Homogeneous spaces and Wadge theory

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joint work with Raphaël Carroy and Andrea Medini

Arctic Set Theory Workshop 4

- How I got interested in general topology
- Our main tool: Wadge theory
- The beauty of Hausdorff operations
- Putting everything together
- Open questions and future goals

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All our topological spaces will be separable and metrizable. A *homeomorphism* between two spaces X and Y is a bijective continuous function f such that the inverse f^{-1} is continuous as well.

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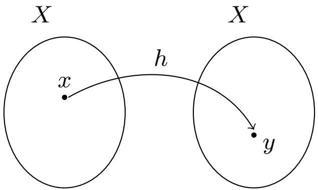

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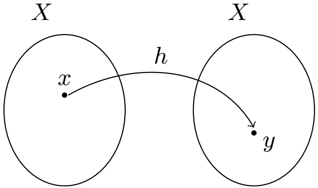

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Fact

X is a locally compact zero-dimensional homogeneous space iff X is discrete, $X \approx 2^\omega$, or $X \approx \omega \times 2^\omega$.

We will therefore focus on non-locally compact (equivalently, nowhere compact) zero-dimensional homogeneous spaces.

h-homogeneity

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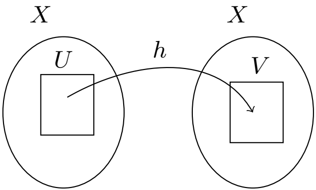

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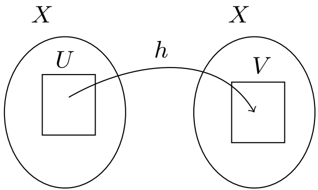

Examples of h-homogeneous spaces: $\mathbb{Q}, 2^\omega, \omega^\omega$, any product of zero-dimensional h-homogeneous spaces (Medini, 2011)

h-homogeneity versus homogeneity

Theorem (Folklore)

Assume that X is a zero-dimensional space. If X is h-homogeneous, then X is homogeneous.

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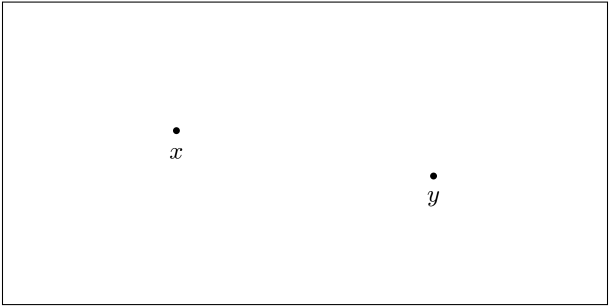

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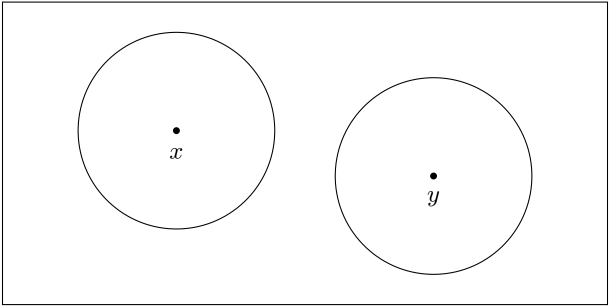

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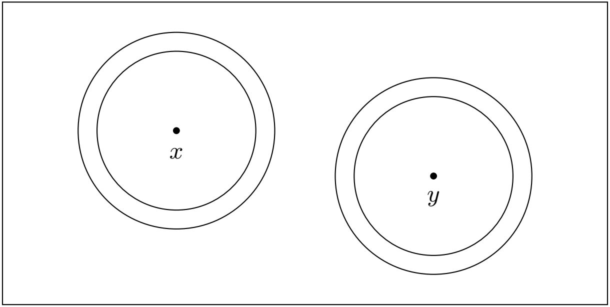

