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Mathematical Logic

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A note on extensions of infinitary logic*

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Abstract. We show that a strong form of the so called Lindström's Theorem [4] fails to generalize to extensions of $L_{\kappa\omega}$ and $L_{\kappa\kappa}$: For weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to κ. With an additional set-theoretic assumption, there is no strongest extension of $L_{\kappa\kappa}$ with the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to $< \kappa$.

By a well-known theorem of Lindström [4], first order logic $L_{\omega\omega}$ is the strongest logic which satisifies the compactness theorem and the downward Löwenheim-Skolem theorem. For weakly compact κ, the infinitary logic $L_{\kappa\omega}$ satisfies both the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to κ. In [1] Jon Barwise pointed out that $L_{\kappa\omega}$ is not maximal with respect to these properties, and asked what is the strongest logic based on a weakly compact cardinal κ which still satisfies the (κ, κ)-compactness property and some other natural conditions suggested by κ. We prove (Corollary 5) that for weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ)-compactness property and the Löwenheim-Skolem theorem down to κ. This shows that there is no extension of $L_{\kappa\omega}$ which would satisfy the most obvious generalization of Lindström's Theorem. A stronger result (Theorem 11) is proved under an additional assumption.

We use the notation and terminology of [2, Chapter II] as much as possible. We will work with concrete logics such as first order logic $L_{\omega\omega}$, infinitary logic $L_{\kappa\lambda}$ and their extensions $L_{\omega\omega}(\{Q_i : i \in I\})$ and $L_{\kappa\lambda}(\{Q_i : i \in I\})$ by generalized quantifiers. Therefore it is not at all critical which definition of a logic one uses as long as these logics are included and some basic closure properties are respected. We use $\mathcal{L} \leq \mathcal{L}'$ to denote the sublogic relation. Let $\mathcal{P}$ be a property of logics. A logic $\mathcal{L}^*$ is *strongest extension of* $\mathcal{L}$ *with* $\mathcal{P}$, if

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1. $\mathcal{L} \leq \mathcal{L}^*$,
2. $\mathcal{L}^*$ has property $\mathcal{P}$,

and whenever $\mathcal{L}$ is a sublogic $\mathcal{L}'$ and $\mathcal{L}'$ has property $\mathcal{P}$, then $\mathcal{L}' \leq \mathcal{L}^*$.

Let $\mathcal{L}$ be a logic and κ and λ infinite cardinals. $\mathcal{L}$ is (κ, λ)-*compact* if for all $\Phi \subseteq \mathcal{L}$ of power κ, if each subset of Φ of cardinality $< \lambda$ has a model, then Φ has a model. $\mathcal{L}$ is κ-*compact* if it is (κ, ω)-compact. $\mathcal{L}$ is *weakly* κ-*compact* if $\mathcal{L}$ is (κ, κ)-compact. $\mathcal{L}$ is *fully compact* if it is κ-compact for all κ. $\mathcal{L}$ has the *Löwenheim-Skolem property down to* κ, denoted by LS(κ) if every $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $\leq \kappa$. If every sentence $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $< \kappa$, we say that $\mathcal{L}$ satisfies LS$(< \kappa)$. The order-type of the well-ordering R is denoted by otp(R).

Theorem 1. *[4] The logic $L_{\omega\omega}$ is strongest extension of $L_{\omega\omega}$ with $\aleph_0$-compactness and LS$(\aleph_0)$.*

Let C be a class of cardinals. Let

$$\mathfrak{A} \models \mathrm{Q}^{\mathrm{cf}}_C xy\phi(x, y, \vec{z}) \iff \{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}$$
$$\text{is a linear order with cofinality in } C.$$

By [9], $L_{\omega\omega}(\mathrm{Q}^{\mathrm{cf}}_C)$ is always fully compact. For C an interval we use the notation $\mathrm{Q}^{\mathrm{cf}}_{[\kappa,\lambda)}$ and $\mathrm{Q}^{\mathrm{cf}}_{[\kappa,\lambda]}$.

