Abstract Logic

Lecture 4

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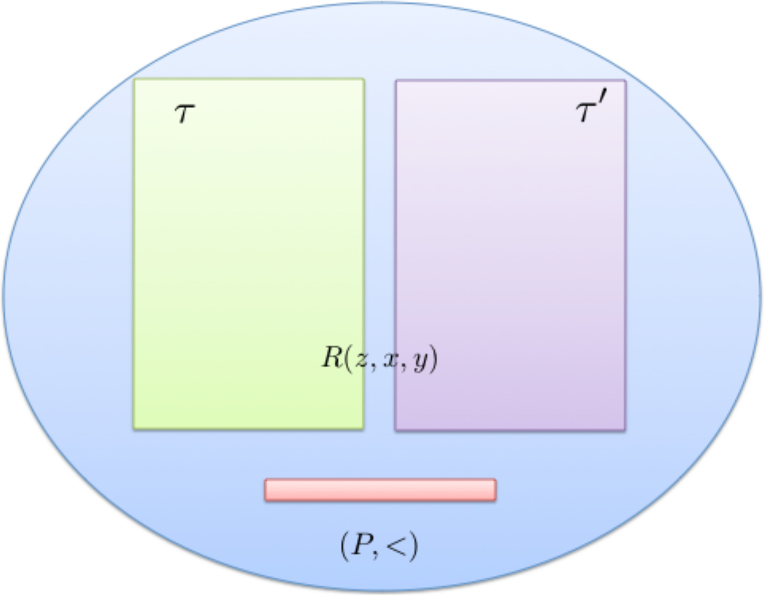
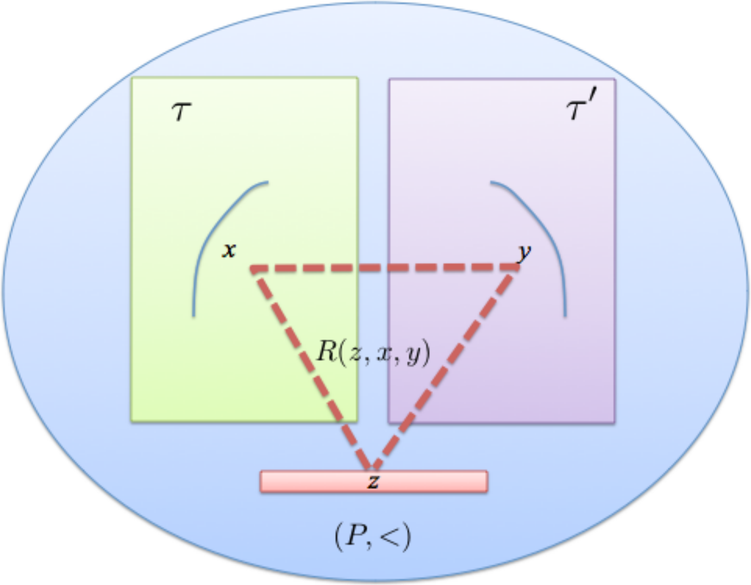
Full proof of Lindström Theorem

- Recall that we have a finite vocabulary τ and a disjoint copy of it τ'
- We have a ternary predicate R coding a back-and-forth set.
- We have a unary predicate P and an order relation $<$ on P, and a symbol 0 for the smallest element of the order.
- We have a ternary predicate $S(x, y, z)$ with the meaning "element number x in the sequence y is z". We denote this z be $y(x)$.