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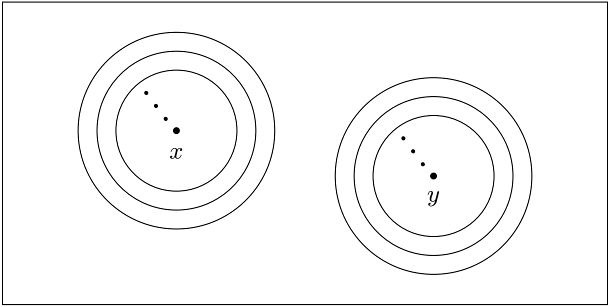

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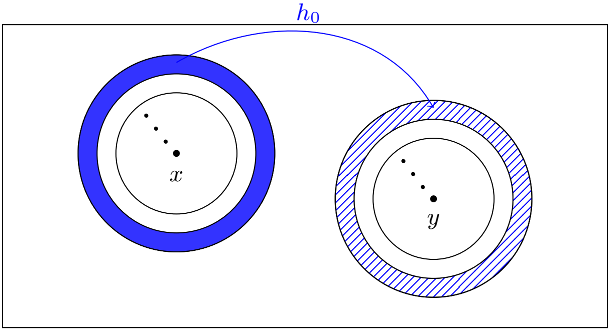

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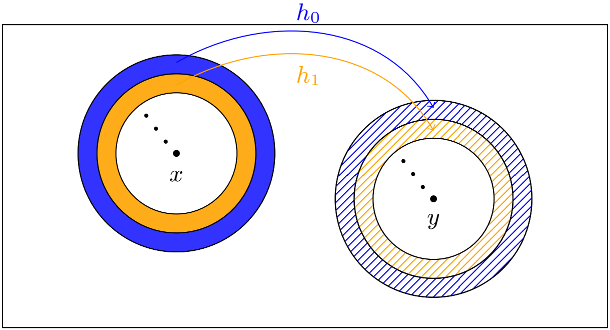

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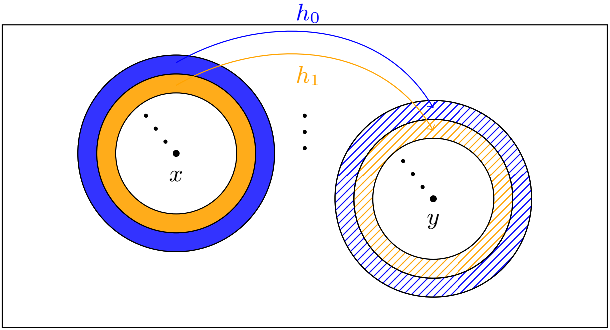

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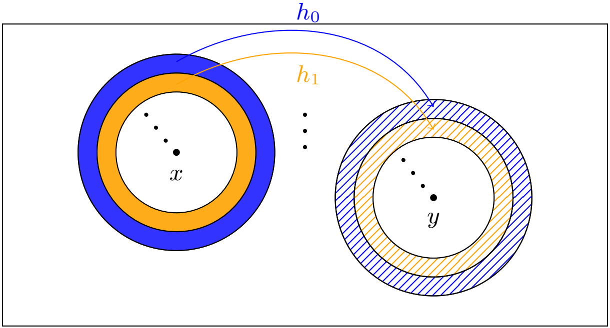

Now $\bigcup_{n\in\omega}(h_n \cup h_n^{-1})$ can be extended to a homeomorphism $h\colon X \to X$ such that $h(x) = y$ and $h^{-1}(y) = x$. $\square$

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But the converse does not hold in general.

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Question

Can we say more under projective determinacy (PD) or AD?

Yes!

Theorem (Carroy – Medini – M)

- (PD) *A projective non-locally-compact subspace of* 2^ω *is homogeneous if and only if it is h-homogeneous.*
- (AD+DC) *A non-locally-compact subspace of* 2^ω *is homogeneous if and only if it is h-homogeneous.*

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Main ideas to extend van Engelen's result beyond Borel spaces:

- The proof relies on an extremely refined classification of subsets of a Polish zero-dimensional space: the Wadge quasi-order.
- Need an understanding of the induced Wadge hierarchy in ω^ω beyond Borel classes.
- Need a method of transferring these results from ω^ω to 2^ω.
- Want to apply a theorem of Steel in 2^ω.

