Abstract Logic

Lecture 3

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Infinitary logic

Definition

The infinitary logic $L_{\omega_1\omega}$ is the extension of first order logic by allowing, in addition to $\neg, \wedge, \vee, \exists, \forall$ also countable conjunctions and disjunctions:

$$\bigwedge_{n\in\mathbb{N}} \phi_n, \bigvee_{n\in\mathbb{N}} \phi_n.$$

Skolem functions

Definition

Given a model M and $\phi(x_1,,,.x_n,y) \in L_{\omega_1\omega}$, a **Skolem function** of $\phi(x_1,,,.x_n,y)$ on M is any function $f : M^n \to M$ such that

$$\forall a_1,...,a_n \in M(M \models \exists x\phi(a_1,...,a_n,x) \leftrightarrow \phi(a_1,...,a_n,f(a_1,...,a_n)).$$

Suppose $\phi \in L_{\omega_1\omega}$. A **set of Skolem functions for ϕ on M** is any set $\mathbb{F}$ of functions on M such that every subformula of ϕ has a Skolem function in $\mathbb{F}$.

Lemma

For every M and every $\phi \in L_{\omega_1\omega}$ there is a countable set of Skolem functions for ϕ on M.

Löwenheim-Skolem Theorem for $L_{\omega_1\omega}$

Theorem

$L_{\omega_1\omega}$ satisfies LS_ω, even more: if M is any model and $\phi \in L_{\omega_1\omega}$ such that $M \models L_{\omega_1\omega}$, then there is a countable submodel M' of M such that $M' \models \phi$.

Proof.

W.l.o.g. the vocabulary of ϕ is countable. Let $\mathbb{F}$ be a countable set of Skolem functions for ϕ on M. Let M' be the smallest $\mathbb{F}$-closed substructure of M. Then M' is countable. It remains to prove by induction on subformulas $\psi(x_1, ..., x_n)$ of ϕ that for all $a_1, ..., a_n \in M'$:

$$M' \models \psi(a_1, ..., a_n) \iff M \models \psi(a_1, ..., a_n).$$

This is routine. □

What does it mean to have "many" countable subsets?

Definition

Suppose A is a set. A set X of countable subsets of A is **closed unbounded (club)** if:

1. Every countable subset of A is contained in some element of X.
2. If $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ are in X, then so is $\bigcup_n B_n$.

- There are club sets: $\{B \subseteq A : B \text{ countable}\}$.
- Some unbounded sets are not club: $\{B \subseteq \mathbb{R} : B \text{ is closed}\}$
- The intersection of two (countably many) club sets is club.

Countable submodels are "everywhere"

Theorem

If M is any model and $\phi \in L_{\omega_1\omega}$ such that $M \models \phi$, then there is a closed unbounded set of universes of countable submodels M' of M such that $M' \models \phi$.

Proof.

Sets of the form $\{B \subseteq A : B \text{ is closed under } \mathbb{F}\}$, where $\mathbb{F}$ is a countable set of (finitary) functions on M, are always club. □

Other approaches

- Henkin model construction. Can be used also to prove Completeness Theorem and Interpolation Theorem. Model Existence Game, consistency property, Hintikka set, forcing.
- Set theory: "ϕ has a model" is Σ_1 in set theory with ϕ as a parameter. Now use Levy Reflection (i.e. $HC \prec_1 V$).
- Reduction to omitting types. For any countably fragment L of $L_{\omega_1\omega}$ there is a countable set of types (introduced by M. Morley) such that any sentence in the fragment can be translated into first order logic on models (S^*) that omit these types. So $(S, L, \models) \leq_{\theta,\pi,f} (S^*, FO, \models)$ and now we can use the LS_ω of first order logic on models that omit countably many types.

Generalized quantifier Q_1

Definition

The logic $L(Q_1)$ is the extension of first order logic by the *generalized quantifier*

$$Q_1 x\phi(x, \vec{y}).$$

Semantics is defined as follows:

$$M \models Q_1 x\phi(x, \vec{b}) \iff \{a \in M : M \models \phi(a, \vec{b}\} \text{ is uncountable.}$$

(Recall: "A is countable" means "A can be indexed by natural numbers as $A = \{a_n : n \in \mathbb{N}\}$.)
Clearly, $L(Q_1)$ is a standard abstract logic.

Countable compactness of $L(Q_1)$

Definition

A general abstract logic $L = (S, F, \models)$ is **countably compact** if the following condition is satisfied: Suppose $T \subseteq F$ is *countable* and for all finite $T_0 \subseteq T$ there is $M \in S$ such that $M \models T_0$. Then there is $M \in S$ such that $M \models T$.

Theorem (Fuhrken, Vaught)

The logic $L(Q_1)$ is countably compact.