τ
τ'
$R(z, x, y)$
$(P, <)$



τ
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x
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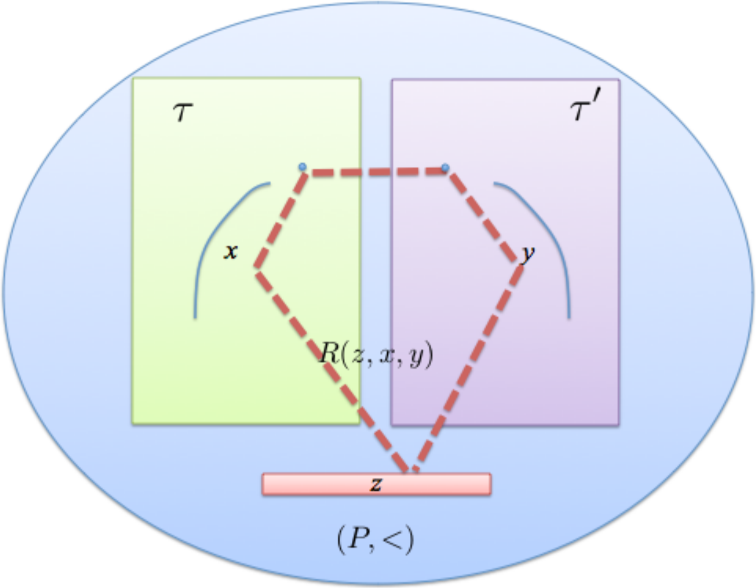
$R(z, x, y)$
z
$(P, <)$


τ
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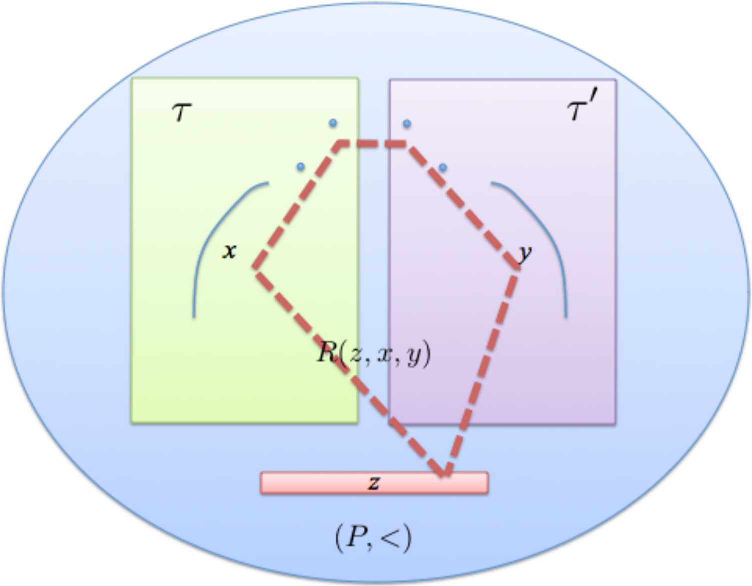

τ
τ'
x
y
$R(z, x, y)$
z
$(P, <)$

The sentence ϕ

Let ψ be the following sentence of L:

1. $\phi \wedge \neg\phi'$
2. $\exists x \exists y R(0, x, y)$
3. $\forall x(P(x) \to (\exists y(P(y) \wedge x < y) \to \exists y(P(y) \wedge x < y \wedge \forall z((P(z) \wedge x < z) \to (y < z \vee y = z)))))$
4. $\forall z \forall x \forall y \forall s \exists r \exists x' \exists y'((P(z) \wedge R(z, x, y)) \to \forall w((P(z) \wedge z < w) \to \forall s \exists r \exists x' \exists y'(\forall t(t \neq w \to (x(t) = x'(t) \wedge y'(t) = y(t)) \wedge S(z, x', s) \wedge S(z, y', r) \wedge R(w, x', y'))$
5. $\forall z \forall x \forall y \forall s \exists r \exists x' \exists y'((P(z) \wedge R(z, x, y)) \to \forall w((P(z) \wedge z < w) \to \forall r \exists s \exists x' \exists y'(\forall t(t \neq w \to (x(t) = x'(t) \wedge y'(t) = y(t)) \wedge S(z, x', s) \wedge S(z, y', r) \wedge R(w, x', y'))$
6. $\forall z \forall x \forall y(R(z, x, y) \to \forall t_1 ... \forall t_n((P(t_1) \wedge ... \wedge P(t_n) \wedge t_1 < z \wedge ... \wedge t_n < z) \to (\phi(x(t_1), ..., x(t_n)) \leftrightarrow \phi'(y(t_1), ..., y(t_n)))))$ for atomic formulas $\phi(x_1, ..., x_n)$ of τ.

