Arch. math. Logik 22 (1982), 141–158

ON ORDERINGS OF THE FAMILY OF ALL LOGICS*,[1]

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The purpose of this paper is to examine the structural complexity of the sublogic relation between abstract logics.
Let $\mathscr{L}^{(n)}$ denote n^{th} order logic. Then $\mathscr{L}^{(n)}$ is a proper sublogic of $\mathscr{L}^{(n+1)}$ for each $n<\omega$, and we have the chain

$$\mathscr{L}^{(1)}<\mathscr{L}^{(2)}<\ldots<\mathscr{L}^{(n)}<\ldots.$$

The question naturally arises, what other kinds of chains or partial orderings we can have among sufficiently regular abstract logics. Can one have a family of logics ordered in the order type of the rationals (or perhaps the reals)?
In Chapter 2 we prove that any distributive lattice, and therefore any partial ordering, can be embedded into the sublogic relation among what we call normal logics. Chapter 3 is devoted to a sublogic relation defined using PC-classes rather than elementary classes. In the last chapter we restrict ourselves to the very special logics of the form $\mathscr{L}(Q_1,\ldots,Q_n)$. The situation becomes more problematic but we can still prove that any countable partial ordering is embeddable into the sublogic relation of these logics.
We confine ourselves to single-sorted structures, but many of the results are equally true of many-sorted structures.

1. Introduction

The basic idea of an abstract logic is very general and easily formulated: we have a class $\mathscr{A}$ of objects called alphabets and a mapping $\mathscr{L}$ which relates every element t of $\mathscr{A}$ with a triple

$$\mathscr{L}_t=\langle\mathscr{S}_t,\mathscr{F}_t,\mathscr{T}_t\rangle,\quad\text{where}\quad\mathscr{T}_t\subseteq\mathscr{S}_t\times\mathscr{F}_t.$$

Intuitively,

$\mathscr{S}_t$ is the class of $\mathscr{L}_t$-structures,

$\mathscr{F}_t$ is the class of $\mathscr{L}_t$-sentences,

$\mathscr{T}_t$ is the $\mathscr{L}_t$-satisfaction relation.

* Eingegangen am 11.2.1980.
[1] Some of the results of this paper were contained in the Ph.D thesis of the first author. He wishes to thank his supervisor Professor A. Lachlan for his help and encouragement.

This definition is all one needs to formulate such fundamental properties of abstract logics as compactness and its generalizations (see [7]). But already the formulation of the various Löwenheim-Skolem properties is unclear due to the lack of any notion of cardinality of a model.

It is usually assumed that the alphabets are sets of predicate-, function-, and constant-symbols, and that $\mathscr{S}_t$ is the class of all first order structures over the alphabet t. This simplifies matters, but rules out all non-classical logics. For a treatment covering non-classical logics as well, see [5]. However, once this step of considering usual first order structures is taken, a variety of model theoretic notions become available. What is more important, the general idea of an abstract logic, sketched above, can be turned into a rigorous definition, which enables us to carry out non-trivial and interesting model theoretic constructions on abstract logics. Standard references for such definitions are [3] and [1]. A discussion on the mutual relationships between these definitions in contained in [8].

In a context where the alphabets and structures are fixed, the following natural sublogic-relation suggests itself:

$$\mathscr{L} \leqq \mathscr{L}' \quad \text{if and only if} \quad \forall t \in \mathscr{A}(\mathscr{F}_t \subseteqq \mathscr{F}_t' \text{ and } \mathscr{T}_t = \mathscr{T}_t' \cap (\mathscr{S}_t \times \mathscr{F}_t)).$$

The study of this relation is the main purpose of this paper.

The properties of the relation $\leqq$ certainly depend heavily on the particular definition of an abstract logic one uses. A too general definition would allow non-standard completely uninteresting logics to mess up the ordering, whereas a too narrow definition might exclude logics which are relevant for the ordering. We shall start with a fairly general class of logics we call *normal*. Normality is all one uses in such standard results of abstract model theory as Lindström's theorem. As an example of a more restricted class of logics, we consider the class of logics generated by a finite number of generalized (first-order) quantifiers, the so called *finitely generated* logics. In sharp contrast to normal logics, these logics have a simple and well-defined syntax. Their ordering, as well, turns out to be very different from that of normal logics.

There are obvious set-theoretical difficulties in any attempt to give a rigorous definition of abstract logics and their ordering. It is not only that an abstract logic itself is a map relating sets with classes, but the ordering $\leqq$ is defined between these maps. The reason why difficulties hardly ever arise in practice is that the embeddings of one logic into another have a nice uniform nature.

In the present context we overcome the difficulty by simply switching to class theory, more exactly to the Mostowski-Kelley-Morse theory of classes.

In class theory families of classes can be coded into classes in the following well-known way: If $\mathscr{K}$ is a class, the *domain* of $\mathscr{K}$, $\mathrm{dom}(\mathscr{K})$, is the class $\{x|\exists y(\langle x,y\rangle \in \mathscr{K})\}$. If x is any set, $\mathscr{K}_x$ is used to denote the class $\{y|\langle x,y\rangle \in \mathscr{K}\}$. Now we can think of $\mathscr{K}$ as a code of the family $\{\mathscr{K}_x | x \in \mathrm{dom}(\mathscr{K})\}$ of classes. Accordingly we use the notation

$$\begin{array}{ll} \mathscr{M} \in \mathscr{K} & \text{for} \quad \exists x(\mathscr{M} = \mathscr{K}_x) \\ \mathscr{K} \subseteqq_{\mathrm{ext}} \mathscr{K}' & \text{for} \quad \forall \mathscr{X}(\mathscr{X} \in \mathscr{K} \rightarrow \mathscr{X} \in \mathscr{K}'). \end{array}$$

We shall now proceed to the definition of an abstract logic. Our presentation closely follows [3].

