Second order logic and set theory

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Abstract

Both second order logic and set theory can be used as a foundation for mathematics, that is, as a formal language in which propositions of mathematics can be expressed and proved. We take it upon ourselves in this paper to compare the two approaches, second order logic on one hand and set theory on the other hand, evaluating their merits and weaknesses. We argue that we should think of first order set theory as a very high order logic.

1 Axiomatizations

Towards the end of the 19th century mathematics had become so evolved that questions about its foundations emerged. These questions were mainly about calculus and the use of set theory. Richard Dedekind had already in 1858 sensed that something was lacking:

> As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences.
>
> ... that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should

> find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis.
>
> ... It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity.[1] [4, page 9].

This was the impetus that led Dedekind in 1888 to introduce his famous axiomatisation of number theory [5] and other concepts such as the continuum using what is now known as the method of Dedekind cuts, another typically second order concept.

Dedekind's was not an axiomatisation of number theory in the modern sense, but rather a structural analysis of natural numbers. It was Peano who in 1889 formulated axioms of number theory in the modern sense. Be it Dedekind's analysis or Peano's axiomatisation, the main non-trivial axiom (or principle) is the Induction Axiom:

$$\forall X([X(0) \wedge \forall y(X(y) \to X(y+1))] \to \forall y X(y)), \tag{1}$$

which is still today the paradigmatic example of a *second order* sentence. It is called a second order sentence because the quantifier $\forall X$ quantifies over *sets* X of numbers rather than over individuals i.e. numbers.

Another typical second order axiom is the Completeness Axiom for linear orders

$$\begin{aligned}&\forall X([\exists y X(y) \wedge \exists z \forall y(X(y) \to y \leq z)] \to \\ &\exists z \forall y(\exists u(X(u) \wedge y \leq u) \vee z \leq y)),\end{aligned} \tag{2}$$

which says that every non-empty set of elements of the linear order which has an upper bound has a least upper bound. This is second order because it has the bound variable X which ranges over all subsets of the domain of the linear order, rather than just over the individuals that are linearly ordered.

Second order logic was quite dominating in early axiomatisations of mathematical concepts, although they were not called "second order" at the time. Hilbert's 1898 axiomatisation of geometry [10] was second order, as was Huntington's 1902 axiomatisation of the continuum as an ordered set [13]. Finally, Zermelo's axiomatisation of set theory [24] was also second order, although the later Zermelo-Fraenkel axiomatization of set theory is first order.

[1] English translation [6]

2 Categoricity

The early users of second order logic, such as Hilbert, Ackermann, Bernays, Huntington and Veblen, paid substantial attention to a concept that had a lasting effect on our understanding of second order logic, namely categoricity. An axiom system Γ is said to be *categorical* if it has, up to isomorphism, at most one model[2], i.e.

$$\forall \mathfrak{M}, \mathfrak{M}'((\mathfrak{M} \models \Gamma \wedge \mathfrak{M}' \models \Gamma) \rightarrow \mathfrak{M} \cong \mathfrak{M}'). \tag{3}$$

Logicians went to pains to show, usually successfully, that their (second order) axiom systems were categorical. In particular, the ordered structure of the natural numbers, the complete separable Archimedian field of reals numbers, the complex field, as well as practically all commonly occurring mathematical structures have a categorical second order axiomatisation.

A nice property of a categorical axiom system is (semantical) *completeness*: Any sentence ϕ in the language in which the axiom system is written is decided by Γ in the following sense. Either every model of Γ satisfies ϕ or every model of Γ satisfies $\neg\phi$. In other words, either ϕ or $\neg\phi$ is a (semantical) logical consequence of Γ. The reason is very simple: Γ has, up to isomorphism, only one model M. Since isomorphism preserves truth in second order logic, ϕ or $\neg\phi$ is a (semantical) logical consequence of Γ according to whether ϕ is true in M or false in M.

If ϕ is a second order sentence we define

$$\mathrm{Mod}(\phi) = \{\mathfrak{M} : \mathfrak{M} \models \phi\} \tag{4}$$

If ϕ characterises $\mathfrak{M}$ up to isomorphism, then, up to isomorphism,

$$\mathrm{Mod}(\phi) = \{\mathfrak{M}\}.$$

A kind of "formalist" view of second order logic would insist that it is *only* ϕ that exists in any meaningful sense, while $\mathfrak{M}$ itself, the structure that we are really interested in, cannot (according to this view) be meaningfully claimed to exist, as it is infinite. The reason why $\mathrm{Mod}(\phi)$ seems to have a

[2]A *vocabulary* consists of a set of constant, relation and function symbols, each with their own arity. A *first order structure (or model)* for a vocabulary L is a non-empty set endowed with interpretations of the constant, relation and function symbols of L. We use $\mathfrak{M}$, $\mathfrak{N}$, etc to denote first order structures. The truth of a sentence ϕ of first order logic, or any other logic, in a model $\mathfrak{M}$ is denoted $\mathfrak{M} \models \phi$.

stronger claim to "existence" than $\mathfrak{M}$ itself is that it, unlike $\mathfrak{M}$, is completely determined by the finite string of symbols ϕ. One may even take the next step and declare that the "existence" of $\mathfrak{M}$ *means* simply the existence of ϕ combined with the fact that ϕ is categorical. What is problematic with this line of thinking is that to check whether ϕ is categorical one has to appeal to the the infinity of infinite models of ϕ.

