Abstract Logic

Lecture 5

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May 18, 2010

The Uncountable Omitting Types Theorem

For simplicity we consider logics obtained from FO be adding (perhaps infinitely many) generalized quantifiers. All known logics, like FO, $L_{\omega_1\omega}$ and the second order logic are of this form.

Definition

Suppose L is an abstract logic. A pair $(\Phi, \Psi(x))$, where Φ is a set of sentences of L and $\Psi(x)$ is a set of formulas of L with one free variable x, satisfies the λ-**Omitting Types Condition**, $\lambda - OTC$, if

- Φ has a model,
- $|\Phi| \leq \lambda$ and $|\Psi(x)| = \lambda$,
- For every $\Theta(x)$ such that $|\Theta(x)| < \lambda$, if $\Phi \cup \{\exists x \bigwedge \Theta(x)\}$ has a model, then there is $\psi(x) \in \Psi(x)$ such that $\Phi \cup \{\exists x(\bigwedge \Theta(x) \wedge \neg\psi(x))\}$ has a model.

Definition

An abstract logic L has the λ-**Omitting Types Property**, $\lambda - OTP$, if for any $(\Phi, \Psi(x))$ satisfying the $\lambda - OTC$ the set $\Phi \cup \{\forall x \bigvee_{\psi(x)\in\Psi(x)} \neg\psi(x)\}$ has a model (i.e. $\Psi(x)$ is "omitted").

$L_{\omega_1\omega}$ satisfies $\omega - OTP$. FO satisfies $\lambda - OTP$ for all infinite λ.

Why does FO satisfy $\lambda - OTP$?

Let us recall the proof. Suppose $(\Phi, \Psi(x))$ has $\lambda - OTC$. Let c_α, $\alpha < \lambda$, be new constants. Let ϕ_α,$\alpha < \lambda$, be a list of all sentences with possible the new constants. We make a Henkin style construction of an increasing sequence T_α, $\alpha < \lambda$, of consistent theories so that

- For each $\alpha = \nu + 3n$, ν limit, there is some $\psi_\alpha(x) \in \Psi(x)$ so that $\neg\psi_\alpha(c_{\nu+n}) \in T_{\alpha+1}$,
- For each $\alpha = \nu + 3n + 1$, ν limit, if $\phi_{\nu+n}$ is of the form $\exists x\psi(x)$, then there is some c_ξ so that $\psi(c_\xi) \in T_{\alpha+1}$,
- For each $\alpha = \nu + 3n + 2$, ν limit, either $\phi_{\nu+n} \in T_{\alpha+1}$ or $\neg\phi_{\nu+n} \in T_{\alpha+1}$.

$T_0 = \Phi$. Suppose T_α has been defined, and $\alpha = \nu + 3n$, ν limit. We claim that there is some $\psi(x) \in \Psi(x)$ so that $T_\alpha \cup \{\psi_\alpha(c_{\nu+n})\}$ is consistent. The condition $\lambda - OTC$ guarantees that such a $\psi(x) \in \Psi(x)$ exists.

Another characterization of FO

Theorem (Lindström [3])

If L satisfies $\lambda - OTP$ for some $\lambda > \omega$, then $L \equiv_{inf} FO$, i.e. every sentence of L is equivalent in infinite models to a first order sentence.

LS follows from OTP

Proposition

Suppose L has the $\lambda - OTP$, $|\Phi| \leq \lambda$ and Φ has a model. Then Φ has a model of size $\leq \lambda$.

Proof.

Let $\Psi = \{\neg x = c_\alpha : \alpha < \lambda\}$, where c_α, $\alpha < \lambda$, are new constants. Clearly, $(\phi, \Psi(x))$ satisfies the $\lambda - OTC$. Hence there is a model M of Φ such that $M \models \forall x(\bigvee_\alpha x = c_\alpha)$. □

Getting models of cofinality λ

Lemma

Suppose λ is regular, L has the $\lambda - OTP$, $|\Phi| \leq \lambda$ and Φ has a model M such that $(P, <)^M$ is a linear order without last element. Then Φ has a model N such that $(P, <)^N$ is a linear order of cofinality λ.

Proof: Let c_α, $\alpha < \lambda$, be new constants. Let

$$\Phi^* = \Phi \cup \{\text{“}(P, <) \text{ is a linear order w/o last element.}\} \cup$$

$$\{P(c_\alpha) \wedge c_\alpha < c_\beta : \alpha < \beta < \lambda\}.$$

Clearly Φ^* has a model. Now

$$(\Phi^*, \{P(x)\} \cup \{c_\alpha \leq x : \alpha < \lambda\})$$

satisfies the $\lambda - OTC$.