The logic frame $L(Q_1)$

Definition (Keisler)

The **axioms** of $L(Q_1)$ are the following:

1. Axioms of FO.
2. (Monotonicity) $\forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (Q_1 x \phi(x) \rightarrow Q_1 x \psi(x))$. ("Subset of a countable set is countable")
3. (Non-triviality) $\neg Q_1 x(x = y \vee x = z)$ ("2 is not uncountable")
4. (Change of variable) $Q_1 x \phi(x) \rightarrow Q_1 y \phi(y)$
5. (Keisler's Axiom) $Q_1 y \exists x \phi \rightarrow (\exists x Q_1 y \phi \vee Q_1 x \exists y \phi)$ ("Countable union of countable sets is countable"))

The Inference Rules are the same as for FO.

Completeness Theorem of $L(Q_1)$

Theorem (Keisler [3])

A countable theory in $L(Q_1)$ has a model if and only if it is consistent with the axioms and rules of $L(Q_1)$.

Corollary

$L(Q_1)$ is countably compact.

Definition

A **weak model** is a pair (M, q), where M is a model and $q \subseteq \mathbb{P}(M)$ such that the axioms are true if Q_1 is interpreted as follows:

$$M \models Q_1 x \phi(x, \vec{b}) \iff \{a \in M : M \models \phi(a, \vec{b}\} \in q.$$

Thus q is a "fake" version of being uncountable.

Weak Completeness Theorem

Theorem

A countable theory in $L(Q_1)$ has a countable weak model if and only if it is consistent with the axioms and rules of $L(Q_1)$.

Proof.

The Henkin style proof of the Weak Completeness Theorem is entirely standard. □

Proof of the Completeness Theorem of $L(Q_1)$

- The trick now is to start from a weak model (M, q) and extend it step by step to a real model.
- This is done with an elementary chain of weak models.
- The length of the chain is ω_1.
- In the chain every formula $\phi(x)$ that the theory "thinks" should be satisfied by uncountably many elements (i.e. $(M, q) \models Q_1 x \phi(x)$) is extended ω_1 many times.
- In the chain every formula $\psi(x)$ that the theory "thinks" should be satisfied by only countably many elements (i.e. $(M, q) \models \neg Q_1 x \psi(x)$) is kept all the time fixed.
- The key element of this construction is the Omitting Types Theorem, which allows one to extends one predicate while keeping another predicate fixed.

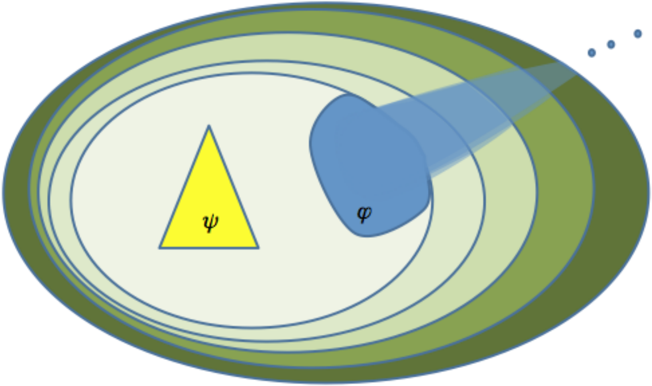

Figure: ψ is kept fixed while ϕ is extended.

Remarks

- Countable compactness of $L(Q_1)$ can be proved also by reduction to two-cardinal models (introduced by G. Fuhrken). Any sentence of $L(Q_1)$ can be translated into first order logic on so called two cardinal models, which are uncountable models (S^*) that have a special unary predicate P which is always countable in all models. So $(S, L(Q_1), \models) \leq_{\theta,\pi,f} (S^*, FO, \models)$ and now we can use compactness of first order logic on two cardinal models. That compactness was proved by Vaught and is a not too difficult model theoretic construction using so called Vaughtian Pairs, elementary chains and homogeneous models.
- There is a stronger logic, stationary logic, which is also countably compact. In many cases the countable compactness of particular extensions of first order logic by generalized quantifiers is closely related to such set theoretical principles as GCH, $\Box_\lambda$, $\diamondsuit$, etc.

Recap

- FO is compact and satisfies LS_ω. (i.e. LS_ω^Σ for the canonical Σ)
- $L_{\omega_1\omega}$ satisfies LS_ω but is not countably compact.
- $L(Q_1)$ is countably compact but fails to satisfy $L(Q_1)$.
- We now proceed to prove that there is **no** proper extension of FO that is both countably compact and satisfies LS_ω.
- This is the famous Lindström Theorem.

Lindström Theorem

Theorem (Lindström[4])

If FO $\leq$, *L is compact and satisfies* LS_ω, *then* $L \equiv FO$.