The project of axiomatising mathematical structures with second order sentences was so successful in the first quarter of the 20th century that Carnap even proposed that *every* mathematical structure is determined, up to isomorphism, by its second order theory. There are trivial cardinality reasons why this could not possibly hold for models of size 2^ω and bigger. There are simply not enough second order theories in comparison with the number of non-isomorphic models. However, the proposal of Carnap is trivially true for finite models, and not entirely unreasonable for countable models. In fact Ajtai [1] showed that it is consistent with ZFC, provided ZFC itself is consistent, that any two countable structures in a finite vocabulary that satisfy the same second order sentences are isomorphic. Ajtai also showed that it is consistent with ZFC, provided ZFC itself is consistent, that there are two countable structures in a finite vocabulary that satisfy the same second order sentences without being isomorphic. Thus Carnap's conjecture (for countable models) is independent of ZFC.

3 Incompleteness

When Gödel proved his Completeness and Incompleteness Theorems for *first* order logic, a puzzling situation existed for a while concerning *second* order logic: Let θ be the second order axiomatisation of the natural numbers, using (1). The sentence θ is categorical hence complete, that is, for every sentence ϕ of the language of number theory

$$\theta \models \phi \text{ or } \theta \models \neg\phi. \tag{5}$$

On the other hand, Gödel's First Incompleteness Theorem says that if an effective deductive system is given for (first or) second order logic, then there are sentences ϕ that

$$\theta \nvdash \phi \text{ and } \theta \nvdash \neg\phi. \tag{6}$$

The puzzling situation arose when this was contrasted with Gödel's Completeness Theorem which states that for the Hilbert-Ackermann [11] notion of derivability for first order logic

$$\theta \vdash \phi \text{ iff } \theta \models \phi. \tag{7}$$

The conditions (5), (6) and (7) seem to be in utter contradiction with each other. Moreover, Gödel's theorem seemed to be quite general, especially the Incompleteness Theorem, so it was not clear which one of (5)-(7) is the one that fails, for one has to fail. What failed turned out to be (7): for the kind of semantics that (5) is based on there cannot be an effective deductive system which would satisfy (7). The condition (6) is not a problem as it only talks about derivability. By changing the semantics to so-called Henkin semantics we obtain (7) but then we lose (5), or we can keep our semantics and thereby maintain (5), but then we have to give up on (7). Let us first focus on the latter choice, i.e. emphasising (5), having (6) and abandoning (7). This is the topic of the next section.

4 Proof theory of second order logic

The deductive system of second order logic, presented first explicitly in Hilbert-Ackermann [11], is based on the obvious extension of axioms and rules of first order logic added with the Comprehension Axioms, defined as follows: Suppose $\phi(x_1, \ldots, x_n)$ is a second order formula with $x_1, \ldots, x_n$ among its free individual variables and the second order variable R is *not* free in ϕ. Then the following formula is a Comprehension Axiom:

$$\exists R \forall x_1 \ldots x_n (\phi(x_1, \ldots, x_n) \leftrightarrow R(x_1, \ldots, x_n)). \tag{8}$$

A priori (8) might appear very strong as it stipulates the existence of a relation. However, such an R exists simply because it is definable:

$$R = \{(a_1 \ldots a_n) \in M^n : \mathfrak{M} \models \phi(a_1, \ldots, a_n)\}.$$

Hilbert-Ackermann [11] add to the proof system of second order logic also two different Axioms of Choice. The first is

$$AC : \forall x_1 \ldots x_m \exists x_{m+1} R(x_1, \ldots, x_{m+1})$$
$$\rightarrow \exists F \forall x_1 \ldots \forall x_m R(x_1, \ldots, x_m, F(x_1 \ldots x_m))$$

and the second is

$$AC' : \forall x_1 \dots x_m \exists F \varphi \to \exists F' \forall x_1 \dots x_m \varphi',$$

where the formula φ' is obtained from the formula φ by replacing everywhere $F(t_1, \dots, t_k)$ by $F'(t_1 \dots t_k, x_1, \dots, x_m)$.

The first Axiom of Choice AC says intuitively that if a set

$$\{a_{n+1} \in M : \mathfrak{M} \models R(a_1, \dots, a_n, a_{n+1})\}$$

is non-empty, we have a function which picks an element a_{n+1} from the set, using the parameters $a_1, \dots, a_n$ as arguments. The second Axiom of Choice AC' is a kind of second order choice: We have for all $a_1, \dots, a_n$ a function F with the property ϕ, so in fact F depends on $a_1, \dots, a_n$ and should be denoted $F_{a_1 \dots a_n}$. What AC' says now is that we can collect the functions $F_{a_1 \dots a_n}$ together to form just one function F' of higher arity such that

$$F'(a_1, \dots, a_{n+1}) = F_{a_1 \dots a_n}(a_{n+1}).$$

Although F' appears to be definable, this is not the case, as the mapping

$$(a_1, \dots, a_n) \mapsto F_{a_1 \dots a_n}$$

may be highly undefinable.

Both AC and AC' are non-trivial axioms, just as Axiom of Choice of set theory is. The power of these axioms in second order logic is weaker than the power of Axiom of Choice in set theory, because second order logic is limited to the domain of the model. Often in set theory one applies the Axiom of Choice to an auxiliary larger set obtained by means of the Power Set Axiom, or even a combination of the Power Set Axiom and the Replacement Axiom. Nothing like that works in second order logic. Still AC and AC' are essential tools of second order logic.

The given axiom system of second order logic is designed so that it is sound, i.e. preserves truth. But it does not satisfy the Completeness Theorem, for as we already observed in Section 3, with the method of Gödel's Incompleteness Theorem we can easily construct a second order number theoretic sentence ϕ, a "Gödel sentence", such that (6) holds for the second order characterization θ of natural numbers. We can use stronger theories such as ZFC to prove the statement ϕ, so its truth-value is not a puzzle. However, we can write in the vocabulary of the reals also a second order sentence ψ

which is true of the reals if and only if the Continuum Hypothesis (CH) holds. Nobody knows at present whether CH is true or not and in particular, the axioms of second order logic are not able to decide CH. According to the classical point of view the reals satisfy CH or its negation, we just do not know which. We only know that the current axioms of second order logic are not sufficiently strong to decide it. The same is true of Souslin's Hypothesis and many other set-theoretic statements.

