Abstract Logic

Lecture 2

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May 17, 2010

Vocabularies and structures

- We now go to the other extreme from the first lecture: we assume the class of structures S is the class of ordinary first order structures, the logics are closed under conjunction and negation.
- We denote abstract logics henceforth $(F, \models)$ because S is always the same. Now there is a canonical cardinality schema: $\Sigma(M) =$ the cardinality of the universe of the first order structure M.
- We limit ourselves to relational vocabularies (with constants) in order to make things even simpler. This is unessential.
- Vocabularies are denoted τ, τ' etc.

Operations on structures

- Interpretation of a relation symbol R in a model M is denoted R^M.
- Recall that structures have **reducts** $M \restriction \tau$ and expansions. Substructure relation: $M \subseteq N$. If $M \subseteq N$, then we write $M \preceq_L N$, if $(M, a)_{a \in M} \equiv_L (N, a)_{a \in M}$.
- A **name changer** is a mapping π from one vocabulary τ to another τ' which preserves arity. If π is a name changer, then for every τ-structure M we have the corresponding τ'-structure $\pi(M)$ obtained in the obvious way.

Standard abstract logics

Definition

- A **(standard) abstract logic** is an abstract logic $L = (F, \models)$ satisfying the following conditions:

  1. (Isomorphism Condition) If $\phi \in F$, then for all $M, M' \in S$: $M \cong M'$ implies $M \models \phi \iff M' \models \phi$.
  2. (Name Changing Condition) If $\phi \in F$ and $\pi : \tau \to \tau'$ is a name changer then there is $F' \in F$ such that for all $M \in S$: $M \models \phi \iff \pi(M) \models \phi'$.
  3. (Occurrence Condition) If $\phi \in F$ then there is τ such that for all M: $M \restriction \tau \models \phi \iff M \models \phi$
  4. (Boolean Condition) The logic is closed under conjunction and negation.

Example

First order logic, denoted FO. Others in a moment. $M \equiv N$ means $M \equiv_{FO} N$. Similarly other notation.

Definition

An abstract logic has the **Łoś property** if for every set $\{M_i : i \in I\}$ of structures of the same vocabulary and every ultrafilter D on I there is a structure M such that for all $\phi \in F$:

$$M \models \phi \iff \{i \in I : M_i \models \phi\} \in D.$$

If M can be chosen to be $\prod_i M_i/D$ we talk about **strong Łoś property**.

- First order logic has the strong Łoś property.
- An abstract logic is compact if and only if it has the Łoś property. [Compactness clearly implies that $\{\phi : \{i \in I : M_i \models \phi\} \in D\}$ has a model. Conversely, if every finite subset i of T has a model M_i, and D extends $\{X_i\} : i \in I$, where I is the set of all finite subsets of T and $X_i = \{j \in I : M_j \models i\}$, then the model M given by the Łoś property satisfies T.]

Ultraproduct characterization of first order logic

Theorem (Essentially Keisler and Shelah)

If $FO \leq L$ and L satisfies the strong Łoś property, then $FO \equiv L$.

Proof.

Since Łoś property implies compactness, it suffices to prove that $L \leq_w FO$. So suppose $M \equiv N$. By Shelah's Isomorphism Theorem ([1]) there is an index set I and an ultrafilter D such that $M^I/D \cong N^I/D$. By strong Łoś Property, $M \equiv_L M^I/D$ and $N \equiv_L N^I/D$. By the Isomorphism Property $M^I/D \equiv_L N^I/D$. Thus $M \equiv_L N$. □

Tarski Union Property

Recall the union of a chain of models $\bigcup_n M_n$.

Definition

A logic $L = (F, \models)$ has the **Tarski Union Property** if for all $M_0 \preceq_L M_1 \preceq_L \ldots$ we have $M_m \preceq_L \bigcup_n M_n$ for all $m \in \mathbb{N}$.

- For first order logic: By induction on $\phi(x_1, \ldots, x_k)$: If m is such that $a_1, \ldots, a_k \in M_m$, then $M_m \models \phi(a_1, \ldots, a_k) \iff \bigcup_n M_n \models \phi(a_1, \ldots, a_k)$.

Helpful Lemma

Lemma (Lindström)

Suppose FO $<$ L and L is compact, then for some $\phi \in F$ there are $M \preceq N$ such that $M \models \phi$ but $N \models \neg\phi$.

Proof.

Recall that we previously got only that $M \equiv N$, not that $M \preceq N$. However, we start with these structures. So we assume $M \models \phi$ and $N \models \neg\phi$. Let M^* be the expansion of M by giving a name for each element of M. Let $T = Th(M^*) \cup \{\neg\phi\}$. Suppose $T_0 \subseteq Th(M^*)$ is finite. Let $\theta(c_1, ..., c_n)$, where $c_1, ..., c_n$ are the new constants, be the (first order) conjunction of T_0. Let ψ be the sentence $\exists\vec{x}\theta(\vec{x})$. Since $M \models \exists\vec{x}\theta(\vec{x})$, $N \models \exists\vec{x}\theta(\vec{x}) \wedge \neg\phi$. Thus T has a model by compactness, and we are done. □

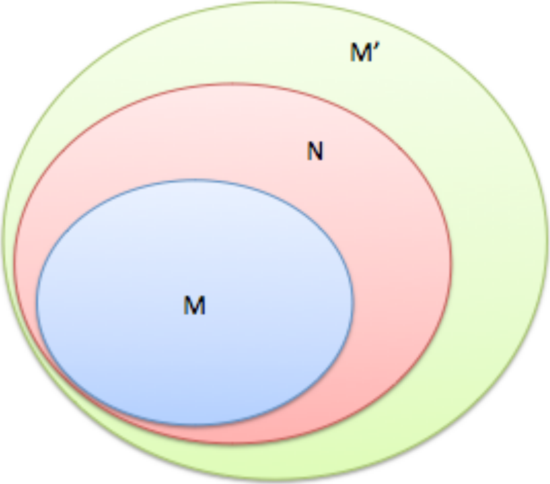
Second helpful Lemma

Lemma (Lindström)

Suppose L is compact, $FO \leq L$, and $M \preceq N$. Then there is M' such that $M \preceq_L M'$ and $N \preceq M'$.