Proposition 2. *There is no strongest κ-compact extension of $L_{\omega\omega}$. In fact:*

1. *there are fully compact logics $\mathcal{L}_n$, $n < \omega$, such that $\mathcal{L}_n \leq \mathcal{L}_{n+1}$ for all $n < \omega$, but no $\aleph_0$-compact logic extends each $\mathcal{L}_n$.*
2. *There is an $\aleph_0$-compact logic $\mathcal{L}_1$ and a fully compact logic $\mathcal{L}_2$ such that no $\aleph_0$-compact logic extends both $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Proof. Let $\mathcal{L}_n = L_{\omega\omega}(\{\mathrm{Q}^{\mathrm{cf}}_{[\aleph_\omega,\infty]}\} \cup \{\mathrm{Q}^{\mathrm{cf}}_{\aleph_l} : l < n\})$. By [9], each $\mathcal{L}_n$ is fully compact. Clearly, no $\aleph_0$-compact logic can extend each $\mathcal{L}_n$.

For the second claim, let $\mathcal{L}_1$ be the logic $L_{\omega\omega}(Q_1)$, where Q_1 is the quantifier "there exists uncountable many" introduced by Mostowski [8]. This logic is $\aleph_0$-compact [3], see [2, Chapter IV] for more recent results. Let $\mathcal{L}_2$ be the logic $L_{\omega\omega}(Q_B)$, where Q_B is the quantifier "there is a branch" introduced by Shelah [10]. More exactly,

$$Q_B xytu M(x)T(y)(t \leq u)$$

if and only if $\leq_T$ is a partial order of $T \subseteq M$ and there are D,$\leq_D$, f and B such that:

1. $\leq_D$ is a total order of $D \subseteq M$
2. $f : \langle T, \leq_T\rangle \to \langle D, \leq_D\rangle$ is strictly increasing
3. $\forall s \in D \exists p \in T(f(p) = s)$
4. $B \subseteq T$ is totally ordered by $\leq_T$
5. $\forall b \in B((p \in T \& p \leq_T b) \to p \in B)$
6. $\forall s \in D \exists b \in B(s \leq_D f(b))$.

The reader is referred to [10] for a proof of the full compactness of $\mathcal{L}_2$.

Suppose there were an $\aleph_0$-compact logic $\mathcal{L}$ containing both $\mathcal{L}_1$ and $\mathcal{L}_2$ as a sublogic. It is easy to see that the class of countable well-orders can be expressed as a relativized pseudoelementary class in $\mathcal{L}$. This contradicts $\aleph_0$-compactness of $\mathcal{L}$. □

Lauri Hella pointed out that by elaborating the proof of claim (2) of the above proposition, we can make $\mathcal{L}_1$ fully compact. It was proved in [11] that, assuming GCH, there is no strongest extension of $\mathcal{L}_{\omega\omega}$ which is $\aleph_0$-compact. Our proof of (1) of the above proposition essentially occurs in a note, based on a suggestion of Paolo Lipparini, added after Theorem 8 of [11].

Proposition 3. *Suppose $\kappa > \aleph_0$. There is no strongest extension of $L_{\kappa^+\omega}$ with $LS(\kappa)$.*

Proof. Let $\mathcal{L}_1 = L_{\kappa^+\omega}(\mathrm{Q}^{\mathrm{cf}}_{\aleph_0})$ and $\mathcal{L}_2 = L_{\kappa^+\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. By using standard arguments with elementary chains of submodels, it is easy to see that both $\mathcal{L}_1$ and $\mathcal{L}_2$ have $\mathrm{LS}(\kappa)$, but the consistent sentence

$$R \text{ is a linear order with no last element } \wedge$$

$$\neg\mathrm{Q}^{\mathrm{cf}}_{\aleph_0} xyR(x,y) \wedge \neg\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]} xyR(x,y)$$

has no models of size $\leq \kappa$. □

It was proved in [11] that there is no strongest extension of $\mathcal{L}_{\omega\omega}$ with $\mathrm{LS}(\omega)$.