The incompleteness of the axioms of second order logic with respect to the semantics $\mathfrak{M} \models \phi$ is somewhat similar to the incompleteness of the Zermelo-Fraenkel axioms of set theory, ZFC. There are many mathematically meaningful statements that cannot be decided either by ZFC or by means of the given axioms of second order logic, as we will see below when we investigate Henkin models. In both cases the proof of the impossibility uses Cohen's method of forcing. In addition there is the consequence of Gödel's Incompleteness Theorem: some number theoretic statements cannot be decided. This is true both in set theory with respect to ZFC, and in second order logic with respect to any sentence characterising the natural numbers.

There is a quick technical way to derive the failure of completeness: The set of Gödel numbers of second order sentences that are (semantic) logical consequences of the categorical characterisation of the natural numbers, i.e. are true in the natural numbers, is non-arithmetical by Tarski's Undefinability of Truth Theorem, while the set of theorems of any effective axiom system is recursively enumerable. So these two sets cannot be the same.

We now turn to the alternative of changing the semantics to so-called Henkin semantics in order to obtain (7) even if we lose (5).

5 Henkin models

The problem of the incompleteness of the Hilbert-Ackermann deductive system for second order logic was solved by Henkin [9] in a decisive and bold way which has become a standard technique in other logics too. He modified the concept of a structure by allowing the second order variables to range over a *limited* set of possibilities, rather than over all possibilities. This is somewhat analogous to the situation in set theory, where we cannot (yet) decide the truth value of the CH but we can study models of set theory in which the set variables range over a limited collection of sets rather than over all sets. Technically speaking these models are transitive sets that satisfy the

ZFC-axioms. Some of them satisfy the CH, some don't.

A *Henkin model* is a pair $(\mathfrak{M}, \mathcal{G})$, where $\mathfrak{M}$ is a usual first order structure and $\mathcal{G}$ is a set of subsets, relations and functions on M. Note that we do not limit the subsets to be unary, and we allow relations as well as functions. In *monadic second order logic* $\mathcal{G}$ is assumed to contain unary predicates only.

We can readily define the concept

$$(\mathfrak{M}, \mathcal{G}) \models \phi$$

for second order ϕ by induction on ϕ by stipulating

$$(\mathfrak{M}, \mathcal{G}) \models \exists R \phi(R)$$

$$\iff$$

$$(\mathfrak{M}, \mathcal{G}) \models \phi(S) \text{ for some interpretation } S \in \mathcal{G} \text{ of } R.$$

The intended meaning of "$\exists R$" is that there is a relation R on the domain of the model, and any relation can serve as R, even the most complicated and abstract R, even an R which is by no means definable. Naturally this intended meaning of "$\exists R$" can in some cases go deeply into set-theoretical questions about existence of this or that kind of a relation on a given set. The situation with $(\mathfrak{M}, \mathcal{G}) \models \exists R \phi(R)$ is different. We only ask whether there is a relation R in $\mathcal{G}$. Of course, $\mathcal{G}$ may be complicated itself, such as $\mathcal{P}(M)$, but it can also consist of just the kind of relations we are interested in and that we can deal with. Truth in a full model is not absolute in the sense of set theory, but truth in a given Henkin model can easily seen to be absolute. [3]

There is an additional assumption on Henkin models: We assume that every Henkin model satisfies the Axioms of Comprehension, and the Axioms of Choice. As a consequence of the Axioms of Comprehension, in every Henkin model $(\mathfrak{M}, \mathcal{G})$ we have $M \in \mathcal{G}$ and also all the distinguished constants, relations and functions of $\mathfrak{M}$, i.e. the interpretations of the symbols in the vocabulary of $\mathfrak{M}$, are in $\mathcal{G}$.

We take the convention that if $\mathcal{G}$ is *any* set whatsoever, however big or small, we define $(\mathfrak{M}, \mathcal{G})$ to be $(\mathfrak{M}, \mathcal{G} \upharpoonright M)$, where $\mathcal{G} \upharpoonright M$ is defined as follows:

[3] A formula $\phi(x_1, \ldots, x_n)$ of set theory is *absolute* if for any two models M and M' of ZFC, M' an end-extension of M (e.g. both transitive and $M \subseteq M'$), if $a_1, \ldots, a_n \in M$, then $M \models \phi(a_1, \ldots, a_n)$ if and only if $M' \models \phi(a_1, \ldots, a_n)$.

$$
\begin{aligned}
\mathcal{G} \restriction M &= \{A \cap M : A \in \mathcal{G}\} \cup \\
&\quad \{A \cap M^n : A \in \mathcal{G} \text{ an } n\text{-ary relation}\} \cup \\
&\quad \{f \restriction M^n : f \in \mathcal{G} \text{ an } n\text{-ary function}\},
\end{aligned}
$$

provided that then $(\mathfrak{M}, \mathcal{G})$ satisfies the Comprehesnion Axioms, i.e. is a Henkin model. So if $\mathcal{G}$ is a set such that $(\mathfrak{M}, \mathcal{G})$ is a Henkin model, then M and all the structural elements of $\mathfrak{M}$ are in $\mathcal{G}$. We write

$$\mathfrak{M} \in \mathcal{G}$$

whenever this happens, in order to indicate that everything that $\mathfrak{M}$ is built up from is in $\mathcal{G}$.