Proof.

Let $T = Th_L(M^*) \cup Th(N^*)$. We show that T has a model. Let $\psi(c_1, ..., c_n)$ be a finite conjunction of elements of $Th(N^*)$, where only constants from $N \setminus M$ are displayed. Then $\exists x_1 ... \exists x_n \psi(x_1, ..., x_n)$ is true in N, hence in M. Thus $Th_L(M^*) \cup \{\psi(c_1, ..., c_n)\}$ can be satisfied in M by interpreting the constants suitably. □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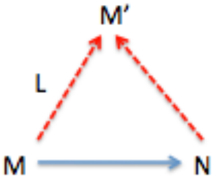
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Tarski Union characterization of *FO*

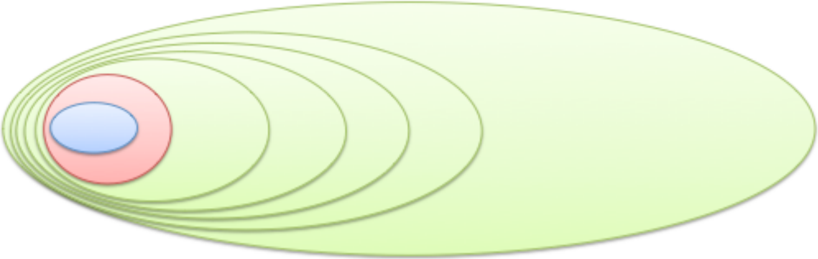
Theorem (Lindström)

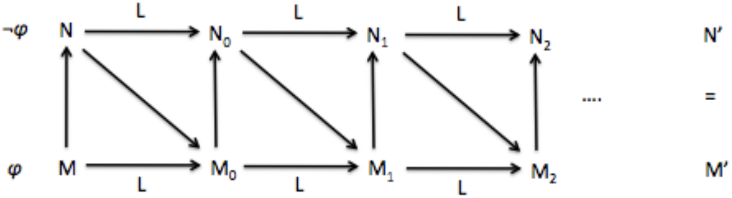
If $FO \leq L$ and L satisfies the Compactness Theorem and the Tarski Union Property, then $FO \equiv L$.

Proof.

- Suppose $FO < L$. By the first helpful lemma we have $M \preceq N$ such that $M \models \phi$ and $N \models \neg\phi$. Now we start a sequence of applications of the second helpful lemma.
- If we apply the second helpful lemma to $M \preceq N$, we get M_0 such that $M \preceq_L M_0$ and $N \preceq M_0$.
- If we apply the second helpful lemma to $N \preceq M_0$, we get N_0 such that $N \preceq_L N_0$ and $M_0 \preceq N_0$.
- If we apply the second helpful lemma to $M_0 \preceq N_0$, we get M_1 such that $M_0 \preceq_L M_1$ and $N_0 \preceq M_1$. Etc.
- Let $M' = \bigcup_n M_n$ and $N' = \bigcup_n N_n$. By the Tarski Union Property, $M' \models \phi$ and $N' \models \neg\phi$. But $M' = N'$, a contradiction. So $FO \equiv L$.

□

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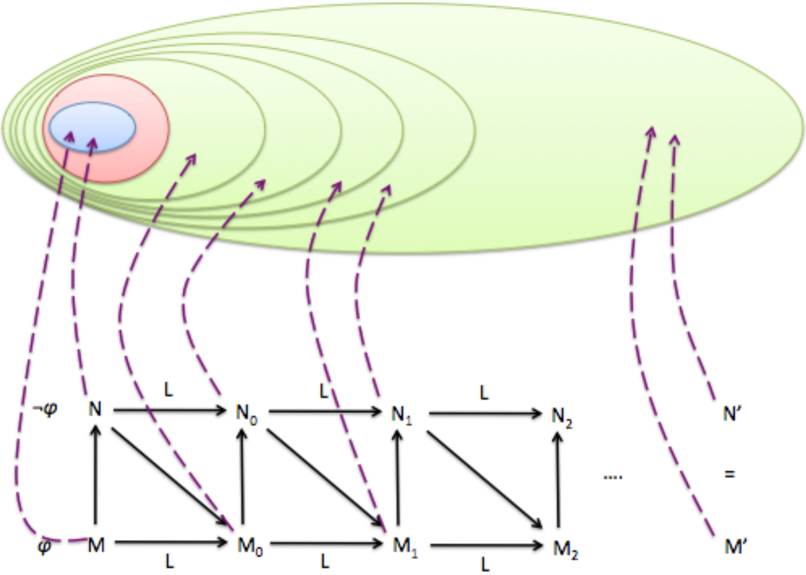

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Saharon Shelah.
Every two elementarily equivalent models have isomorphic ultrapowers.
Israel J. Math., 10:224–233, 1971.