Therefore Φ^* has a model M such that

$$M \models \forall x(P(x) \rightarrow \bigvee_{\alpha<\lambda} x < c_\alpha).$$

It is obvious that $(P, <)^M$ has cofinality λ. QED

Upward LS from OTP

Lemma

Suppose λ is regular, L has the λ-OTP, $|\Phi| \leq \lambda$ and Φ has an infinite model. Then Φ has a model of cardinality λ.

Proof.

Suppose M is an infinite model of Φ. Expand M be a linear order $(P, <)$ with last element. My the previous lemma Φ has a model in which $(P, <)$ has cofinality λ. So Φ has a model of cardinality $\geq \lambda$. By another previous lemma Φ has a model of cardinality exactly λ. □

Compactness from OTP

Definition

A logic is λ-**compact** if every finitely consistent set T of sentences such that $|T| \leq \lambda$ has a model. We say "in infinite models" if only infinite models are considered.

Lemma

If λ is regular and L has the λ-OTP, then L is μ-compact for all $\mu < \lambda$.

We use below the following construction: Given τ, let τ^+ be obtained from τ by replacing each $R(x_1, ..., x_n)$ by $R^+(x, x_1, ..., x_n)$. SImilarly ϕ^+ and M^+.

Proof.

We use induction on ν. Let us assume L is μ-compact for all $\mu < \nu$ (holds trivially if $\nu = \omega$). Suppose $T = \{\phi_\xi : \xi < \nu\}$ is finitely consistent. By Induction Hypothesis, $\{\phi_\xi : \xi \leq \eta\}$ has an infinite model for each $\eta < \nu$. By the above lemma we may assume these models are of cardinality λ. We take some new symbols and let

$$\Psi = \{P(c_\xi) \wedge \forall x(c_\xi \leq x \rightarrow \phi_\xi^+(x)) : \xi < \nu\}.$$

By the above lemma Ψ has a model M so that $(P, <)$ has cofinality λ. Since $\nu < \lambda$, there is $a \in P^M$ so that all the constants c_ξ, $\xi < \nu$, are below a. From a we can get a model where each ϕ_ξ, $\xi < \nu$, i.e. T itself, is true. □

OTP characterizes *FO*

Theorem (Lindström)

If L satisfies $\lambda - OTP$ for some $\lambda > \omega$, then $L \equiv FO$, i.e. every sentence of L is equivalent in infinite models to a first order sentence.

Proof

Proof:
Let us assume we have $M \equiv N$ such that $M \models \phi$ and $N \models \neg\phi$ for some ϕ in L. W.l.o.g. M and N have a finite vocabulary τ. Let τ' be a disjoint copy of τ. Note that M and N are infinite. By the above lemma, w.l.o.g. M and N have cardinality λ. We take new constants c_ξ, c'_{xi}, $\xi < \lambda$. Let

$$\Psi = \{\phi, \neg\phi'\} \cup \{\psi(c_{\xi_1}, ...c_{\xi_n}) \leftrightarrow \psi'(c'_{\xi_1}, ...c'_{\xi_n}) : \psi \text{ atomic}, \xi_1, ..., \xi_n < \lambda\}.$$

We show that

$$(\Psi, \{x \neq c_\xi \vee x \neq c'_\eta : \xi, \eta < \lambda\})$$

has the λ-OTC.

- Clearly, Ψ has a model.
- Now let $|\Theta(x)| < \lambda$ such that $\Psi \cup \{\exists x \bigwedge \Theta(x)\}$ has a model K. Suppose c is an element of K that satisfies $\Theta(x)$.
- Let $X = \{\xi : c_\xi$ or c'_ξ occurs in $\Theta(x)\}$. Note that $|X| < \lambda$. Pick $\eta < \eta'$ from $\lambda \setminus X$.
- By using $M \equiv N$ and the fact that L is $|X|$-compact, one can construct a model for $\Psi \cup \Theta(c^0_\eta) \cup \{c_\eta = c'_{\eta'}\}$.

Finishing the proof

- By the λ-OTP we have a model K of Ψ such that $K \models \forall x \bigvee_{\xi,\xi'<\lambda}(x = c_\xi = c'_{\xi'})$.
- Thus
$$c_\xi^K \mapsto {c'}_\xi^K$$
is an isomorphism between the τ-part and the τ'-part of K.
- This contradicts the fact that the τ-part satisfies ϕ and the τ'-part satisfies $\neg\phi$. QED

Absolute logics (Barwise)

Definition (Barwise [2])

Let T be a true set theory extending the Kripke-Platek axioms KP. An abstract logic $L = (F, \models)$ is **absolute relative to** T if there are Σ_1-predicates $R(x, y)$ and $S(x, y, z)$ and a Π_1-predicate $P(x, y, z)$ such that

1. For all ϕ: $\phi \in F$ if and only if $R(\phi, L)$.
2. For all $\phi \in F$ and all L-structures M, $M \models \phi$ if and only if $S(M, \phi, L)$.
3. The following is a theorem of T: For all languages z, all z-structures M and all ϕ such that $R(\phi, z)$, $S(M, \phi, z)$ if and only if $P(M, \phi, z)$.