There is a canonical family of Henkin models, namely the *full* Henkin models $(M, \mathcal{G})$, where $\mathcal{G}$ is the set $\mathcal{P}^*(M)$ of *all* subsets, relations and functions on M. Since $\mathcal{P}^*(M)$ is uniquely determined by $\mathfrak{M}$ we denote $(\mathfrak{M}, \mathcal{P}^*(M))$ simply by $\mathfrak{M}$. So a non-Henkin model $\mathfrak{M}$ can be thought of as a Henkin model by identifying it with $(\mathfrak{M}, \mathcal{P}^*(M))$. We call semantics based on Henkin models the *Henkin semantics*, and the original semantics based on full models the *full semantics.*

The point of Henkin models is that they satisfy the Completeness Theorem:

Theorem 1 ([9]) *The following are equivalent for all second order sentences θ and ϕ:*

1. $\theta \vdash \phi$

2. *Every Henkin model of θ is a Henkin model of ϕ.*

Henkin's original proof was for type theory. A modern proof can be written along the lines of a "Henkin-style" proof of the Completeness Theorem of first order logic. Respectively, we get the Compactness Theorem[4]as well as the downward and the upward versions of the Löwenheim-Skolem Theorem.[5]

The way to think about Henkin models is that they fill the gaps left by full models, as irrational numbers fill the gaps left by rational numbers.

[4]Every set of second order sentences, every finite subset of which has a Henkin model, has itself a Henkin model.

[5]If a second order sentence has a Henkin model $(\mathfrak{M}, \mathcal{G})$ with M infinite, it has Henkin models $(\mathfrak{M}, \mathcal{G})$ with M any infinite cardinal.

They are needed in order to get a smooth theory of second order logic. They manifest "paradoxical" phenomena that we do not see among full models. For example we get non-standard models of number theory, countable models of the axioms of real numbers, etc. The categorical sentences characterising mathematical structures among full models now suddenly have also other "models", namely Henkin models, and they come in all infinite cardinalities.

If ϕ is a second order sentence we define

$$Mod_H(\phi) = \{(\mathfrak{M}, \mathcal{G}) : (\mathfrak{M}, \mathcal{G}) \models \phi\} \tag{9}$$

Obviously, $\mathrm{Mod}(\phi) \subseteq Mod_H(\phi)$. If ϕ characterises $\mathfrak{M}$ up to isomorphism, then

$$\mathfrak{M} \in \mathrm{Mod}(\phi) \subseteq Mod_H(\phi).$$

In fact, in all non-trivial cases[6] $Mod_H(\phi) \neq \{\mathfrak{M}\}$. It is instructive to think of $Mod_H(\phi)$ as a "cloud" around $\mathfrak{M}$, because the Henkin models in $Mod_H(\phi)$ in a sense "blur" our image of $\mathfrak{M}$. One of them is the "real" $\mathfrak{M}$ but by means of deductions in second order logic we cannot tell which. Because of the inherent weakness of formal systems, going back to Skolem and Gödel, infinite structures are shrouded by Henkin models and cannot be gotten perfectly into focus by means of deductions.

Categoricity is lost in the passage from full models to Henkin models. No infinite Henkin model is characterisable in second order logic, i.e. for no $(\mathfrak{M}, \mathcal{G})$ with infinite M is there second order θ such that

$$(\mathfrak{M}, \mathcal{G}) \models \theta \text{ and } \forall (\mathfrak{M}', \mathcal{G}')((\mathfrak{M}', \mathcal{G}') \models \theta \to \mathfrak{M}' \cong \mathfrak{M}) \tag{10}$$

Respectively, no second order sentence θ with an infinite Henkin model is categorical even in the weak sense that

$$\forall (\mathfrak{M}, \mathcal{G}), (\mathfrak{M}', \mathcal{G}')(((\mathfrak{M}, \mathcal{G}) \models \theta \wedge (\mathfrak{M}', \mathcal{G}') \models \theta) \to \mathfrak{M} \cong \mathfrak{M}'). \tag{11}$$

Both facts are consequences of the upward Löwenheim-Skolem Theorem.[7]

However, in the next section we introduce a form of categoricity which holds also for Henkin models in interesting ways.

[6]By the upward Löwenheim-Skolem Theorem, if M is infinite, $Mod_H(\phi)$ has Henkin models of all infinite cardinalities.

[7]By the upward Löwenheim-Skolem Theorem any second order sentence with an infinite Henkin model has Henkin models in all infinite cardinalities.

It is important to bear in mind that Henkin models $(\mathfrak{M}, \mathcal{G})$ are not in *themselves* interesting mathematical structures. Apart from the full model, none of them have any claim to fame. Their importance is in their auxiliary role in explaining what can be proved and what cannot. This is in harmony with the fact that second order sentences are not categorical in Henkin semantics.

It is true that some are more interesting than others, for example, if N is a transitive model (set or class) of set theory and

$$N \models \text{“}(\mathfrak{M}, \mathcal{G}) \text{ is a full Henkin model”},$$

then $(\mathfrak{M}, \mathcal{G})$ is a Henkin model[8], albeit not necessarily a full one. Let us then say that $(\mathfrak{M}, \mathcal{G})$ is N*-full*. Certainly the L-full models are quite interesting, where L is Gödel's universe of constructible sets. Indeed, the N-full models for various N give rise to a wealth of interesting and useful Henkin models. Note that in order to build a Henkin model for second order logic we here build a model for the entire set theory. This is emblematic of the situation that the best way to understand second order logic is to understand set theory.