Let ψ be the following sentence of L:

1. ϕ holds in the τ-vocabulary and ϕ' in the τ'-vocabulary.
2. At time 0 some sequences (perhaps empty sequences) are equivalent.
3. If an element of P has a successor, it has an immediate successor.
4. If at time z we have two equivalent sequences x and y, and w is a later moment of time, and we extend x to a sequence x' with some new elements, then we can likewise extend y to y' and then x' and y' are equivalent at time w.
5. The same as above but in the other direction.
6. If sequences x and y are equivalent at some moment of time then their components form a partial isomorphism between the τ-part and the τ'-part.

Crucial lemmas

Let $Th_n(M)$ be the (finite) set of first order τ-formulas with quantifiers rank $\leq n$ that are true in M.

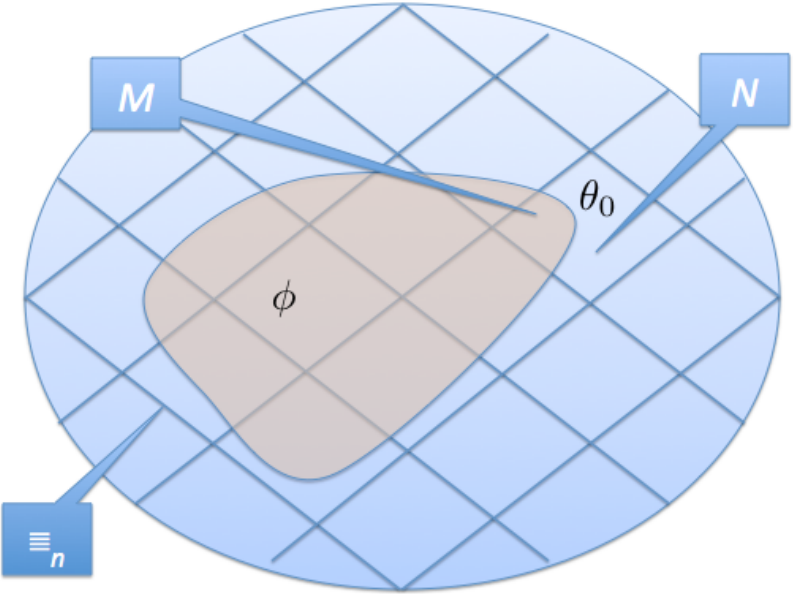
Lemma

For every n we can find a model M of ψ such that the length of $(P, <)^M$ is exactly n.

Proof: Let us fix n. We can choose M_0 so that $\phi \wedge \bigwedge Th_n(M_0)$ is not first order, for if each of the finitely many $\phi \wedge \bigwedge Th_n(M_0)$ for different M_0 is first order, so is ϕ as their disjunction. Let $\theta_0 = \bigwedge Th_n(M_0)$. We can choose $M \models \phi \wedge \theta_0$ so that M is infinite. Otherwise:

- $\phi \wedge \theta_0$ has no infinite models. But surely $\phi \wedge \theta_0$ has arbitrarily large finite models, for otherwise there is an natural number n_0 such that $\phi \wedge \theta_0$ has no models of size $\geq n_0$. From this it would follow that $\phi \wedge \theta_0$ is first order, a contradiction. So $\phi \wedge \theta_0$ has arbitrarily large finite models, but no infinite ones. This contradicts compactness.

Since we assume LS_ω, we can choose M to be countably infinite. Note that also $\neg\phi \wedge \theta_0$ is non-first order. So a similar argument gives a countably infinite $N \models \neg\phi \wedge \theta_0$. Thus M and N are countably infinite, $M \equiv_n N$, $M \models \phi$ and $N \models \neg\phi$. W.l.o.g. both have $\mathbb{N}$ are their universe. The Lemma follows now easily.


M
N
θ_0
ϕ
$\equiv_n$

Second crucial lemma

Lemma

ψ has no models M such that $(P, <)^M$ is infinite.

