

Abstract Logic

Lecture 5

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The Uncountable Omitting Types Theorem

For simplicity we consider logics obtained from FO by adding (perhaps infinitely many) generalized quantifiers. All known logics, like FO , $L_{\omega_1\omega}$ and the second order logic are of this form.

Definition

Suppose L is an abstract logic. A pair $(\Phi, \Psi(x))$, where Φ is a set of sentences of L and $\Psi(x)$ is a set of formulas of L with one free variable x , satisfies the λ -**Omitting Types Condition**, λ -*OTC*, if

- Φ has a model,
- $|\Phi| \leq \lambda$ and $|\Psi(x)| = \lambda$,
- For every $\Theta(x)$ such that $|\Theta(x)| < \lambda$, if $\Phi \cup \{\exists x \wedge \Theta(x)\}$ has a model, then there is $\psi(x) \in \Psi(x)$ such that $\Phi \cup \{\exists x (\wedge \Theta(x) \wedge \neg \psi(x))\}$ has a model.

Definition

An abstract logic L has the λ -**Omitting Types Property**, λ -*OTP*, if for any $(\Phi, \Psi(x))$ satisfying the λ -*OTC* the set $\Phi \cup \{\forall x \bigvee_{\psi(x) \in \Psi(x)} \neg \psi(x)\}$ has a model (i.e. $\Psi(x)$ is “omitted”).

$L_{\omega_1\omega}$ satisfies ω -*OTP*. *FO* satisfies λ -*OTP* for all infinite λ .

Why does FO satisfy $\lambda - OTP$?

Let us recall the proof. Suppose $(\Phi, \Psi(x))$ has $\lambda - OTC$. Let c_α , $\alpha < \lambda$, be new constants. Let ϕ_α , $\alpha < \lambda$, be a list of all sentences with possible the new constants. We make a Henkin style construction of an increasing sequence T_α , $\alpha < \lambda$, of consistent theories so that

- For each $\alpha = \nu + 3n$, ν limit, there is some $\psi_\alpha(x) \in \Psi(x)$ so that $\neg\psi_\alpha(c_{\nu+n}) \in T_{\alpha+1}$,
- For each $\alpha = \nu + 3n + 1$, ν limit, if $\phi_{\nu+n}$ is of the form $\exists x\psi(x)$, then there is some c_ξ so that $\psi(c_\xi) \in T_{\alpha+1}$,
- For each $\alpha = \nu + 3n + 2$, ν limit, either $\phi_{\nu+n} \in T_{\alpha+1}$ or $\neg\phi_{\nu+n} \in T_{\alpha+1}$.

$T_0 = \Phi$. Suppose T_α has been defined, and $\alpha = \nu + 3n$, ν limit. We claim that there is some $\psi(x) \in \Psi(x)$ so that $T_\alpha \cup \{\psi_\alpha(c_{\nu+n})\}$ is consistent. The condition $\lambda - OTC$ guarantees that such a $\psi(x) \in \Psi(x)$ exists.

Theorem (Lindström [3])

If L satisfies $\lambda - OTP$ for some $\lambda > \omega$, then $L \equiv_{inf} FO$, i.e. every sentence of L is equivalent in infinite models to a first order sentence.

Proposition

Suppose L has the λ -*OTP*, $|\Phi| \leq \lambda$ and Φ has a model. Then Φ has a model of size $\leq \lambda$.

Proof.

Let $\Psi = \{\neg x = c_\alpha : \alpha < \lambda\}$, where c_α , $\alpha < \lambda$, are new constants. Clearly, $(\phi, \Psi(x))$ satisfies the λ -*OTC*. Hence there is a model M of Φ such that $M \models \forall x (\bigvee_\alpha x = c_\alpha)$. □

Getting models of cofinality λ

Lemma

Suppose λ is regular, L has the λ – OTP, $|\Phi| \leq \lambda$ and Φ has a model M such that $(P, <)^M$ is a linear order without last element. Then Φ has a model N such that $(P, <)^N$ is a linear order of cofinality λ .

Proof: Let c_α , $\alpha < \lambda$, be new constants. Let

$$\Phi^* = \Phi \cup \{“(P, <) \text{ is a linear order w/o last element.}”\} \cup \\ \{P(c_\alpha) \wedge c_\alpha < c_\beta : \alpha < \beta < \lambda\}.$$

Clearly Φ^* has a model. Now

$$(\Phi^*, \{P(x)\} \cup \{c_\alpha \leq x : \alpha < \lambda\})$$

satisfies the λ – OTC.