Lemma 4. *Suppose κ is weakly compact. Then $L_{\kappa\omega}(Q^{cf}_{\{\aleph_0\}})$ and $\mathcal{L}_{\kappa\omega}(Q^{cf}_{[\aleph_1,\kappa]})$ are weakly κ-compact. Moreover, if $\kappa > \omega$, these logics satisfy $LS(\kappa)$.*

Proof. The claim concerning $\mathrm{LS}(\kappa)$ is proved with a standard elementary chain argument. We prove the weak compactness of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. The case of $L_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{\{\aleph_0\}})$ is similar, but easier. For this end, suppose T is a set of sentences of $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ and $|T| = \kappa$. We may assume $T \subseteq \kappa$. If $\alpha < \kappa$, then we assume that there is a model $\mathfrak{M}_\alpha \models T \cap \alpha$. In view of $\mathrm{LS}(\kappa)$, it is not a loss of generality to assume that $\mathfrak{M}_\alpha = \langle \kappa, R_\alpha \rangle$, where $R_\alpha \subseteq \kappa \times \kappa$. Let $R(\alpha, \beta, \gamma) \iff R_\alpha(\beta, \gamma)$. By weak compactness there is a transitive M of cardinality κ such that

$$\langle H(\kappa), \epsilon, T, R \rangle \prec_{L_{\kappa\kappa}} \langle M, \epsilon, T^*, R^* \rangle$$

and $\kappa \in M$. Let $\mathfrak{M} = \langle M, S \rangle$, where $S(x,y) \iff R^*(\kappa, x, y)$. We claim that $\mathfrak{M} \models T$. We need only worry about the cofinality-quantifier. Cofinalities $< \kappa$ can be expressed in $L_{\kappa\kappa}$, so they are preserved both ways. Therefore also cofinality κ is preserved, and no other cofinalities can occur as the models have cardinality κ. □

Since the logics $L_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{\aleph_0}))$ and $\mathcal{L}_{\kappa\omega}(\mathrm{Q}^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ cannot both be sublogics of a logic with $\mathrm{LS}(\kappa)$, we get from the above lemma:

Corollary 5. *Suppose $\kappa > \omega$ is weakly compact. Then there is no strongest weakly κ-compact extension of $L_{\kappa\omega}$ with $LS(\kappa)$.*

The logic $L_{\kappa\omega}$ actually satisfies the property LS($<\kappa$) which is stronger than LS(κ). To prove a result like the above corollary for the property LS($<\kappa$) we have to work a little harder. At the same time we extend the proof to extensions of $L_{\kappa\kappa}$. Here the cofinality quantifiers Q^{cf}_C will not help as $Q^{cf}_{\{\lambda\}}$ is definable in $L_{\kappa\kappa}$ for $\lambda < \kappa$. Therefore we use more refined order-type quantifiers.

Definition 6. *Let $L_{\kappa\lambda}(Q)$ denote the formal extension of $L_{\kappa\lambda}$ by the generalized quantifier symbol $Qxy\phi(x, y, \vec{z})$. If $\mathcal{Y}$ is a class of ordinals, we get a logic $L_{\kappa\lambda}(Q, \mathcal{Y})$ from $L_{\kappa\lambda}(Q)$ by defining the semantics by*

$$\mathfrak{A} \models Qxy\phi(x, y, \vec{c}) \iff otp(\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}) \in \mathcal{Y}.$$

If $\phi \in L_{\kappa\lambda}(Q, \mathcal{Y})$ and $\mathfrak{A} \models \phi$, we say that $\mathfrak{A} \models \phi$ holds in the $\mathcal{Y}$-interpretation.

If $\mathfrak{A}$ is a model, then

$$o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda)$$

is the supremum of all $\mathrm{otp}(\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\})$ where $\phi \in L_{\kappa\lambda}(\mathcal{Y})$, $\vec{c} \in A^{<\lambda}$ and $\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}$ is well-ordered.