Proof.

Suppose such an M existed. W.l.o.g. M is countably infinite. Let $M_0 = M \restriction \tau$ and $M_1' = M \restriction \tau'$. Let M_1 be the translation of M_1' back to the vocabulary τ'. Now look at $c_0 = 0^M$. Since $(P, <)^M$ is infinite, there are elements in P^M that are greater than c_0. Hence there is the least such, call it c_1. By continuing in this way we get an infinite ascending sequence $c_0 < c_1 < c_2 < \ldots$ in P^M. Define I_n on M by $(a_1, \ldots, a_n) I_n (b_1, \ldots, b_n)$ iff there are a and b in M such that $M \models R(c_n, a, b)$ and $a(i) = a_i$ and $b(i) = b_i$ for $i = c_i$, $i = 0, \ldots, n-1$. Clearly $(I_n) n \in \mathbb{N}$ is an infinte back-and-forth sequence between M_0 and M_1. Hence $M_0 \cong M_1$, a contradiction since $M_0 \models \phi$ and $M_1 \models \neg\phi$. □

Finishing the proof of Lindström Theorem

Proof.

Recall that we have an abstract logic L such that $FO \leq L$, L is compact and L satisfies the LS_ω. We noted that L must satisfy the Finite Occurrence Condition. If $FO < L$, then the construction of the sentence ψ above leads to a contradiction. □

Observation about the proof

We can reformulate the construction behind the above proof as follows:

Theorem (Lindström)

If $FO < L$, L satisfies the Finite Occurrence Condition and the LS_ω, then there is a sentence $\psi \in L$ with a unary predicate P so that for every $n \in \mathbb{N}$ there is a model M of ψ with P^M of size n, but there are no models M of ψ with P^M infinite.

There was one place where we used compactness in the middle of the proof. It was to derive a contradiction from the assumption that some sentence in L has arbitrarily large finite model but no infinite ones. But in this case the Theorem is trivially true!

Corollary

Corollary (Lindström)

Suppose If $FO \leq L$, L is "strong" and satisfies the Finite Occurrence Condition, ULS_ω and LS_ω. Then $L \equiv FO$.

Proof.

Suppose such an L satisfies $FO < L$. Let ψ be as above. We can write another sentence ψ_ω with $<'$ $<'$ is a linear order of the whole model and for every a the sentence ψ is true with $P(z)$ and $x < y$ replaced everywhere by $z <' a$ and $x <' y \wedge y <' a$. For this to be possible we need the assumption "strong". Now ψ_ω has a countably infinite model. But in any infinite model every element can have only finitely many $<'$-predecessors. Hence ψ_ω cannot have any uncountable models. □

Another formulation

Theorem (Lindström)

First order logic is the maximal logic in which every sentence which has models of some infinite cardinality has models in every infinite cardinality. This is a strong cardinal absoluteness of first oder logic. It follows that every proper extension of first order logic has to make a distinction between some infinite cardinalities.

Axiomatizability

Theorem (Mostowski)

Suppose $FO \leq L(Q)$, Q unary, $L(Q)$ is axiomatizable and satisfies LS_ω. Then $L(Q) \equiv FO$.

What about non-unary?

Theorem (Lindström)

Suppose $FO \leq L$, L satisfies the Finite Occurrence Condition, L is axiomatizable, and L satisfies LS_ω. Then $L \equiv FO$.

Proof.

Suppose $FO < L$. Then we can construct ψ as above. Let τ be the finite vocabulary of ψ and let $\tau' = \{R\}$, R binary, $R \notin \tau$. For any first order τ'-sentence ϕ let $\phi^{(P)}$ be the relativization of ϕ to the predicate P. Then the following conditions are equivalent:

1. ϕ is valid in finite τ'-models.
2. $\psi \to \phi^{(P)}$ is valid $\tau \cup \tau'$- models.