As was said earlier, the axioms of second order logic are not able to decide CH. Let us see how this can be seen in the light of classical independence proofs in set theory. We know that CH is true in the universe L of constructible sets. Let $(\mathfrak{M}, \mathcal{G})$ be the L-full model of the second order axioms θ of the real-numbers as a completely ordered separable Archimedian field. Thus, $(\mathfrak{M}, \mathcal{G}) \in L$ and

$$L \models \text{“}(\mathfrak{M}, \mathcal{G}) \text{ is the unique model of } \theta\text{”}.$$

Since L satisfies CH, the second order sentence ϕ expressing this on the reals satisfies

$$L \models \text{“}(\mathfrak{M}, \mathcal{G}) \models \phi\text{”}.$$

By the absoluteness of second order truth in a Henkin model, $(\mathfrak{M}, \mathcal{G}) \models \phi$. As a consequence the axioms of second order logic cannot prove $\theta \to \neg\phi$.

[8] First of all, $\mathcal{G}$ is clearly a set of relations and functions on M. To prove that $(\mathfrak{M}, \mathcal{G})$ satisfies the Comprehension Axioms, suppose $\phi(x_1, \ldots, x_n)$ is a second order formula in which the second order variable R does not occur free. The second order formula $\phi(x_1, \ldots, x_n)$ over M can be written in the first order language of set theory as $\phi^*(x_1, \ldots, x_n, M)$. By the Separation Axiom of ZFC, the set of tuples of elements of M satisfying $\phi^*(x_1, \ldots, x_n, M)$ is an element of N.

Next we show that the axioms of second order logic cannot prove $\theta \to \phi$. By the famous result of Paul Cohen, as further developed by Robert Solovay and Dana Scott, there is a complete Boolean algebra $\mathbb{B}$ such that the truth value of $\neg CH$ in the $\mathbb{B}$-valued universe of sets[9] $V^{\mathbb{B}}$ is one. As a consequence[10] there is a $\mathbb{B}$-valued Henkin model for $\neg CH$. As the Boolean valued Henkin models satisfy the Comprehension axioms and the Axioms of Choice, we conclude that $\theta \to \phi$ cannot be proved.

Set theorists study transitive models of ZFC even though they are really interested in the "full" models[11] V_α, where "normal" mathematics takes place. Second order logicians study Henkin models even though they are really interested in the full models, i.e. the structures "normal" mathematics is based on.

6 Internal categoricity

We have reached the following situation in our analysis of second order logic:

1. Important mathematical structures such as natural numbers, real numbers, complex numbers, etc can be characterized up to isomorphism in the full semantics, but truth in full models cannot be effectively axiomatized.

2. Truth in all Henkin structures can be effectively axiomatised, but no infinite Henkin model can be characterized up to isomorphism.

We now introduce a form of categoricity which holds for Henkin structures in important cases and agrees with the usual concept of categoricity in the case of full Henkin models.

We say that a second order sentence θ is *internally categorical* [22] if

$$\begin{aligned} \forall(\mathfrak{M},\mathcal{G}), \mathfrak{M}_0 \in \mathcal{G}, \mathfrak{M}_1 \in \mathcal{G}(((\mathfrak{M}_0,\mathcal{G}) \models \theta \wedge (\mathfrak{M}_1,\mathcal{G}) \models \theta) \\ \to \exists f \in \mathcal{G}(f : \mathfrak{M}_0 \cong \mathfrak{M}_1)). \end{aligned} \tag{12}$$

This is called "internal" categoricity because the models $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are internal to $(\mathfrak{M},\mathcal{G})$, i.e. elements of $\mathcal{G}$, and furthermore the isomorphism f can be found "internally" i.e. from $\mathcal{G}$ itself.

[9] See e.g. [2].

[10] See [15] for details.

[11] $V_0 = \emptyset, V_{\alpha+1} = \mathcal{P}(V_\alpha), V_\nu = \bigcup_{\beta<\nu} V_\beta$, for ν limit.

It is important to notice that internal categoricity is *stronger* than usual categoricity, because if $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are two models of an internally categorical θ, then we need only choose a transitive model N of (a suitable finite part of) ZFC such that all subsets, relations and functions on $M_0 \cup M_1$ are in N and then apply (12) to the N-full model $(N, \mathcal{G})$. Note that here we treat N as a model of the empty vocabulary, which is all we need.

In fact, internal categoricity is a kind of *provable* categoricity, because if the vocabulary of θ is just $\{R\}$, for simplicity, then (12) is equivalent to

$$\vdash \forall M_0, R_0 \forall M_1, R_1((\theta(R_0)^{(M_0)} \wedge \theta(R_1)^{(M_0)}) \rightarrow \exists F \phi_{iso}(F, M_0, R_0, M_1, R_1)),$$

where $\theta(R_i)^{(M_i)}$ denotes the relativisation of $\theta(R_i)$ to M_i, and the formula $\phi_{iso}(F, M_0, R_0, M_1, R_1)$ is the second order formula which says that F is an isomorphism between (M_0, R_0) and (M_1, R_1).

To verify the categoricity of a second order sentence one has to go through infinite structures in a way which essentially calls for set theory. The situation with internal categoricity is totally different. To verify the internal categoricity of a second order sentence one just has to produce a proof which means essentially going through natural numbers. So there is a dramatic difference. And still internal categoricity is stronger than categoricity. So it would be foolish to establish categoricity if one could establish even internal categoricity.

The main observation about internal categoricity is that the classical examples of categorical sentences are all internally categorical:

Theorem 2 ([23]) *The received second order sentences characterising the structures* $(\mathbb{N}, <)$ *and* $(\mathbb{R}, <, +, \cdot, 0, 1)$ *are internally categorical.*

The concept of internal categoricity provides a bridge between full semantics and Henkin semantics. It works in the same way in both cases and shows that the full semantics is a limit case of Henkin semantics but does not have a monopoly when it comes to categoricity.