Characterization of elementary equivalence

Two models are *n*-**equivalent**, denoted $M \equiv_n N$, if they satisfy the same first order sentences of quantifier rank $\leq n$.

Theorem (Ehrenfeucht [1],Fraïssé [2])

Two models M and N are n-equivalent if and only if there are relations $I_i, i \leq n$, such that

- *If $(a_1, ..., a_i)I_i(b_1, ..., b_i)$, then $a_1, ..., a_i \in M$ and $b_1, ..., b_i \in N$*
- $()I_0()$
- *If $(a_1, ..., a_i)I_i(b_1, ..., b_i)$ then for all $a_{i+1} \in M$ $(b_{i+1} \in N)$ there is $b_{i+1} \in N$ $(a_{i+1} \in M)$ such that $(a_1, ..., a_{i+1})I_{i+1}(b_1, ..., b_{i+1})$.*
- *If $(a_1, ..., a_i)I_i(b_1, ..., b_i)$, then for all atomic formulas $\phi(v_1, ..., v_i)$ we have $M \models \phi(a_1, ..., a_i)$ if and only if $N \models \phi(b_1, ..., b_i)$.*

Proof

Proof.

Suppose first $M \equiv_n N$. Define for each $i \leq n$:

$$(a_1, \ldots, a_i) I_i (b_1, \ldots, b_i) \iff (M, a_1, \ldots, a_i) \equiv_{n-i} (N, b_1, \ldots, b_i).$$

It is easy to see that this is as required. Conversely, suppose such relations $(I_i)_{i<n}$ exist. Then one can prove by induction on $i \leq n$, going down from n all the way to 0, that

$$(a_1, \ldots, a_i) I_i (b_1, \ldots, b_i) \Rightarrow (M, a_1, \ldots, a_i) \equiv_{n-i} (N, b_1, \ldots, b_i).$$

When n reaches 0, we have $() I_0 ()$ and hence $M \equiv N$. □

Partial Isomorphism

The above sequences $(I_i)_{i \leq n}$ are called **back-and-forth sequences of length** $n + 1$

Definition

If a sequence $(I_i)_{i<\omega}$ exists so that $(I_i)_{i \leq n}$ is a back-and-forth sequence of length $n + 1$ for each n, then we say that M and N are **partially isomorphic**, in symbols $M \simeq_p N$. Then $(I_i)_{i<\omega}$ is called an **infinite back-and-forth sequence**.

Proposition

Countable partially isomorphic models are isomorphic.

This is a back-and-forth argument. Let $M = \{a_n : n \in \mathbb{N}\}$ and $N = \{b_n : n \in \mathbb{N}\}$.

1. Since $()I_0()$, there is $b'_0 \in N$ such that $(a_0)I_1(b'_0)$.
2. Since $(a_0)I_1(b'_0)$, there is $a'_0 \in M$ such that $(a_0, a'_0)I_2(b'_0, b_0)$.
3. Since $(a_0, a'_0)I_2(b'_0, b_0)$, there is $b'_1 \in N$ such that $(a_0, a'_0, a_1)I_3(b'_0, b_0, b'_1)$.
4. Since $(a_0, a'_0, a_1)I_3(b'_0, b_0, b'_1)$, there is $a'_1 \in M$ such that $(a_0, a'_0, a_1, a'_1)I_4(b'_0, b_0, b'_1, b_1)$.
5. Etc. We get sequences $\{a'_n : n \in \mathbb{N}\}$ and $\{b'_n : n \in \mathbb{N}\}$ such that for all n
$$(a_0, a'_0, ..., a_{n-1}, a'_{n-1})I_{2n}(b'_0, b_0, ..., b'_{n-1}, b_{n-1})$$
6. The mapping $a_i \mapsto b'_i$ is an isomorphism $M \to N$. It is onto: Let us look at b_i. We have chosen a "partner" a'_i for it. For some j we have $a_j = a'_i$. So $b'_j = b_i$, that is, the element a_j is mapped to b_i.

First: An easier version of Lindström Theorem

- Assume $FO < L$, L is compact and it satisfies LS_ω for countable sets of sentences (rather than just for single sentences). Assume also all vocabularies are finite.
- It suffices to prove $L \leq_w FO$. So assume $M \equiv N$ but $M \not\equiv_L N$, say $M \models \phi$ and $N \models \neg\phi$. For every n there is a back-and-forth sequence $(I_i)_{i \leq n}$ for (M, N).
- Let us formulate the claim that there is an infinite sequence $(I_i)_{i \leq n}$ for a model of ϕ and a model of $\neg\phi$ into a set T of L-sentences using new predicate symbols P_i, $i \in \mathbb{N}$.

