Abstract Logic

Lecture 1

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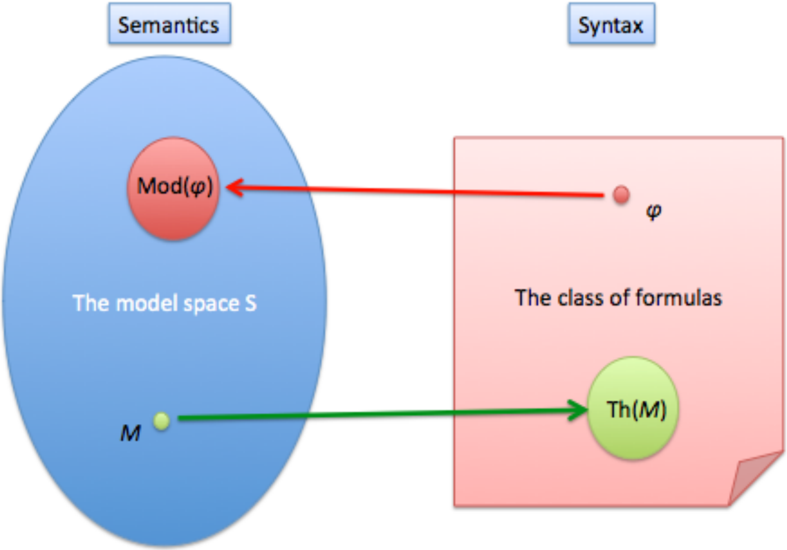
Abstract logic - definition

Definition

An *abstract logic* is a triple $L = (S, F, \models)$ where $\models \subseteq S \times F$. Elements of the class S are called the structures of L, elements of F are called the sentences of L, and the relation $\models$ is called the satisfaction relation of L.

$M \models \phi$		M is a model of ϕ	ϕ is true in M
$Mod_L(\phi)$	$=$	$\{M \in S : M \models \phi\}$	a model class of L
$Th_L(M)$	$=$	$\{\phi \in F : M \models \phi\}$	a complete theory in L
$\phi \equiv_L \psi$	$\iff$	$Mod_L(\phi) = Mod_L(\psi)$	logical equivalence of L
$M \equiv_L M'$	$\iff$	$Th_L(M) = Th_L(M')$	elementary equivalence in L



Different models spaces

Different classes of formulas

Connection to topology

We can think of S as a space and the sets $\{M \in S : M \models \phi\}$, where $\phi \in F$, as a sub-basis of a topology. We (usually) get a topological space $Top(L)$.

Table: Topology and logic

Topology	Logic
Space (of models)	S
Sub-basis of open sets	F
$M \in \phi$	$M \models \phi$
Metric, norm, etc	Syntax
Metrization results	Lindström's Theorem

Example

- S is the class of **all** first order structures, F is the class of all first order sentences of all possible vocabularies, and $M \models \phi$ has the usual meaning.
- S is the class of **all finite** first order structures, F is the class of all n-variable first order sentences of all possible vocabularies, and $M \models \phi$ has the usual meaning.

Examples

Structures	Sentences
Valuations -double valuations -three-valued	Propositional logic Relevance logic Paraconsistent logic
Relational structures -monadic -ordered -finite -pseudofinite -topological -Banach space -Borel -recursive -ω-models	Predicate logic -with two variables -guarded fragment -with infinitary operations -with generalized quantifiers -higher order -positive bounded, etc -logic of measure and category
Kripke structures -transitive reflexive -equivalence relation -etc	Intuitionistic logic Modal logic -S4,S5, etc
Many-valued structures	Many-valued logic, fuzzy logic
Games	Linear logic

Compactness

When $M \in S$ and $T \subseteq S$, we write $M \models T$, if $M \models \phi$ for all $\phi \in T$.

Definition

An abstract logic $L = (S, F, \models)$ is **compact** if the following condition is satisfied: Suppose $T \subseteq F$ is such that for all finite $T_0 \subseteq T$ there is $M \in S$ such that $M \models T_0$. Then there is $M \in S$ such that $M \models T$.

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Definition

(Marta Garcia-Matos, V.) An abstract logic $L = (S, F, \models)$ is **dual compact** if the following condition is satisfied: Suppose $T \subseteq F$ is such that every $M \in S$ is a model of some $\phi \in T$, then there is finite $T_0 \subseteq T$ such that every $M \in S$ is a model of some $\phi \in T_0$.

We then also say that L satisfies the *(dual) compactness Theorem*.

Definition

A logic $L = (S, F, \models)$ is **closed under conjunction** if for all $\phi, \psi \in F$ there is $\theta \in F$ such that

$$\forall M \in S(M \models \theta \iff M \models \phi \wedge M \models \psi).$$

We then denote θ by $\phi \wedge \psi$. Similarly for disjunction and negation.