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Wadge reducibility

Definition

For $A \subseteq X$ and $B \subseteq Y$, we say that A *Wadge (or continuously) reduces to* B if there is a continuous function $f\colon X \to Y$ such that $x \in A \Leftrightarrow f(x) \in B$. For $X = Y = \omega^\omega$, we write $A \leq_W B$.

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This yields (under $\mathrm{AD} + \mathrm{DC}$) a very nice hierarchy of subsets of ω^ω.

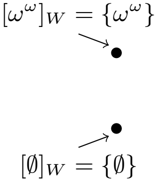
Theorem (Wadge, Martin – Monk)

Assuming AD *and* DC, $\leq_W$ *satisfies the* semi-well-ordering principle*:*

- *(Wadge) For any $A, B \subseteq \omega^\omega$ either $A \leq_W B$ or $B \leq_W \omega^\omega \setminus A$.*
- *(Martin – Monk) The quasi-order $\leq_W$ is well-founded.*

Picture of the Wadge hierarchy

Recall: $A \leq_W B$ iff there is a continuous function $f\colon \omega^\omega \to \omega^\omega$ such that $x \in A \Leftrightarrow f(x) \in B$. Write $[A]_W = \{B \mid B \leq_W A\}$.



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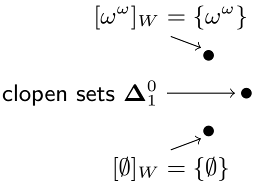

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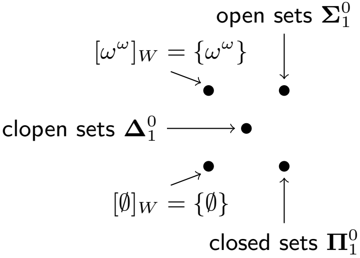

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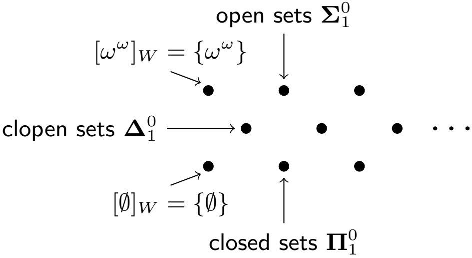

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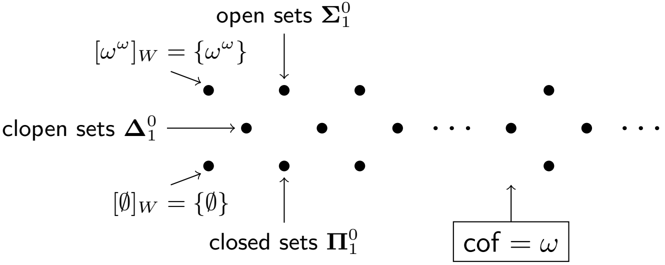

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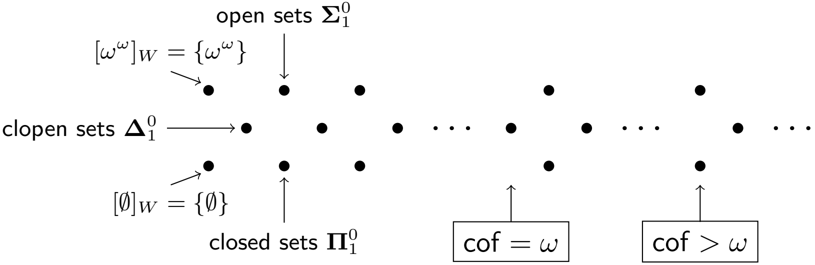

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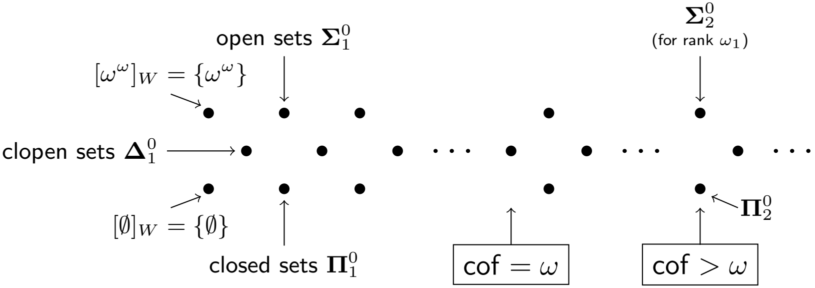

