Lindström's Theorem — Problems

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Problems

(1) Show that $L(Q_1)$ satisfies the following axioms:

1. (Monotonicity)
   $\forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (Q_1 x\phi(x) \rightarrow Q_1 x\psi(x))$.
2. (Non-triviality) $\neg Q_1 x(x = y \vee x = z)$
3. (Keisler's Axiom) $Q_1 y\exists x\phi \rightarrow (\exists x Q_1 y\phi \vee Q_1 x\exists y\phi)$

(Actually, these axioms are complete, i.e. there is a Completeness Theorem for $L(Q_1)$ based on these axioms.)

Problems

(2) Show that an abstract logic L is compact if and only if the following condition holds: If $T \subseteq F$ is such that every model satisfies some $\phi \in T$, then there is a finite $T_0 \subseteq T$ such that every model satisfies some $\phi \in T_0$.

(3) Suppose $L_{\omega\omega} \leq L$. Show that L has the Löwenheim-Skolem Property if and only if the following condition holds: If $T \subseteq F$ and $\phi \in F$ are such that every countable model of T is a model of ϕ, then every model of T is a model of ϕ.

(4) Assume as given that $L(Q_1)$ is compact and $L_{\omega_1\omega}$ has the Löwenheim-Skolem Property. Show that $L_{\omega_1\omega} \not\leq L(Q_1)$ and $L(Q_1) \not\leq L_{\omega_1\omega}$.

(5) An abstract logic L has the **Finite Occurrence Property** if for every $\phi \in F$ there is a finite τ such that for all $M \in S$: $M \models \phi \iff M \restriction \tau \models \phi$. Show that a compact logic always satisfies the Finite Occurrence Property. (Hint: Suppose $\phi \in F$. Write ϕ in two different but similar vocabularies using a name changer. Write down axioms saying that the vocabularies are interpreted identically. Add ϕ in one vocabulary and $\neg\phi$ in the other. You get a theory which has no models. Now use compactness.)

Additional problems

(6) An abstract logic has the **Łoś property** if for every set $\{M_i : i \in I\}$ of structures of the same vocabulary and every ultrafilter D on I there is a structure M such that for all $\phi \in F$:

$$M \models \phi \iff \{i \in I : M_i \models \phi\} \in D.$$

Show that an abstract logic is compact if and only if it has the Łoś property. (Hint: Given $T \subseteq F$, let I be the set of all finite subsets of T.)
Note that you do not need so called ultraproducts in this exercise.

Additional problems

(7) Suppose $L_{\omega\omega} < L$ and L is compact. Show that for some $\phi \in F$ there are $M \preceq N$ such that $M \models \phi$ but $N \models \neg\phi$. (Hint: In the lecture notes we got such M and N that $M \equiv N$, $M \models \phi$ and $N \models \neg\phi$. Let M^* be the expansion of M by giving a name for each element of M. Let $T = Th(M^*) \cup \{\neg\phi\}$. Now T has a model by compactness, and we are done.)

(8) Suppose L is compact, $L_{\omega\omega} \leq L$, and $M \preceq N$. Show that there is M' such that $M \preceq_L M'$ and $N \preceq M'$. Here $M \preceq_L M'$ means that $M \subseteq N$ and for all $a_1, ..., a_n \in M$ the models $(M, a_1, ..., a_n)$ and $(M', a_1, ..., a_n)$ satisfy exactly the same L-sentences. (Hint: Show that $T = Th_L(M^*) \cup Th(N^*)$ has a model.)

Additional problems

(9) A logic $L = (F, \models)$ has the **Tarski Union Property** if for all $M_0 \preceq_L M_1 \preceq_L \ldots$ we have $M_m \preceq_L \bigcup_n M_n$ for all $m \in \mathbb{N}$.
Show that if $L_{\omega\omega} \leq L$ and L satisfies the Compactness Theorem and the Tarski Union Property, then $L_{\omega\omega} \equiv L$.
(Hint: Suppose $L_{\omega\omega} < L$. Use previous exercises to find M and N such that $M \preceq N$, $M \models \phi$ and $N \models \neg\phi$. Then find M_0 such that $M \preceq_L M_0$ and $N \preceq M_0$. Then find N_0 such that $N \preceq_L N_0$ and $M_0 \preceq N_0$. Then find M_1 such that $M_0 \preceq_L M_1$ and $N_0 \preceq M_1$. Etc. Then finish the proof by using the Tarski Union Property.)