, so $h : \theta \to \lambda_0$.
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1. \( \vec{g} \) is \( <^* \)-increasing;
2. for all \( \gamma \in S^\lambda_{>\theta} \), there is a club \( D_\gamma \) in \( \gamma \) such that, for all \( \beta \in D_\gamma \), \( g_\beta < g_\gamma \);
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Therefore, $\vec{g}$ has an exact upper bound, i.e. a $<^*$-upper bound $h$ such that, for every $h' <^* h$, there is $\beta < \lambda^+$ such that $h' <^* g_\beta$. Moreover, we may assume $\text{cf}(h)(i) > \theta$ for all $i < \theta$, so $h : \theta \to \lambda_0$. For $i < \theta$, let $\gamma_i = h(i)$. We claim that this works.
Proof sketch (cont.)

If not, then there is $\alpha < \lambda^+$ and an unbounded $A \subseteq \theta$ such that, for all $i \in A$, $\gamma_i \not\in C_\alpha$.
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$$h'(i) = \begin{cases} 0 & \text{if } i \not\in A \\ \max(C_\alpha \cap \gamma_i) & \text{if } i \in A \end{cases}$$
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\( h' < h \), so there is \( \beta < \lambda^+ \) such that \( h' \prec^* g_\beta \). But then there is \( \gamma < \lambda^+ \) such that \( f_\beta^{C_\alpha} < f_\gamma \).
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$$\max(C_\alpha \cap h(i)) < g_\beta(i) < h(i) < \min(C_\alpha \setminus g_\beta(i)) < g_\gamma(i).$$
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$h' < h$, so there is $\beta < \lambda^+$ such that $h' <^* g_\beta$. But then there is $\gamma < \lambda^+$ such that $f_{_\beta}^{C_\alpha} < f_\gamma$. Now, for all sufficiently large $i \in A$, we have

$$\max(C_\alpha \cap h(i)) < g_\beta(i) < h(i) < \min(C_\alpha \setminus g_\beta(i)) < g_\gamma(i).$$

In particular, $h$ is not a $<^*$-upper bound for $\vec{g}$. Contradiction!
III: Diagonal sequences
Definition
Suppose that $\theta$ is a regular cardinal and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals.
## Diagonal clubs

<table>
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<tr>
<th>Definition</th>
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<td>1. A <em>diagonal club in</em> $\vec{\mu}$ is a sequence $\langle C_i \mid i &lt; \theta \rangle$ such that, for all $i &lt; \theta$, $C_i$ is club in $\mu_i$.</td>
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Definition

Suppose that $\theta$ is a regular cardinal and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals.

1. A *diagonal club in* $\vec{\mu}$ is a sequence $\langle C_i \mid i < \theta \rangle$ such that, for all $i < \theta$, $C_i$ is club in $\mu_i$.

2. If $\kappa \leq \mu_0$ is a regular cardinal, then a *diagonal club in* $\mathcal{P}_\kappa(\vec{\mu})$ is a sequence $\langle D_i \mid i < \theta \rangle$ such that, for all $i < \theta$, $D_i$ is club in $\mathcal{P}_\kappa(\mu_i)$.
Diagonal ordinal sequences

Theorem

Suppose that:

1. \( V \) is an inner model of \( W \);
2. in \( V \), \( \mu \) is a singular cardinal of cofinality \( \theta \);
3. there is \( \kappa < \mu \) such that every \( V \)-regular cardinal in \( [\kappa, \mu) \) has cofinality \( \theta \) in \( W \);
4. in \( W \), \( (\mu^+) \) remains a cardinal and \( \theta^{+2} < |\mu| \).
Diagonal ordinal sequences

Theorem

Suppose that:

1. $V$ is an inner model of $W$;
2. in $V$, $\mu$ is a singular cardinal of cofinality $\theta$;
3. there is $\kappa < \mu$ such that every $V$-regular cardinal in $[\kappa, \mu)$ has cofinality $\theta$ in $W$;
4. in $W$, $(\mu^+)^V$ remains a cardinal and $\theta^{+2} < |\mu|$.

Then there are:

- an increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle \in V$, cofinal in $\mu$;
Diagonal ordinal sequences

Theorem

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2. in $V$, $\mu$ is a singular cardinal of cofinality $\theta$;
3. there is $\kappa < \mu$ such that every $V$-regular cardinal in $[\kappa, \mu)$ has cofinality $\theta$ in $W$;
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Then there are:

- an increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle \in V$, cofinal in $\mu$;
- a function $g \in \prod_{i<\theta} \mu_i$ in $W$.
Theorem

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such that, for every $\langle C_i \mid i < \theta \rangle \in V$ that is a diagonal club in $\vec{\mu}$, for all sufficiently large $i < \theta$, $g(i) \in C_i$. 
Generalized diagonal sequences

Theorem

Suppose that:

1. $V$ is an inner model of $W$;
2. in $V$, $\text{cf}(\mu) = \theta < \kappa = \text{cf}(\kappa) < \mu$ are cardinals, with $\mu$ strong limit;
3. in $V$, $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals, cofinal in $\mu$, with $\kappa \leq \mu_0$;
4. in $W$, there is a $\subseteq$-increasing sequence $\langle x_i \mid i < \theta \rangle$ from $(\mathcal{P}_\kappa(\mu))^V$ such that $\bigcup_{i<\theta} x_i = \mu$;
5. in $W$, $(\mu^+)^V$ remains a cardinal and $\mu \geq 2^\theta$;
6. in $V$, $\langle \vec{D}(\alpha) \mid \alpha < \mu^+ \rangle$ is a sequence of diagonal clubs in $\mathcal{P}_\kappa(\vec{\mu})$. 
Generalized diagonal sequences

Theorem

Suppose that:

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4. in $W$, there is a $\subseteq$-increasing sequence $\langle x_i \mid i < \theta \rangle$ from $(P_\kappa(\mu))^V$ such that $\bigcup_{i<\theta} x_i = \mu$;
5. in $W$, $(\mu^+)^V$ remains a cardinal and $\mu \geq 2^\theta$;
6. in $V$, $\langle \vec{D}(\alpha) \mid \alpha < \mu^+ \rangle$ is a sequence of diagonal clubs in $P_\kappa(\vec{\mu})$.

Then, in $W$, there is $\langle y_i \mid i < \theta \rangle$ such that, for all $\alpha < \mu^+$ and all sufficiently large $i < \theta$, $y_i \in D(\alpha)_i$. 
References

Ari Meir Brodsky and Assaf Rinot, More notions of forcing add a Souslin tree, Preprint.


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Thank you!