7 Set theory

The approach of set theory to mathematics differs from that of second order logic in one fundamental aspect. While second order logic focuses on *one* mathematical structure at a time, set theory builds up directly one so powerful a hierarchy that all necessary mathematical structures can be seen as parts of it. In principle this hierarchy starts from some individuals, or *urelements*, but experience has shown that the urelements are unnecessary.

A possible way to compare second order logic and set theory might be the following: Start with a structure $\mathfrak{M}$. We have second order logic for this structure by considering the power-set of the underlying domain. Likewise, we have third, fourth, fifth, etc order logics for this structure by iterating taking power-sets. Let us continue to the transfinite. What results is a kind of transfinite theory of types. Now throw away the distinction between variables of different orders and let variables assume their order from the context. Then observe that there are so many sets around that the original $\mathfrak{M}$ is not needed any more. The whole construction can be started from the empty set and still all the structures needed in ordinary mathematics would be there.

Gödel presents the relationship between the theory of types and set theory very much in this vain:

> It may seem as if another solution were afforded by the system of axioms for the theory of aggregates, as presented by Zermelo, Fraenkel and von Neumann, but it turns out that this system of axioms is nothing else but a natural generalisation of the theory of types, or rather, it is what becomes of the theory of types if certain superfluous restrictions are removed. [8, 1933o]

When mathematics is developed in set theory, everything is constructed by means of the membership relation $\in$ only. One first defines the concept of an ordinal: a transitive set of transitive sets. The set $\mathbb{N}$ of natural numbers is defined as the smallest non-zero limit ordinal, i.e. the smallest non-zero ordinal which is not of the form $x \cup \{x\}$. Thus there is a simple formula $\phi_0(x)$ of set theory, with just $\in$, such that

$$\forall x(x = \mathbb{N} \iff \phi_0(x)).$$

The quantifier $\forall x$ of this equivalence, as well as all the quantifiers inside $\phi_0(x)$ are supposed to range over all sets. The universe of set theory is assumed to be closed under such basic constructions as Cartesian products and power-sets. Thus we can construct the integers $\mathbb{Z}$ as pairs of natural numbers, and the rationals $\mathbb{Q}$ as pairs of integers. Then we have $\phi_1(x)$ and $\phi_2(x)$ such that

$$\forall x(x = \mathbb{Z} \iff \phi_1(x)) \text{ and } \forall x(x = \mathbb{Q} \iff \phi_2(x)).$$

After this we follow Dedekind and construct the reals $\mathbb{R}$ as Dedekind cuts of rationals, and again we have a formula $\phi_3(x)$ such that

$$\forall x(x = \mathbb{R} \iff \phi_3(x)).$$

In set theory we do not define central mathematical structures, such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, up to isomorphism, as in second order logic, but up to identity.

Second order logic is perhaps correct to defining these structures up to isomorphism only, for mathematicians do not want to know what objects the numbers are. It is their structure that matters. If too much is known, it may complicate matters unnecessarily. So set theory seems to go astray here. But the point of set theory is *not* to insist that the numbers in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ *really* are what the formulas $\phi_0(x), \ldots, \phi_3(x)$ say, only that this is one convenient way, and there are numerous isomorphic ways. The advantage of defining the number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the way we have done is that every object in set theory becomes built up by means of membership $\in$ only, and then we can prove things about all sets by transfinite induction on $\in$.

So the advantage of set theory is that everything is built up in a uniform way, and can be handled uniformly. It is actually a problem in second order logic that we can work in one structure and then work in another structure, but how to bring these two things together. It is possible to transfer structures with isomorphisms so that they become substructures of each other, but after doing this for a while, one asks, why not assume that everything we need is part of one big structure, and then use second order logic on that big structure. Well, that is the idea of set theory. The universe of sets is the one big structure and every set that we may ever need is an element of it.

8 Hierarchies

Hierarchies based on quantifier structure are a useful method to compare definability of concepts in different systems. Second order logic has an obvious quantifier-rank hierarchy:

Σ^1_1	Existential second order quantifiers followed by a first order formula
Π^1_1	Universal second order quantifiers followed by a first order formula
$\vdots$	$\vdots$
Σ^1_{n+1}	existential second order quantifiers followed by a Π^1_n-formula
Π^1_{n+1}	universal second order quantifiers followed by a Σ^1_n-formula

We use the above concepts always up to logical equivalence. The logical equivalences we usually use are all provable from the Hilbert-Ackermann axioms.

For example, (1) and (2) are Π^1_1-formulas.

These classes of formulas have some obvious closure properties, up to logical equivalence: They are all closed, up to logical equivalence, under $\wedge$, $\vee$, and first order quantifiers. The hierarchy is proper:

$$\Sigma^1_n \not\subseteq \Pi^1_n, \Pi^1_n \not\subseteq \Sigma^1_n.$$

This was first proved by Kuratowski in early descriptive set theory by means of the following trick: By suitable coding one can construct in number theory a *universal* Σ^1_n formula $\Phi_n(x,y)$, that is, for every Σ^1_n-formula $\phi(y)$ there is $a \in \mathbb{N}$ such that

$$(\mathbb{N},+,\cdot,0,1)) \models \forall y(\phi(y) \leftrightarrow \Phi_n(a,y)).$$

Suppose now $\neg\Phi_n(x,x)$ were logically equivalent to a Σ^1_n formula. Then there would be $a \in \mathbb{N}$ such that

$$(\mathbb{N},+,\cdot,0,1)) \models \forall y(\neg\Phi_n(y,y) \leftrightarrow \Phi_n(a,y)).$$

Letting $y = a$ leads to a contradiction. Since $\neg\Phi_n(x,x)$ is Π^1_n up to logical equivalence, we get the result $\Pi^1_n \not\subseteq \Sigma^1_n$. Analogously, $\Sigma^1_n \not\subseteq \Pi^1_n$. As a consequence, it makes sense to define

$$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n,$$

and then Δ^1_n is closed under $\wedge$, $\vee$, $\neg$ and first order quantifiers. Craig's Interpolation Theorem can be formulated in an equivalent form as follows:

> If ϕ and ψ are Σ^1_n-sentences with no models in common, then there is a first order sentence θ such that $\phi \models \theta$ and moreover θ and ψ have no models in common.