Lemma 7. *Suppose $\kappa \geq \lambda$, $\phi \in L_{\kappa\lambda}(Q)$, $\mathfrak{A}$ is a model, $\vec{a} \in A^{<\lambda}$, and $\mathcal{Y}' \cap o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda) = \mathcal{Y}$. Then $\mathfrak{A} \models \phi(\vec{a})$ in the $\mathcal{Y}$-interpretation if and only if $\mathfrak{A} \models \phi(\vec{a})$ in the $\mathcal{Y}'$-interpretation.*

Proof. This is a straightforward induction of the length of the formula ϕ. □

Lemma 8.
1. *Suppose $\kappa > \omega$, $\phi \in L_{\kappa\kappa}(Q)$, and ϕ has a model $\mathfrak{A}$ in the $\mathcal{Y}$-interpretation. Then there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq 2^\kappa$ and $\mathcal{Y}' \subseteq (2^\kappa)^+$ such that $\mathcal{Y}' \cap \kappa = \mathcal{Y}$ and $\mathfrak{B} \models \phi$ in the $\mathcal{Y}'$-interpretation.*
2. *Suppose $\kappa = \kappa^{<\kappa}$, $T \subseteq L_{\kappa\kappa}(Q)$, $|T| \leq \kappa$ and T has a model $\mathfrak{A}$ in the $\mathcal{Y}$-interpretation. Then for all $\xi < \kappa^+$ there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq \kappa$ and $\mathcal{Y}' \subseteq \kappa^+$ such that $\mathcal{Y} \cap \xi = \mathcal{Y}' \cap \xi$ and $\mathfrak{B} \models T$ in the $\mathcal{Y}'$-interpretation.*

Proof. We may assume $|A| \geq 2^\kappa$. Let us expand $\mathfrak{A}$ by

1. A well-ordering $\prec$ the order-type of which exceed all the order-types of well-orderings definable by subformulas of ϕ with parameters in A.
2. A new predicate P which contains those elements d of A for which $\mathrm{otp}(\{\langle a, b\rangle : a \prec b \prec d\}) \in \mathcal{Y}$.
3. A prediacte F which codes an isomorphism from each well-ordering, definable by a subformula of ϕ with parameters in A, onto an initial segment of $\prec$.

Let $\langle \mathfrak{A}, \prec, P, F\rangle$ be the expanded structure and $\langle \mathfrak{B}, \prec^*, P^*, F^*\rangle$ an $L_{\kappa\kappa}$-elementary substructure of it of cardinality $\leq 2^\kappa$. Let

$$\mathcal{Y}' = \{\mathrm{otp}(\{\langle a, b\rangle \in B^2 : a \prec^* b \prec^* d\} : d \in P^*\}$$

It is easy to see that $\mathfrak{B} \models \phi$ in the $\mathcal{Y}'$-interpretation. The second claim is proved similarly. □

Let π be a canonical bijection from triples of ordinals to ordinals such that $\pi[\kappa^3] = \kappa$ for each infinite cardinal κ. We say that a pair (δ_1, Z_1) *codes* a pair (δ_2, Z_2) if δ_1, δ_2 are ordinals, $Z_1 \subseteq \delta_1$, $Z_2 \subseteq \delta_2$ and there is a bijection $f : \delta_2 \to \delta_1$ such that

1. δ_1 is closed under π
2. $\pi(0, \alpha, \beta) \in Z_1 \iff f(\alpha) < f(\beta)$
3. $\pi(1, 0, \alpha) \in Z_1 \iff f(\alpha) \in Z_2$.

Suppose κ is an uncountable cardinal. The *weakly compact ideal* on κ is the ideal of subsets of κ generated by the sets $\{\alpha : (V(\alpha), \epsilon, A \cap V(\alpha)) \models \neg\phi\}$, where $A \subseteq V(\kappa)$ and ϕ is a Π^1_1-sentence such that $(V(\kappa), \epsilon, A \cap V(\kappa)) \models \phi$.

Definition 9. *A cardinal κ satisfies $\diamondsuit$(WC) if it is weakly compact and there is a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ such that*

1. $A_\alpha \subseteq \alpha$ for $\alpha < \kappa$.
2. $(\forall A \subseteq \kappa)(\{\lambda < \kappa : A_\lambda = A \cap \lambda\} \notin \mathcal{I})$, where $\mathcal{I}$ is the weakly compact ideal on κ.