Therefore Φ^* has a model M such that

$$M \models \forall x(P(x) \rightarrow \bigvee_{\alpha < \lambda} x < c_\alpha).$$

It is obvious that $(P, <)^M$ has cofinality λ . QED

Lemma

Suppose λ is regular, L has the λ -OTP, $|\Phi| \leq \lambda$ and Φ has an infinite model. Then Φ has a model of cardinality λ .

Proof.

Suppose M is an infinite model of Φ . Expand M to be a linear order $(P, <)$ with last element. By the previous lemma Φ has a model in which $(P, <)$ has cofinality λ . So Φ has a model of cardinality $\geq \lambda$. By another previous lemma Φ has a model of cardinality exactly λ . \square

Definition

A logic is λ -**compact** if every finitely consistent set T of sentences such that $|T| \leq \lambda$ has a model. We say “in infinite models” if only infinite models are considered.

Lemma

If λ is regular and L has the λ -OTP, then L is μ -compact for all $\mu < \lambda$.

We use below the following construction: Given τ , let τ^+ be obtained from τ by replacing each $R(x_1, \dots, x_n)$ by $R^+(x, x_1, \dots, x_n)$. Similarly ϕ^+ and M^+ .

Proof.

We use induction on ν . Let us assume L is μ -compact for all $\mu < \nu$ (holds trivially if $\nu = \omega$). Suppose $T = \{\phi_\xi : \xi < \nu\}$ is finitely consistent. By Induction Hypothesis, $\{\phi_\xi : \xi \leq \eta\}$ has an infinite model for each $\eta < \nu$. By the above lemma we may assume these models are of cardinality λ . We take some new symbols and let

$$\Psi = \{P(c_\xi) \wedge \forall x(c_\xi \leq x \rightarrow \phi_\xi^+(x)) : \xi < \nu\}.$$

By the above lemma Ψ has a model M so that $(P, <)$ has cofinality λ . Since $\nu < \lambda$, there is $a \in P^M$ so that all the constants c_ξ , $\xi < \nu$, are below a . From a we can get a model where each ϕ_ξ , $\xi < \nu$, i.e. T itself, is true. □

Theorem (Lindström)

If L satisfies λ -OTP for some $\lambda > \omega$, then $L \equiv FO$, i.e. every sentence of L is equivalent in infinite models to a first order sentence.

Proof:

Let us assume we have $M \equiv N$ such that $M \models \phi$ and $N \models \neg\phi$ for some ϕ in L . W.l.o.g. M and N have a finite vocabulary τ . Let τ' be a disjoint copy of τ . Note that M and N are infinite. By the above lemma, w.l.o.g. M and N have cardinality λ . We take new constants c_ξ, c'_{ξ_i} , $\xi < \lambda$. Let

$$\Psi = \{\phi, \neg\phi'\} \cup \{\psi(c_{\xi_1}, \dots, c_{\xi_n}) \leftrightarrow \psi'(c'_{\xi_1}, \dots, c'_{\xi_n}) : \psi \text{ atomic}, \xi_1, \dots, \xi_n < \lambda\}.$$

We show that

$$(\Psi, \{x \neq c_\xi \vee x \neq c'_\eta : \xi, \eta < \lambda\})$$

has the λ -OTC.

- Clearly, Ψ has a model.
- Now let $|\Theta(x)| < \lambda$ such that $\Psi \cup \{\exists x \wedge \Theta(x)\}$ has a model K . Suppose c is an element of K that satisfies $\Theta(x)$.
- Let $X = \{\xi : c_\xi \text{ or } c'_\xi \text{ occurs in } \Theta(x)\}$. Note that $|X| < \lambda$. Pick $\eta < \eta'$ from $\lambda \setminus X$.
- By using $M \equiv N$ and the fact that L is $|X|$ -compact, one can construct a model for $\Psi \cup \Theta(c_\eta^0) \cup \{c_\eta = c'_{\eta'}\}$.

- By the λ -OTP we have a model K of Ψ such that $K \models \forall x \bigvee_{\xi, \xi' < \lambda} (x = c_\xi = c'_{\xi'})$.

- Thus

$$c_\xi^K \mapsto c'_{\xi^K}$$

is an isomorphism between the τ -part and the τ' -part of K .

- This contradicts the fact that the τ -part satisfies ϕ and the τ' -part satisfies $\neg\phi$. QED

Definition (Barwise [2])

Let T be a true set theory extending the Kripke-Platek axioms KP . An abstract logic $L = (F, \models)$ is **absolute relative to T** if there are Σ_1 -predicates $R(x, y)$ and $S(x, y, z)$ and a Π_1 -predicate $P(x, y, z)$ such that

- 1 For all ϕ : $\phi \in F$ if and only if $R(\phi, L)$.
- 2 For all $\phi \in F$ and all L -structures M , $M \models \phi$ if and only if $S(M, \phi, L)$.
- 3 The following is a theorem of T : For all languages z , all z -structures M and all ϕ such that $R(\phi, z)$, $S(M, \phi, z)$ if and only if $P(M, \phi, z)$.