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Lemma

- *If the logic is closed under conjunction then it is dual compact if and only if the topological space $Top(L)$ is compact.*
- *If the logic is closed under negation, then it is compact if and only if it is dual compact.*
- *Existential second order is compact but not dual compact.*

Examples

Example

- S is the class of all pairs (M, s) where M is a Kripke structure and s is a node of M, F is the class of all sentences of modal logic, and $(M, s) \models \phi$ has the usual meaning.
- The same for intuitionistic logic.
- S is the class of valuations $v : \mathbb{N} \to \{0, 1\}$, F is the class of propositional formulas, and $v \models \phi$ means that $v(\phi) = 1$.

Examples

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Sublogic

- Intuitively, L is a sublogic of L' if every model class of L is a model class of L'.
- Assume $S = S'$. Then this is equivalent to:

$$\forall \phi \in F \exists \psi \in F' \forall M \in S (M \models \phi \iff M \models \psi).$$

- We then say that L is a **sublogic** of L', in symbols $L \leq L'$. If $L \leq L'$ and $L' \leq L$, then we say that the logics are equivalent, $L \equiv L'$.

General sublogic

Definition (Garcia-Matos and V.)

An abstract logic $L = (S, F, \models)$ is a *general sublogic* of another abstract logic $L' = (S', F', \models')$, if there are a sentence $\theta \in F'$ and functions $\pi : S' \to S$ ("projection") and $f : F \to F'$ ("translation") such that

1. $\forall A \in S \exists A' \in S' (\pi(A') = A \text{ and } A' \models' \theta)$
2. $\forall \phi \in F \forall A' \in S' (A' \models \theta \to (A' \models' f(\phi) \iff \pi(A') \models \phi))$.

We then write

$$L \leq_{\theta,\pi,f} L'.$$

The idea is that the structures in S' are richer than the structures in S, and therefore we need the projection π. The role of θ is to cut out structures in S' that are meaningless from the point of view of L.

Projections

Structure A	Projection $\pi(A)$	θ expresses
Ordered structure $(M, <)$	Structure M	Axioms of order
First order structure	Its reduct to a valuation	
First order structure	Its unary reduct	
Binary structure	Kripke structure	Transitivity (e.g.)
Structure with unary predicates	Many-sorted structure	
Finite structure	Structure itself	
First order structure	Full Henkin model	

Figure: Typical projections

Translations

Sentence ϕ	Translation $F(\phi)$
$p_0 \wedge \neg p_1$	$P_0() \wedge \neg P_1()$
$\Box p_0$	$\forall y(R(c,y) \to P_0(c))$
$\forall x P_0(x)$	$\neg \exists x \neg P_0(x)$
$Q_0 x P_0(x)$	$\exists_{\text{fin}} X \forall x(P_0(x) \to X(x))$

Figure: Typical translations

General sublogic relations

Sublogic	Logic
Propositional logic	Predicate logic
Monadic logic	Predicate logic
Predicate logic without $=$	Predicate logic
Intuitionistic logic	Predicate logic
Modal logic	Predicate logic
$L(Q_0)$	Weak second order

Figure: Some general sublogics

Compactness in general sublogics

Lemma

If $L \leq_{\theta,\pi,f} L'$ *and* L' *is compact, then so is* L*.*

Proof.

Suppose $T \subseteq F$ is given and if $T_0 \subseteq T$ is finite, then there is $M \in S$ such that $M \models T_0$. Let $T' = \{f(\phi) : \phi \in T\}$. Suppose $T'_0 \subseteq T'$ is finite, say $T'_0 = \{f(\phi) : \phi \in T_0\}$, where $T_0 \subseteq T$ is finite. There is $A \in S$ such that $A \models T_0$. Let $A' \in S'$ such that $\pi(A') = A$ and $A' \models' \theta$. Then $A' \models' T'_0$. By the compactness of L' there is $A' \in S'$ such that $A' \models' T' \cup \{\theta\}$. Thus $\pi(A') \models T$. □

Compactification

- A **compactification** of a logic L is a compact logic L' such that $L' \leq L$ and there is no compact logic L'' such that $L' < L'' \leq L$.
- Can we compute compactifications of well-known incompact logics?
- Lindström's Theorem says that first order logic is a compactification of any extension of first order logic with the Löwenheim Property.
- E.g. $L(Q_0)$.

Definition

An abstract logic $L = (S, F, \models)$ is a **weak sublogic** of another abstract logic $L' = (S, F', \models')$, in symbols

$$L \leq_w L',$$

if

$$\forall A, B \in S(A \equiv_{L'} B \Rightarrow A \equiv_L B).$$

Note that if $L \leq L'$, then $L \leq_w L'$.

Lemma (Lindström)

Suppose L is closed under conjunction and negation, L' is closed under negation, $L \leq L'$, $L' \leq_w L$ and L' is compact. Then $L \equiv L'$.