Let the vocabulary of ϕ be τ (Occurrence Condition!). Let us take a disjoint copy τ' of τ. So there is a name changer $\pi : \tau \to \tau'$. For any atomic formula ψ of the vocabulary τ let ψ' be the τ'-translation of ψ. Let ϕ' be the translation of ϕ into the vocabulary τ' (Name Changer Condition!). We now write a theory T in the vocabulary $\tau \cup \tau'$. Recall that we can form negations and conjunctions in L (Boolean Condition):

1. $\phi \wedge \neg\phi'$
2. $P_0()$
3. $\forall x_1 ... \forall x_i \forall y_1 ... \forall y_i (P_i(x_1, ..., x_i, y_1, ..., y_i) \to \forall x_{i+1} \exists y_{i+1} P_{i+1}(x_1, ..., x_{i+1}, y_1, ..., y_{i+1})$, for $i \in \mathbb{N}$
4. $\forall x_1 ... \forall x_i \forall y_1 ... \forall y_i (P_i(x_1, ..., x_i, y_1, ..., y_i) \to \forall y_{i+1} \exists x_{i+1} P_{i+1}(x_1, ..., x_{i+1}, y_1, ..., y_{i+1})$, for $i \in \mathbb{N}$
5. $\forall x_1 ... \forall x_i \forall y_1 ... \forall y_i (P_i(x_1, ..., x_i, y_1, ..., y_i) \to (\phi(x_1, ..., x_i) \leftrightarrow \phi'(y_1, ..., y_i))$ for $i \in \mathbb{N}$ and for atomic formulas $\phi(x_1, ..., x_n)$ of τ.

Finishing the proof

1. Every finite subset of T can be satisfied using M and N.
2. By compactness the whole T can be satisfied.
3. By the strong form of LS_ω we can satisfy T with a countable model M^*.
4. Let $M_0 = M^* \upharpoonright \tau$ and $M_1' = M^* \upharpoonright \tau'$. Let M_1 be the translation of M_1' back to the vocabulary τ. Then $M_0 \models \phi$ and $M_1 \models \neg\phi$.
5. Let us define in M: $(a_1, \ldots, a_i) I_i (b_1, \ldots, b_i)$ if $M \models P_i(a_1, \ldots, a_i, b_1, \ldots, b_i)$. Now $(I_i)_{i \in \mathbb{N}}$ is an infinite back-and-forth sequence for (M_0, M_1).
6. Since M_0 and M_1 are countable, $M_0 \cong M_1$, so we get a contradiction with the (Isomorphism Condition).

□

Finite Occurrence Property

Definition

An abstract logic L has the **Finite Occurrence Property** if for every $\phi \in F$ there is a finite τ such that for all $M \in S$: $M \models \phi \iff M \restriction \tau \models \phi$.

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Lemma

A compact logic satisfies the Finite Occurrence Property.

Proof

Proof.

1. Suppose $\phi \in F$. There is τ such that for all $M \in S$: $M \models \phi \iff M \restriction \tau \models \phi$.
2. Let τ' be a disjoint version of τ obtained by changing each predicate symbol R to R'. Let ϕ' be the name changer translation of ϕ obtained by changing each R to R'.
3. Consider the following set of sentences in $\tau \cup \tau'$:
    1. $\forall x_1 ... \forall x_{k_n}(R(x_1, ..., x_{k_n}) \leftrightarrow R'(x_1, ..., x_{k_n}))$ for all $R \in \tau$
    2. $\neg(\phi \leftrightarrow \phi')$
4. T does not have a model. By compactness some finite part does not have a model. This gives the finite $\tau_0 \subset \tau$ that we want.

□

LS_ω suffices

- To prove that LS_ω suffices one has to reduce the countable theory T with the predicates P_i into one sentence ψ.
- Instead of infinitely many predicate symbols P_i of arity $2i$ we use one binary predicate symbol R. The first argument of R codes i and the second argument of R codes the i-sequences $a_1, ..., a_n$ and $b_1, ..., b_i$. For this we employ a pairing function to code finite sequences.
- We have to use infinite models, but because of the compactness assumption, any sentence with arbitrarily large finite models actually has also infinite models. (In case we have a sentence that has only models of size $\leq n$ for a fixed n, then that sentence has to be FO-definable.)

Contd.

- There is a special linear order $(P, <)$ which tells how long the back-and-forth sequence coded by R is.
- ψ has models with $(P, <)$ arbitrarily large finite linear order.
- ψ cannot have models in which $(P, <)$ has an infinite increasing sequence, because that would give rise to two partially isomorphic models, one satisfying ϕ and the other $\neg\phi$. By LS_ω we could these models to be countable, hence isomorphic.
- We can use ψ to get other characterizations, like ULS_ω & LS_ω, Beth & LS_ω, axiomatizability & LS_ω, etc. (see later).

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