As a consequence Δ^1_1 is just first order logic. However, there is a sense in which Π^1_1, or equivalently Σ^1_1, already has the full power of second order logic, although the above hierarchy result shows that this cannot be literally true. We present the details of this construction (due to [18] and [12]) in some detail for completeness:

To avoid cumbersome notation, we treat only unary second order logic in a vocabulary containing one binary predicate symbol S. Let A, B and E be some fixed new predicate symbols, the first two unary and the third binary. Suppose $\bar{\phi}(z_1, \ldots, z_n, E)$ is obtained from $\phi(R_1, \ldots, R_n)$, where $R_1, \ldots, R_n$ are unary second order variables, by replacing every occurrence of $R_i(y)$ by yEz_i, where $z_1, \ldots, z_n$ are new individual variables. Define a translation $\phi \mapsto \phi^*$ from unary second order logic to first order logic as follows:

- $\phi^* = \phi$ if ϕ is first order
- $(\phi \wedge \psi)^* = \phi^* \wedge \psi^*$, $(\phi \vee \psi)^* = \phi^* \vee \psi^*$, $(\neg\phi)^* = \neg\phi^*$
- $(\exists x\phi)^* = \exists x(A(x) \wedge \phi^*)$, $(\forall x\phi)^* = \forall x(A(x) \to \phi^*)$
- $(\exists R_i\phi)^* = \exists z_i(B(z_i) \wedge \bar{\phi}^*)$, $(\forall R_i\phi)^* = \forall z_i(B(z_i) \to \bar{\phi}^*)$

The idea is that the predicate A is reserved for individuals and the predicate B for subsets of A. The membership-relation is represented by E. Let θ be the Π^1_1-sentence

$$\forall X(\forall y(X(y) \to A(y)) \to \exists z(B(z) \wedge \forall y(yEz \leftrightarrow X(y))))$$

$$\wedge \forall x \forall y((B(x) \wedge B(y) \wedge \forall z(zEx \leftrightarrow zEy)) \to x = y).$$

This sentence says that for every subset X of A there is an element z of B such that, if E were the membership-relation, then z would be exactly X.

Lemma 1 *Suppose ϕ is a second order sentence, $S \subseteq M \times M$ and $(M, S) \models \phi$. Then*

$$(M \cup \mathcal{P}(M), \mathcal{P}(M), M, \in, S) \models \theta \wedge \phi^*.$$

Conversely, if $(M, B, A, E, S) \models \theta \wedge \phi^*$, then there is $(A', S') \cong (A, S)$ such that $2^{|A'|} = |M|$ and $(A', S') \models \phi$.

Theorem 3 *The following are equivalent for any second order ϕ and its first order translation ϕ^*:*

1. *ϕ has a model (of cardinality κ).*
2. *$\theta \wedge \phi^*$ has a model (of cardinality 2^κ).*

As a consequence, checking the validity of a second order sentence ϕ can be recursively reduced to checking the validity of the Σ^1_1-sentence $\theta \to \neg\phi^*$. Likewise checking whether a second order sentence ϕ has a model of cardinality κ can be reduced to asking whether the Π^1_1-sentence $\theta \wedge \phi^*$ has a model of cardinality 2^κ. This means that the Löwenheim number[12] and the Hanf number[13] of the entire second order logic are the same as those of the fragment Π^1_1.

Summing up, upon first inspection the levels $\Sigma^1_n \cup \Pi^1_n$ of the hierarchy of second order formulas grow strictly in expressive power as n increases, but a more careful analysis reveals that already the first level $\Sigma^1_1 \cup \Pi^1_1$ has the power of all the levels $\Sigma^1_n \cup \Pi^1_n$ even if the power is somewhat hidden and needs to be brought to light.

In set theory there is the *Levy-hierarchy* $\Sigma_n \cup \Pi_n$ of formulas [17]. This is a strict hierarchy roughly for the same reason why the hierarchy $\Sigma^1_n \cup \Pi^1_n$ of second order formulas is strict. But there is no known method to reduce the truth of an arbitrary formula to the truth of a formula on the $\Sigma_1 \cup \Pi_1$ level. In fact, if the decision problem, Löwenheim-Skolem number and Hanf number are suitable defined for $\Sigma_n \cup \Pi_n$-formulas of set theory, the decision problem gets more and more complicated, and the numbers get bigger and bigger as n increases [20]. Recall that in the case of second order logic, these concepts attained there maximal complexity already on the first level $\Sigma_1 \cup \Pi_1$.

Let us finally compare the two hierarchies, the hierarchy $\Sigma^1_n \cup \Pi^1_n$ inside second order logic and the hierarchy $\Sigma_n \cup \Pi_n$ in set theory.

Theorem 4 ([20, 21])

1. *The set of Gödel numbers of valid second order sentences is the complete Π_2-set of natural numbers.*
2. *The Löwenheim-Skolem number of second order logic is the supremum of all Π_2-definable ordinals.*

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[12] The smallest cardinal κ such that if a second order sentence has a model at all, it has a model of cardinality $\leq \kappa$.

[13] The smallest cardinal κ such that if a second order sentence has a model of cardinality $\geq \kappa$, it has arbitrarily large models.