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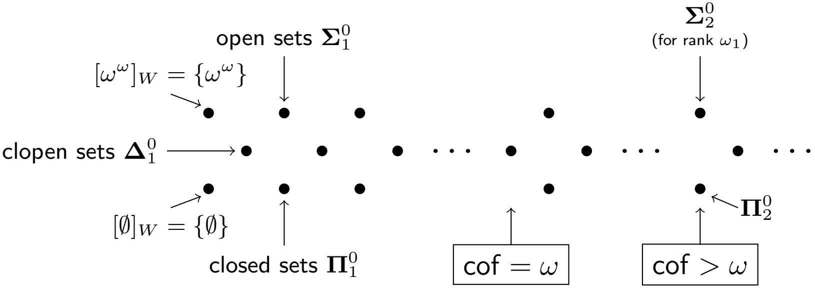

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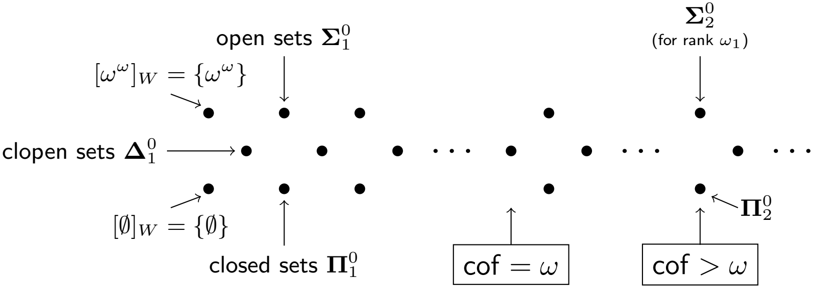

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- The length of the full Wadge hierarchy is

$$\Theta = \sup\{\alpha \mid \exists f (f\colon \mathbb{R} \twoheadrightarrow \alpha)\}.$$

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Definition (Louveau – Saint-Raymond)

The *level* of a pointclass Γ is $\ell(\Gamma) = \sup\{\alpha < \omega_1 \mid \Gamma = \mathrm{PU}_\alpha(\Gamma)\}$.

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Given a pointclass Γ and a countable ordinal α, the α-expansion $\Gamma^{(\alpha)}$ is the class of all preimages of elements of Γ by $\Sigma^0_{1+\alpha}$-measurable functions.

Theorem (Expansion Theorem, Saint-Raymond)

$(\mathrm{AD}+\mathrm{DC})$ *Let Γ be a non-self-dual Wadge pointclass and α a countable ordinal. Then the following are equivalent:*

1. $\ell(\Gamma) \geq \alpha$,
2. $\Gamma = \Lambda^{(\alpha)}$ *for some non-self-dual Wadge class Λ.*

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Hausdorff operations

Given $D \subseteq 2^\omega$ and a sequence of sets $\vec{A} = A_0, A_1, \cdots$, define a set $\mathcal{H}_D(\vec{A})$ as follows:

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- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcap_{i \in I} D_i$.

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$$x \in \mathcal{H}_D(\vec{A}) \leftrightarrow \{i \in \omega \mid x \in A_i\} \in D$$

We call $\mathcal{H}_D$ a *Hausdorff operation*.

Some properties:

- If $D = \{(1)^\omega\}$, then $\mathcal{H}_{\{(1)^\omega\}}(\vec{A})$ is the countable intersection $A_0 \cap A_1 \cap \dots$.
- If D_n is the set of all $s\colon \omega \to 2$ such that $s(n) = 1$ ($s(k)$ for $k \neq n$ arbitrary), then $\mathcal{H}_{D_n}(A_0, A_1, \dots) = A_n$.
- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \dots) = \mathcal{H}_D(A_0, A_1, \dots)$, where $D = \bigcap_{i \in I} D_i$.
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- $X \setminus \mathcal{H}_D(A_0, A_1, \dots) = \mathcal{H}_{2^\omega \setminus D}(A_0, A_1, \dots)$.