We make the following remarks without proof (See note at the end): If κ is measurable $> \omega$, then κ satisfies $\diamondsuit$(WC). If κ is weakly compact $> \omega$, then there is a generic extension which preserves all cardinals and in which κ satisfies $\diamondsuit$(WC). If V=L, then every weakly compact cardinal $> \omega$ satisfies $\diamondsuit$(WC).

Theorem 10. *Suppose $\kappa > \omega$ satisfies $\diamondsuit$(WC) and $2^\kappa = \kappa^+$. Then there is no strongest extension of $L_{\kappa\kappa}$ for which κ is weakly compact and which has LS($< \kappa$).*

Proof. We shall construct two sets $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \kappa^+$ such that the logics $L_{\kappa\kappa}(Q, \mathcal{Y}^i)$ are weakly κ-compact for and these logics satisfy LS($< \kappa$), but no logic containing both $L_{\kappa\kappa}(Q, \mathcal{Y}^1)$ and $L_{\kappa\kappa}(Q, \mathcal{Y}^2)$ satisfies LS($< \kappa$). The sets $\mathcal{Y}^i$ are constructed by induction together with ordinals $\xi^i_\alpha < \kappa^+$ such that:

$$
\begin{array}{ll}
\mathcal{Y}^i = \bigcup_{\alpha<\kappa^+} \mathcal{Y}^i_\alpha & \\
\mathcal{Y}^i_0 = \emptyset & \xi^i_0 = 0 \\
\mathcal{Y}^i_\alpha = \mathcal{Y}^i_\beta \cap \xi^i_\alpha & \text{for } \alpha < \beta \\
\xi^i_\alpha \leq \xi^i_\beta & \text{for } \alpha < \beta \\
\mathcal{Y}^i_\nu = \bigcup_{\alpha<\nu} \mathcal{Y}^i_\alpha, & \xi^i_\nu = \bigcup_{\alpha<\nu} \xi^i_\alpha, \text{ for } \nu = \cup\nu \\
\mathcal{Y}^1_\alpha \cap \mathcal{Y}^2_\alpha = \emptyset & \text{for } \alpha < \kappa
\end{array}
$$

First we define $\mathcal{Y}^i_\alpha$ for $\alpha < \kappa$ in such a way that $L_{\kappa\kappa}(\mathcal{Y}^i)$ will in the end have the property LS($< \kappa$).

Let S_1, S_2 be a partition of the set of cardinals $< \kappa$ into two stationary sets. Let $\{\phi^i_\nu : \nu \in S_i\}$ list all $L_{\kappa\kappa}(Q)$-sentences so that each sentence is listed as ϕ^i_ν for stationary many $\nu \in S_i$.

Suppose $\alpha = \lambda + 1$ and $\xi^i_\lambda = \lambda$. Suppose $\lambda \in S_i$.

Case 1. Suppose that (λ, A_λ) codes some pair (ξ, Z). In this case we let

$$\mathcal{Y}^i_\alpha = \mathcal{Y}^i_\lambda \cup (Z \setminus \lambda), \xi^i_\alpha = \xi$$

$$\mathcal{Y}^{3-i}_\alpha = \mathcal{Y}^{3-i}_\lambda.$$

Case 2. Otherwise we let $\xi^i_\alpha = \lambda$, $\mathcal{Y}^i_\alpha = \mathcal{Y}^i_\lambda$, $\mathcal{Y}^{3-1}_\alpha = \mathcal{Y}^{3-i}_\lambda$.

Suppose then $\alpha = \lambda + 2$, $\xi^i_\lambda = \lambda \in S_i$ and we have defined $\xi^i_{\lambda+1}$ and $\mathcal{Y}^i_{\lambda+1}$.