3. *The Hanf number of second order logic is the supremum of all Σ_2-definable ordinals.*

4. *Every model class which is definable in second order logic is Δ_2-definable in set theory.*

In the light of the above theorem second order logic sits firmly on the $\Sigma_2 \cup \Pi_2$-level of set theoretic definability. This is a reason to consider second order logic weaker than set theory.

9 Foundations of mathematics

Let us first discuss what do I mean when I say that I consider second order logic as a foundation of mathematics. I have in mind the following: Whatever mathematical structures mathematicians have need for can be characterised in second order logic. Then truths about these structures can be identified with logical consequences of those characterisations. If a mathematician doubts the truth of a statement, he or she needs only to figure out what is the structure that the claim is concerned with and then check whether the characterisation of the structure logically implies the claim[14].

There are several points in this scenario which need clarification. First of all, in mathematical practice one sometimes needs to appeal to third order logic, that is, second order is not enough; but I consider this an irrelevant point[15]. Secondly, is it really true that whatever mathematical structures mathematicians have need of can indeed be characterised in second order logic up to isomorphism? After all, there are only countably many possible sentences in second order logic to use for such characterisations. In fact, it is almost impossible to pinpoint a mathematical structure that is (provably in ZFC) not second order characterisable, without resorting to cardinality-type arguments[16].

Finally there is the question what logical consequence means in second order logic. Let us say that ϕ is a *weak logical consequence* of ψ if every model of ψ is a model of ϕ. We noted in Theorem 4 that this relation is

[14] I disregard here statements of the type "Every ordered set has a completion" as they can in principle be restricted, at the cost of losing some elegance, to a suitable universal domain.

[15] My approach can be adapted to include third, fourth, etc order logic.

[16] See [14] for a careful analysis of this situation.

highly complex, which raises the question, is it actually possible to verify logical consequence in this way? There is a clear alternative: ϕ is a *strong logical consequence* of ψ if every Henkin model of ψ is a Henkin model of ϕ, or equivalently, if there is a finite proof of ϕ from ψ and the Comprehension Axioms and the Axioms of Choice.

Strong logical consequence is, of course, much stronger than weak logical consequence. A famous example is the Continuum Hypothesis CH. If $\theta_{\mathbb{R}}$ is a second order characterisation of the continuum[17], then either CH or $\neg CH$ is a weak logical consequence[18] of $\theta_{\mathbb{R}}$, but neither is a strong logical consequence[19]. The good news is that the CH is really exceptional[20] and virtually any logical consequence appearing in mathematics (outside set theory) turns out to be a strong one. Thus the appeal to strong logical consequence, even when weak logical consequence may seem more appealing, solves the otherwise serious complexity problem.

Next I should explain what I mean when I say that I consider *set theory* as a foundation of mathematics. This is easier because nowadays set theory is a much more popular foundation than second order logic. The scenario is as follows: mathematical objects are thought of as sets, whether they look like sets or not. Properties of sets are derived from the axioms of set theory. Thus these axioms (ZFC) are thought to be true in the universe of all sets. In analogy with second order logic there are statements (CH) about sets which are true or false but we do no know which, while we do know that these statements cannot be derived from the ZFC-axioms. In a sense, truth in the universe of sets ("weak" truth) corresponds in second order logic to what we call weak logical consequence, and derivability from ZFC ("strong" truth) corresponds to what we call strong logical consequence.

It should be noted that ("weak") truth in the universe of sets is not a set-theoretical concept[21] but we can usually limit ourselves to some high level of the cumulative hierarchy and then truth *is* definable. This is in analogy with the situation in second order logic where sentences such as "Every linear order has a completion" cannot be given meaning without restricting them to some universal domain.

[17] First presented as a second order sentence in [13].

[18] As pointed out by Kreisel [16].

[19] By results of Gödel [7] and Cohen [3].

[20] However, the independence results in set theory provide numerous examples which behave as CH. The most famous is the Souslin Hypothesis.

[21] By Tarski's Undefinability of Truth argument [19].

Weak logical consequence, and thereby truth in a second order characterisable structure, is a Π_2-concept in set theory. In the Levy-hierarchy of formulas of set theory truth of a lower level is always definable in the next level. Thus there is a Σ_3-formula which defines the truth of Π_2-formulas. Moreover, truth of formulas on any fixed level is reflected by a closed unbounded class of levels V_α of the cumulative hierarchy of sets. The problem with (weak) truth of arbitrary formulas in the entire universe of sets is that we cannot limit the formulas to any fixed level Σ_n of the Levy-hierarchy and we cannot limit truth to any fixed level V_α of the cumulative hierarchy.

In summary, truth in set theory goes beyond set theory and truth in second order logic goes beyond second order logic, but truth in second order logic can be defined in set theory because it is limited to the Π_2-level of the Levy-hierarchy. This is what it means that second order logic is a fragment of set theory.

I have argued that second order logic is really best understood as a fragment of set theory. The apparent problem that second order logic has second order quantifiers while set theory is presented in a first order language, is really only an apparent problem. The underlying first order logic of set theory is first order only if looked upon from outside. But there is no outside position in foundations of mathematics. We are always inside. It is more instructive to think of the first order logic of set theory as a very high order logic. After all, it allows quantification over subsets, sets of subsets, sets of sets of subsets, etc of given parameters.

I claim that when second order logic and set theory are construed as foundations there is essentially no difference in their ability to capture mathematical concepts up to isomorphism. The huge difference between second order logic and first order logic disappears. The matter is different if these logics are used as *tools* in mathematical logic. Then we can observe such big differences as one is compact, the other is not, one is absolute, the other is not, one has Downward Löwenheim-Skolem Theorem, the other does not, etc. But this is mathematical logic, not foundations of mathematics.

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