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In particular, every combination of unions, intersections, and complements can be expressed as a Hausdorff operation.

From ω^ω to 2^ω

For $D \subseteq 2^\omega$, define $\Gamma_D(X)$ as the collection of all subsets of X that are the result of applying $\mathcal{H}_D$ on open sets of X.

Lemma (Relativization Lemma)

Given two spaces X and Y, and $D \subseteq 2^\omega$.

- *If $f\colon X \to Y$ is continuous and $A \in \Gamma_D(Y)$ then $f^{-1}[A] \in \Gamma_D(X)$.*
- *Assume $Y \subseteq X$, then $A \in \Gamma_D(Y)$ if and only if there is $\tilde{A} \in \Gamma_D(X)$ such that $A = \tilde{A} \cap Y$.*

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Theorem (Addison, van Wesep)

$(\mathrm{AD} + \mathrm{DC})$ Γ *is a non-self-dual Wadge class in 2^ω iff $\Gamma = \Gamma_D(2^\omega)$ for some $D \subseteq 2^\omega$.*

This in fact works for all Polish zero-dimensional spaces X instead of 2^ω.

Steel's theorem ...

Let $A \subseteq 2^\omega$ and let Γ be a pointclass. For $s \in 2^{<\omega}$, let $[s] = \{x \in 2^\omega \mid s \subseteq x\}$. Say that

- Γ is *reasonably closed* if it is closed under $\cap\boldsymbol{\Pi}^0_2$ and $\cup\boldsymbol{\Sigma}^0_2$.

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Theorem (Steel, 1980)

$(\mathrm{AD} + \mathrm{DC})$ *Let Γ be a reasonably closed Wadge class of subsets of 2^ω. Take $X, Y \subseteq 2^\omega$ such that*

- *both X and Y are everywhere properly Γ, and*
- *either they are both meager, or both Baire.*

Then there is a homeomorphism $h\colon 2^\omega \to 2^\omega$ such that $h[X] = Y$.

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- So if $X \subseteq 2^\omega$ is homogeneous, $\Gamma = [X]_W$ is reasonably closed and X is everywhere properly Γ, then for any $X \cap [s]$ for $s \in 2^{<\omega}$ we can apply Steel's theorem.
- I.e. X and $X \cap [s]$ are homeomorphic. A result of Terada (1993) yields that X is h-homogeneous.

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Then there is a homeomorphism $h\colon 2^\omega \to 2^\omega$ *such that* $h[X] = Y$.

Corollary

$(\mathrm{AD}+\mathrm{DC})$ *If* $X \subseteq 2^\omega$ *is homogeneous, generates a reasonably closed Wadge class* Γ *and* X *is everywhere properly* Γ, *then* X *is h-homogeneous.*

Good Wadge classes are reasonably closed...

A pointclass Γ in 2^ω is *good* if

- $\Delta D_\omega(\boldsymbol{\Sigma}^0_2) \subseteq \Gamma$,
- Γ is non-self-dual, and
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Let $X \subseteq 2^\omega$. If $X \notin \Delta \mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$ is homogeneous, then $[X]_W$ is good.

(Recall: The case $X \in \Delta \mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$ was analyzed by van Engelen already.)

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$(\mathrm{AD}{+}\mathrm{DC})$ *A non-locally-compact subspace of 2^ω is homogeneous if and only if it is h-homogeneous.*

- How I got interested in general topology
- Our main tool: Wadge theory
- The beauty of Hausdorff operations
- Putting everything together
- **Open questions and future goals**

Van Engelen's characterization of Borel filters

As topological groups, all filters are homogeneous, but there is a characterization for Borel spaces.

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Theorem (van Engelen, 1994)

Let X be a zero-dimensional Borel *space. Then the following are equivalent*

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- *X is homogeneous, meager, homeomorphic to its square, and not locally compact.*

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Question

Can this be generalized to

- all zero-dimensional projective spaces (under PD), or
- all zero-dimensional spaces (under $\mathrm{AD} + \mathrm{DC}$)?

"The Wadge hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity [...]"

(Andretta and Louveau in the Introduction to Cabal Part III)

Thank you for your attention!