Suppose $\phi \in F'$ is given so that $Mod(\phi) \neq Mod(\psi)$ for all $\psi \in F'$. Let T be the set of $\psi \in F$ such that $Mod_{L'}(\phi) \subseteq Mod_L(\psi)$. There is $B \in S$ such that $B \models T \cup \{\neg\phi\}$. Otherwise, by compactness, there are $\psi_1, ..., \psi_n \in F$ such that

$$Mod_L(\psi_1 \wedge ... \wedge \psi_n) = Mod_{L'}(\phi),$$

contrary to the assumption. So B exists. The set

$$\{\psi \in F : B \models \psi\} \cup \{\phi\}$$

has a model $A \in S$. Otherwise, again by compactness, there are $\psi_1, ..., \psi_n \in F$ such that $Mod_{L'}(\phi) \subseteq Mod_L(\neg(\psi_1 \wedge ... \wedge \psi_n))$. Then $\neg(\psi_1 \wedge ... \wedge \psi_n) \in T$. So $B \models \neg(\psi_1 \wedge ... \wedge \psi_n)$, contradiction. Now $A \equiv_L B$ but $A \not\equiv_{L'} B$. QED

Cardinality of models

Definition

A **cardinality schema** for a logic L is any function Σ from S into the class of cardinals. Such a logic has the **Downward Löwenheim-Skolem Property down to** κ, LS_{κ}^{Σ}, if every sentence in F that has a model has a model with $\Sigma(A) \leq \kappa$. $LS_{<\kappa}^{\Sigma}$ means "has a model with $\Sigma(A) < \kappa$". $LS_{<\omega}^{\Sigma}$ is called the **Finite Model Property**. L has the **Upward Löwenheim-Skolem Property at** κ, ULS_{κ}^{Σ}, if every sentence in F that has a model with $\Sigma(A) \geq \omega$ has a model with $\Sigma(A) > \kappa$.

- If $L \leq_{\theta,\pi,f} L'$, L has cardinality schema Σ, L' has cardinality schema Σ', and $\Sigma(\pi(A)) \leq \Sigma'(A)$ for all $A \in S'$, and L' has LS_{κ}^{γ}, then L has $LS_{\kappa}^{\gamma'}$.
- If $L \leq_{\theta,\pi,f} L'$, L has cardinality schema Σ, L' has cardinality schema Σ', and $\Sigma(\pi(A)) \geq \Sigma'(A)$ (and if one is infinite then both are) for all $A \in S'$, and L' has ULS_{κ}^{γ}, then L has $ULS_{\kappa}^{\gamma'}$.

Hanf and Löwenheim numbers

Definition

The Σ-**spectrum** of $\phi \in L$ is $Sp(\phi) = \{\Sigma(A) : A \models \phi, A \in S\}$.

Definition

The **Löwenheim number** of a logic L, ℓ_L, is the smallest κ such that L has the property LS^{Σ}_{κ}. (Equivalently, it is $\sup\{\min Sp(\phi) : \phi \in F, Sp(\phi) \neq \emptyset\}$.)
The **Hanf number** of a logic L, h_L, is the smallest cardinal κ such that if any sentence has a model A with $\Sigma(A) \geq \kappa$, then it has for every λ a model A with $\Sigma(A) \geq \lambda$. (In many important cases it is $\sup\{\sup Sp(\phi) : \phi \in F, Sp(\phi) \text{ bounded}\}$.)

Hanf proved that if F is a set, then ℓ_L and h_L exist. They are important *invariants* of logics.
If $L \leq L'$, then $\ell_L \leq \ell_{L'}$ and $h_L \leq h_{L'}$ (if they exist).

Decision problem

Definition

A **Gödel numbering schema** for a logic L is any (effective) function γ from F into ω. The **decision problem** of such a logic is the set $V_L^\gamma = \{\gamma(\phi) : \phi \in F, \phi \text{ is valid}\}$. A logic with a Gödel numbering schema γ is **axiomatizable** if the set V_L^γ is r.e. The **Satisfiability problem** of such a logic is the set $Sat_L^\gamma = \{\gamma(\phi) : \phi \in F, \phi \text{ has a model}\}$.

- Under self-evident effectivity constraints, $L \leq_{\theta,\pi,f} L'$ implies $Sat_L \leq_1 Sat_{L'}$ (1-1 Turing reducible) via the equivalence

$$\gamma(\phi) \in Sat_L^\gamma \iff \gamma'(\theta \wedge f(\phi)) \in Sat_{L'}^\gamma.$$

- Thus if $L \leq L'$, both are closed under negation and L' is axiomatizable, then so is L.

Axioms

An **axiom** of $(S, F, \models)$ is any element of F. An **inference rule** of $(S, F, \models)$ is any function on F.

Definition

- A **logic frame** if a quadruple $L = (S, F, \models, A)$, where A is a class of axioms and inference rules of $(S, F, \models)$.
- If $L = (S, F, \models, A)$, we write $T \vdash_L \phi$, whenever ϕ can be obtained from the axioms by means of the inference rules.
- $L = (S, F, \models, A)$ satisfies the **Soundness Theorem** if $T \vdash_L \phi$ implies $Mod_L(T) \subseteq Mod_L(\phi)$.
- $L = (S, F, \models, A)$ satisfies the **Completeness Theorem** if $T \vdash_L \phi \iff Mod_L(T) \subseteq Mod_L(\phi)$.

The completeness Theorem of $(S, F, \models, A)$ is very close to the Compactness Theorem of $(S, F, \models)$.