An abstract logic is **strictly absolute** if it is absolute relative to KP (or KP+Infinity).

Absolute logics contd.

Conjunction and other logical operations are assumed to be absolute, too. If L is an abstract logic and A is an admissible set, we use L_A to denote the sub-logic of L consisting of sentences which are elements of A. $L_{H(\kappa)}$ is denoted L_κ.

Example

- The logics FO, $FO(Q_0)$, $L_{\omega_1\omega}$ and $L_{\infty\omega}$ are absolute relative to KP.
- The weak second order logic is strictly absolute.
- The unbounded logic $L_{\infty\omega}(WO)$ is absolute relative to $KP + \Sigma_1$-separation.

Example

- If we add the game quantifier

$$\forall x_1 \exists x_2 \forall x_3 \ldots \bigvee_{n<\omega} \phi_n(x_1, \ldots, x_{p_n}, \vec{y})$$

  and its dual

$$\exists x_1 \forall x_2 \exists x_3 \ldots \bigwedge_{n<\omega} \phi_n(x_1, \ldots, x_{p_n}, \vec{y})$$

  to $L_{\infty\omega}$ an interesting (also unbounded) logic, denoted by $L_{\infty G}$ emerges. This logic is absolute relative to $KP + \Sigma_1-$separation.
- The smallest admissible fragment $(L_{\omega_1\omega})_{HYP}$ of $L_{\omega_1\omega}$ is an interesting absolute logic (usually denoted L_{HYP}).
- The infinite quantifier logic $L_{\omega_1\omega_1}$ and second order logic L^2 are not absolute relative to any true first order set theory T.

Levy Reflection

Theorem (Levy Reflection Principle)

If $\phi(\vec{x})$ is a Σ_1-formula and $\kappa > \omega$ then $\forall \vec{x} \in H(\kappa)(\phi(\vec{x}) \rightarrow H(\kappa) \models \phi(\vec{x}))$.

We can make some immediate observations about absolute logic L by means of Theorem 12:

- If $\phi \in L_{\kappa^+}$ has a model, then it has a model in $H(\kappa^+)$ and therefore a model of cardinality $\leq \kappa$. Thus L_{κ^+} satisfies the Löwenheim-Skolem Theorem down to κ.
- We can prove $L \leq_w L_{\infty\omega}$ almost as quickly: Suppose there is a back-and-forth set E for M and N, but for some $\phi \in L$ we have $M \models \phi$ and $N \not\models \psi$. All this is Σ_1, so by Theorem 12 such objects L, ϕ, M, N, E must exist already in HC. But then M and N are countable, hence isomorphic, a contradiction.

Theorem (Barwise)

If L is a strictly absolute logic and A is an admissible set, then L_A is contained in $(L_{\infty\omega})_A$.

Proof: Suffices to prove this for countable admissible sets: Suppose the claim fails. Thus there is an admissible set A and a sentence $\phi \in L_A$ such that for all $\psi \in (L_{\infty\omega})_A$ there is a model M of $\neg(\phi \leftrightarrow \psi)$. This can be written as a Σ_1-property of A. If an A with this property exists at all, one such exists in HC by Levy Reflection.

Note: If A is a countable admissible set, then L_A satisfies the Craig Interpolation Property.

End of the proof

Suppose L is a strictly absolute logic and $\phi \in A$ is an L-sentence with vocabulary τ (finite). We take a new vocabulary τ' containing τ and the new symbols $\varepsilon, \bar{M}, \bar{\phi}, \bar{L}$. It is possible to write down a sentence Φ of $(L_{\infty\omega})_A$ such that the following conditions are equivalent for any infinite L-structure M:

1. $M \models \phi$
2. There is an expansion M' of M which is a model of Φ and which satisfies $S(\bar{M}, \bar{\phi}, \bar{L})$
3. Every expansion M' of M to a model of Φ satisfies $S(\bar{M}, \bar{\phi}, \bar{L})$

We use the fact that the standard part of a model of KP is again a model of KP. Now the claim follows from the Interpolation Theorem.

We get as a special case the promised characterization of infinitary languages $L_{\kappa\omega}$ for any κ:

Corollary (Barwise [2])

If L is a strictly absolute logic and $\kappa > \omega$, then L_κ is contained in $L_{\kappa\omega}$.

Corollary

$L_{\infty\omega}$ is the largest strictly absolute logic.

First order logic

Theorem (Akkanen, V. [1])

FO is maximal logic that is absolute with respect to KPU^-.

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