Case 3. The sentence ϕ^i_λ has a model in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \subseteq \kappa^+$ with $\mathcal{Y} \cap \xi^i_{\lambda+1} = \mathcal{Y}^i_{\lambda+1}$. By Lemma 8 part 2, ϕ^i_λ has a model $\mathfrak{A}$ of cardinality $< \kappa$ in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \subseteq \kappa$ of cardinality $< \kappa$ with $\mathcal{Y} \cap \xi^i_{\lambda+1} = \mathcal{Y}^i_{\lambda+1}$. Let μ be minimal such that $\phi^i_\lambda \in \mathcal{L}_{\mu\mu}(\mathcal{Y})$. Let $\xi^i_{\lambda+2} = o(\mathfrak{A}, \mathcal{Y}, \mu, \mu)$ and $\mathcal{Y}^i_{\lambda+2} = \mathcal{Y}$. Let $\mathcal{Y}^{3-i}_{\lambda+2} = \mathcal{Y}^{3-i}_{\lambda+1}$.

Case 4. Otherwise $\xi^i_{\lambda+1} = \xi^i_\lambda$, $\mathcal{Y}^i_\alpha = \mathcal{Y}^i_{\lambda+1}$, $\mathcal{Y}^{3-1}_\alpha = \mathcal{Y}^{3-i}_{\lambda+1}$.

Finally for all other $\alpha \le \kappa$ we let ξ^i_α and $\mathcal{Y}^i_\alpha$ be defined canonically.

This ends the construction of $\mathcal{Y}^i_\alpha$ for $\alpha \le \kappa$. Note that $\mathcal{Y}^1_\kappa \cap \mathcal{Y}^2_\kappa = \emptyset$. Moreover, if ϕ^i_ν has a model in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \supseteq \mathcal{Y}^i_\kappa$, then, by construction, ϕ^i_ν has a model of cardinality $< \kappa$ in the $\mathcal{Y}^i_\kappa$-interpretation.

Let $\mathcal{Y}^i_{\kappa+1} = \mathcal{Y}^i_\kappa \cup \{\kappa\}$ and $\xi^i_{\kappa+1} = \kappa + 2$. Next we shall define $\mathcal{Y}^i_\alpha$ and ξ^i_α for $\kappa + 1 < \alpha < \kappa^+$. For this, let $\langle T_\alpha : \kappa < \alpha < \kappa^+\rangle$ enumerate all $L_{\kappa\kappa}(Q)$-theories of cardinality $\le \kappa$ in a language of cardinality $\le \kappa$ which satisfy the condition that every subset of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i_\kappa$-interpretation. Here we use the assumption $2^\kappa = \kappa^+$. We may assume $T_\alpha \subseteq \kappa$ for all α.

Suppose $\mathcal{Y}^i_\beta$ and ξ^i_β have been defined for $\beta < \alpha$. If $\alpha = \cup\alpha$, $\mathcal{Y}^i_\alpha$ and ξ^i_α are defined canonically. So assume $\alpha = \beta + 1$. Let $T : H(\kappa) \to H(\kappa)$ be the function $T(a) = T_\beta \cap a$. If $a \in H(\kappa)$, then $T(a)$ has a model $\mathfrak{B}_a$ in the $\mathcal{Y}^i_\kappa$-interpretation. By construction, we may assume $\mathfrak{B}_a \in H(\kappa)$. Let $B : H(\kappa) \to H(\kappa)$ be the function $B(a) = \mathfrak{B}_a$. Let $Z \subseteq \kappa$ code $(\xi^i_\beta, \mathcal{Y}^i_\beta)$. By $\Diamond$(WC), $W = \{\lambda < \kappa : A_\lambda = Z \cap \lambda\} \in \mathcal{I}^+$, where $\mathcal{I}$ is the weakly compact ideal on κ. Let $A : \kappa \to H(\kappa)$ be the function $A(\alpha) = A_\alpha$. By the definition of $\mathcal{I}$, there are a transitive set M and Y^*, R^* such that

$$\langle H(\kappa), \epsilon, A, W, \mathcal{Y}^i_\kappa, B, T\rangle \prec_{\kappa\kappa} \langle M, \epsilon, A^*, W^*, Y^*, B^*, T^*\rangle$$

and $\kappa \in W^*$. Now $A^*(\kappa) = Z$ and, by construction, $Y^* \cap \xi^i_\beta = \mathcal{Y}^i_\beta$

It is clear now that $B(\kappa)$ is a model of T_α in the Y^*-interpretation. By Lemma 8 there is a model $\mathfrak{B}$ of cardinality $\le \kappa$ of T_β in the Y^{**}-interpretation for some Y^{**} with $Y^{**} \cap \xi^i_\beta = \mathcal{Y}^i_\beta$. Let $\xi^i_\alpha = o(\mathfrak{B}, Y^{**}, \kappa, \kappa)$ and $\mathcal{Y}^i_\alpha = Y^{**} \cap \xi^i_\alpha$.