An abstract logic is **strictly absolute** if it is absolute relative to KP (or $KP + \text{Infinity}$).

Conjunction and other logical operations are assumed to be absolute, too. If L is an abstract logic and A is an admissible set, we use L_A to denote the sub-logic of L consisting of sentences which are elements of A . $L_{H(\kappa)}$ is denoted L_κ .

Example

- The logics FO , $FO(Q_0)$, $L_{\omega_1\omega}$ and $L_{\infty\omega}$ are absolute relative to KP .
- The weak second order logic is strictly absolute.
- The unbounded logic $L_{\infty\omega}(WO)$ is absolute relative to $KP + \Sigma_1$ -separation.

Example

- If we add the game quantifier

$$\forall x_1 \exists x_2 \forall x_3 \dots \bigvee_{n < \omega} \phi_n(x_1, \dots, x_{p_n}, \vec{y})$$

and its dual

$$\exists x_1 \forall x_2 \exists x_3 \dots \bigwedge_{n < \omega} \phi_n(x_1, \dots, x_{p_n}, \vec{y})$$

to $L_{\infty\omega}$ an interesting (also unbounded) logic, denoted by $L_{\infty G}$ emerges. This logic is absolute relative to $KP + \Sigma_1$ -separation.

- The smallest admissible fragment $(L_{\omega_1\omega})_{HYP}$ of $L_{\omega_1\omega}$ is an interesting absolute logic (usually denoted L_{HYP}).
- The infinite quantifier logic $L_{\omega_1\omega_1}$ and second order logic L^2 are not absolute relative to any true first order set theory T .

Theorem (Levy Reflection Principle)

If $\phi(\vec{x})$ is a Σ_1 -formula and $\kappa > \omega$ then $\forall \vec{x} \in H(\kappa)(\phi(\vec{x}) \rightarrow H(\kappa) \models \phi(\vec{x}))$.

We can make some immediate observations about absolute logic L by means of Theorem 12:

- If $\phi \in L_{\kappa^+}$ has a model, then it has a model in $H(\kappa^+)$ and therefore a model of cardinality $\leq \kappa$. Thus L_{κ^+} satisfies the Löwenheim-Skolem Theorem down to κ .
- We can prove $L \leq_w L_{\infty\omega}$ almost as quickly: Suppose there is a back-and-forth set E for M and N , but for some $\phi \in L$ we have $M \models \phi$ and $N \not\models \phi$. All this is Σ_1 , so by Theorem 12 such objects L, ϕ, M, N, E must exist already in HC . But then M and N are countable, hence isomorphic, a contradiction.

Theorem (Barwise)

If L is a strictly absolute logic and A is an admissible set, then L_A is contained in $(L_{\infty\omega})_A$.

Proof: Suffices to prove this for countable admissible sets: Suppose the claim fails. Thus there is an admissible set A and a sentence $\phi \in L_A$ such that for all $\psi \in (L_{\infty\omega})_A$ there is a model M of $\neg(\phi \leftrightarrow \psi)$. This can be written as a Σ_1 -property of A . If an A with this property exists at all, one such exists in HC by Levy Reflection.

Note: If A is a countable admissible set, then L_A satisfies the Craig Interpolation Property.

End of the proof

Suppose L is a strictly absolute logic and $\phi \in A$ is an L -sentence with vocabulary τ (finite). We take a new vocabulary τ' containing τ and the new symbols $\varepsilon, \bar{M}, \bar{\phi}, \bar{L}$. It is possible to write down a sentence Φ of $(L_{\infty\omega})_A$ such that the following conditions are equivalent for any infinite L -structure M :

- 1 $M \models \phi$
- 2 There is an expansion M' of M which is a model of Φ and which satisfies $S(\bar{M}, \bar{\phi}, \bar{L})$
- 3 Every expansion M' of M to a model of Φ satisfies $S(\bar{M}, \bar{\phi}, \bar{L})$

We use the fact that the standard part of a model of KP is again a model of KP. Now the claim follows from the Interpolation Theorem.

We get as a special case the promised characterization of infinitary languages $L_{\kappa\omega}$ for any κ :

Corollary (Barwise [2])

If L is a strictly absolute logic and $\kappa > \omega$, then L_κ is contained in $L_{\kappa\omega}$.

Corollary

$L_{\infty\omega}$ is the largest strictly absolute logic.

Theorem (Akkanen, V. [1])

FO is maximal logic that is absolute with respect to KPU^- .



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