Why? Suppose (1) and let M be any $\tau \cup \tau'$-model of ψ. Then P^M is finite, so $\phi^{(P)}$ holds in $M \restriction \tau'$. Conversely, suppose (2) and suppose M is a finite τ'-model. Let N be a finite τ-model of ψ such that $|P^N| = |M|$. Let N' be a $\tau \cup \tau'$-model so that $N' \restriction \tau = N$ and $(N' \cap P^N) \restriction \tau' = M$. Since $N' \models \psi \to \phi^{(P)}$ and $N' \models \psi$, we have $N' \models \phi^{(P)}$. Hence $M \models \phi$.
We get a contradiction with Trakhtenbrot's Theorem that in the vocabulary of one binary predicate symbol the set of Gödel numbers of first order sentences that are valid in finite models is not r.e. □

Generalized quantifiers

Definition (Lindström)

A **generalized quantifier** is any class K of models of a finite relational vocabulary, closed under isomorphisms. Suppose the vocabulary is (for simplicity) $\{P, R\}$, where P is unary and R is binary. Then we associate with K the **generalized quantifier**

$$M \models Q_K x, yz\phi(x)\psi(y,z) \iff (dom(M), P^*, R^*) \in K,$$

where $P^* = \{a \in M : M \models \phi(a)\}$ and $R^* = \{(a,b) \in M^2 : M \models \psi(a,b)\}$.

In this way we get a variety of extensions of FO. They are all “strong” and satisfy the Finite Occurrence Condition.

Examples

Example

$$M \models Q_0 x \phi(x) \iff \{a : M \models \phi(a)\} \text{ is infinite}$$

$$M \models Q_1 x \phi(x) \iff \{a : M \models \phi(a)\} \text{ is uncountable}$$

$$M \models Ix, y\phi(x)\psi(y) \iff |\{a : M \models \phi(a)\}| = |\{a : M \models \psi(a)\}|$$

$$M \models Wxy\phi(x, y) \iff \{(a, b) : M \models \phi(a, b)\} \text{ is a well-ordering.}$$

$$M \models Gxy\phi(x, y) \iff \{(a, b) : M \models \phi(a, b)\} \text{ is a connected graph.}$$

Beth Theorem

Definition

A logic $L(\vec{Q})$ satisfies the **Beth Definability Property** is for any sentence $\theta(P)$ such that $\theta(P) \wedge \theta(P') \models \forall x_1 ... \forall x_n (P(x_1, ..., x_n) \leftrightarrow P'(x_1, ..., x_n)$, there is a formula ϕ without P such that

$$\theta(P) \models \forall x_1 ... \forall x_n (P(x_1, ..., x_n) \leftrightarrow \phi(x_1, ..., x_n)).$$

Proposition (Mostowski)

Beth Definability Property fails for familiar extensions of FO satisfying LS_ω. Is there a general criterion?

Theorem (Lindström)

Suppose $\vec{Q}$ is a finite sequence of generalized quantifiers such that $L(\vec{Q})$ satisfies both the Beth Definability Property and LS_ω. Then $L(\vec{Q}) \equiv FO$.

The assumption that the logic is of the form $L(\vec{Q})$ cannot be omitted right away because $L_{\omega_1\omega}$ satisfies both the Beth Definability Theorem and LS_ω.

Definability of truth

- Suppose τ is a finite vocabulary containing the vocabulary of arithmetic, and $\tau' = \tau \cup \{P\}$, where P is unary.
- Let $\{\phi_n : n \in \mathbb{N}\}$ be a list of all τ-sentences of $L(\vec{Q})$ such that n is the Gödel number of ϕ_n.
- There is $\theta \in L(\vec{Q})$ in vocabulary τ' such that the following are equivalent for any τ-sentence ϕ of $L(\vec{Q})$ and any $M \models \theta$
  1. $M \restriction \tau \models \phi_n$
  2. $M \models P(\dot{n})$, where $\dot{n}$ is the term denoting n in arithmetic.
- The sentence $\psi_\omega \wedge \theta$ defines P implicitly. If P was explicitly defined by $\psi_\omega \wedge \theta$, we would get a contradiction by an Undefinability of Truth argument.