Finally, let $\mathcal{Y}^i = \bigcup_{\alpha<\kappa^+} \mathcal{Y}^i_\alpha$.

Claim 1. $L_{\kappa\kappa}(\mathcal{Y}^i)$ satisfies the LS$(< \kappa)$-property.

Suppose ϕ is a sentence of $L_{\kappa\kappa}(\mathcal{Y}^i)$ with a model. Let $\lambda \in S_i$ such that $\xi^i_\lambda = \lambda$ and $\phi^i_\lambda = \phi$. By the construction of $\mathcal{Y}^i_{\lambda+2}$ there is a model of ϕ of cardinality $< \kappa$.

Claim 2. $L_{\kappa\kappa}(\mathcal{Y}^i)$ is weakly κ-compact.

Suppose $T \subseteq L_{\kappa\kappa}(\mathcal{Y}^i)$ is given and every subset of T_α of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i$-interpretation. Then $T = T_\alpha$ for some α. By construction, every subset of T_α of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i \cap \kappa$-interpretation. Thus the definition of $\mathcal{Y}^i_\alpha$ is made so that T_α has a model $\mathfrak{B}$ in the $\mathcal{Y}$-interpretation for some $\mathcal{Y}$ such that $\mathcal{Y} \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa) = \mathcal{Y}^i \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa)$. Thus by Lemma 7, $\mathfrak{B} \models T_\alpha$ in the $\mathcal{Y}^i$-interpretation. The Claim is proved.

We can now finish the proof of the theorem. In a logic in which both the quantifier $Q_{\mathcal{Y}^1}$ and $Q_{\mathcal{Y}^2}$ are definable, we can say that the order-type of a well-ordering is in $\mathcal{Y}^1 \cap \mathcal{Y}^2$. Thus such a logic cannot satisfy LS($< \kappa$). □

It is interesting to note that a proof like above would not be possible for the following stronger Löwenheim-Skolem property: A *filter-family* is a family $\mathcal{F} = (\mathcal{F}(A))_{A \neq \emptyset}$, where $\mathcal{F}(A)$ is always a filter on the set A. Luosto [6] defines the concept of a (κ^+, ω)-*neat* filter family. We will not repeat the definition here, its elements are closure under bijections, fineness, κ^+-completeness, normality and upward relativizability (all defined in [6]). Suppose $\mathcal{L}$ is a logic of the form $L_{\kappa\lambda}(\vec{Q})$ for some sequence $\vec{Q}$ of generalized quantifiers. We say that $\mathcal{L}$ has the $\mathcal{F}, \kappa$-*persistency property*, if for all models $\mathfrak{A}$ and $B \in \mathcal{F}(A)$, we have $\mathfrak{A} \restriction B \prec \mathfrak{A}$. Luosto proves that if $\mathcal{L}_1$ and $\mathcal{L}_2$ both satisfy the $\mathcal{F}, \kappa$-persistency property, then there is $\mathcal{L}_3$ such that $\mathcal{L}_1 \leq \mathcal{L}_3$, $\mathcal{L}_2 \leq \mathcal{L}_3$ and $\mathcal{L}_3$ satisfies the $\mathcal{F}, \kappa$-persistency property. Lipparini [5] proves a similar result for families of limit ultrafilters related closely to compactness.

Tapani Hyttinen pointed out that the assumption $2^\kappa = \kappa^+$ is not needed in Theorem 10, if κ is assumed to be measurable.

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Note added in proof: The principle ◇(WC) has been considered in the literature by W. Sun (Arch. Math. Logic 32, 1993), A. Hellsten (Ann. Acad. Sci. Fenn. Math. Diss. 134, 2003) and J. Hamkins (A class of strong diamond principles, 2003).