An element of the set $T=\bigcup_{n<\omega}\omega^n$ is called a *type*. A structure of type t, $\mathfrak{A}$, consists of a domain together with a $t(i)$-ary relation for each $i\in\mathrm{dom}(t)$. The class of all structures of type t is denoted by Mod_t. Type t' is an *extension* of type t, if $\mathrm{dom}(t)\subset\mathrm{dom}(t')$ and $t'|_{\mathrm{dom}(t)}=t$. If $\mathscr{K}$ is a class of structures of type t and t' is an extension of t, then the *free expansion* of $\mathscr{K}$ to type t' is the class $\mathscr{K}^{t'}$ of $\mathfrak{A}\in\mathrm{Mod}_{t'}$ such that the reduct $\mathfrak{A}|_t$ of $\mathfrak{A}$ to t is in $\mathscr{K}$. Two types t and t' are *equivalent*, $t\simeq t'$, if $\mathrm{dom}(t)=\mathrm{dom}(t')=k$ and $t(i)=t'(j_i)$ for some permutation $(j_0,\dots,j_{k-1})$ of k. If $\mathscr{K}$ is a class of structures of type t and $t\simeq t'$, with $t(i)=t'(j_i)$ then $\mathscr{K}^{t,t'}$ is the class

$$\mathscr{K}^{t,t'}=\{\mathfrak{A}\mid\mathfrak{A}=\langle A,R_{j_0},\dots,R_{j_{k-1}}\rangle,\ \langle A,R_0,\dots,R_{k-1}\rangle\in\mathscr{K}\}.$$

A subclass of Mod_t, which is closed under isomorphisms, is called a *model class of type t*. A *family of model classes* is a class $\mathscr{K}$ such that $\mathrm{dom}(\mathscr{K})$ is a set and for each $i\in\mathrm{dom}(\mathscr{K})$, $\mathscr{K}_i$ is a model class.

Definition 1. A *quasilogic* is a class $\mathscr{L}$ with domain T such that L_t is a family of model classes of type t, for each $t\in T$.

If $\mathscr{L}$ is a quasilogic and $t\in T$, we can also think of $\mathscr{L}_t$ as a triple $\langle\mathscr{S}_t,\mathscr{F}_t,\mathscr{T}_t\rangle$ where $\mathscr{S}_t=\mathrm{Mod}_t$, $\mathscr{F}_t=\{\mathscr{K}\mid\mathscr{K}\in\mathscr{L}_t\}$ and $T_t(\mathfrak{A},K)\leftrightarrow\mathfrak{A}\in K$, for $\mathfrak{A}\in\mathscr{S}_t$ and $\mathscr{K}\in\mathscr{F}_t$. Thus our definition is consistent with the general idea presented in the beginning. Note that we do not allow a quasilogic to have a proper class of sentences (as e.g. $\mathscr{L}_{\infty\omega}$). This restriction only plays the role of simplifying the formulation of certain results. A class $\mathscr{L}$ which meets the other parts of Definition 1, except that $\mathrm{dom}(\mathscr{L}_t)$ may be a proper class for some $t\in T$, is called a *class-like quasilogic*.

Abstract logics will be defined as quasilogics satisfying certain simple axioms. Note that any (it seems) property of an abstract logic can already be formulated for quasilogics.

Definition 2. An abstract logic is a quasilogic satisfying the following three conditions:

(1) For all $t\in T$, if $\mathscr{K},\mathscr{M}\in\mathscr{L}_t$, then $\mathscr{K}\cap\mathscr{M}\in\mathscr{L}_t$ and $\mathrm{Mod}_t-\mathscr{K}\in\mathscr{L}_t$.

(2) For every $t\in T$ and for every extension t' of t, $\mathscr{K}\in\mathscr{L}_t$ implies $\mathscr{K}^{t'}\in\mathscr{L}_{t'}$.

(3) For all $t,t'\in T$ such that $t\simeq t'$, $\mathscr{K}\in\mathscr{L}_t$ implies $\mathscr{K}^{t,t'}\in\mathscr{L}_{t'}$.

This definition is equivalent to the definition of a "generalized first order logic" in [3].

The effect of axiom (1) is simply, that $\mathscr{L}$ is closed under the usual finitary propositional operations. Axioms (2) and (3) are the crucial structural axioms that really distinguish an abstract logic from a mere family of families of model classes. Axiom (2) says that the truth of a sentence only depends on the symbols "occurring" in it (if $\mathscr{K}\in\mathscr{L}_t$ we tend to think that only the non-logical symbols determined by t "occur" in $\mathscr{K}$). Axiom (3) says that we can execute the simplest possible substitution operations: replacing of a predicate-symbol by another.

Following common usage, we call a class $\mathscr{K}$ of models an *elementary class of* $\mathscr{L}$, in symbols $\mathscr{K}\in \mathrm{EC}_{\mathscr{L}}$ or $\mathscr{K}\in\mathscr{L}$, if there is a type $t(\mathscr{K})$ such that $\mathscr{K}\in\mathscr{L}_{t(\mathscr{K})}$. We use $\bar{\mathscr{K}}$ to denote $\mathrm{Mod}_{t(\mathscr{K})}-\mathscr{K}$.
Observe that an abstract logic need not contain any nonvoid elementary classes at all: the abstract logic $\mathscr{L}$ such that $\mathscr{L}_t=\emptyset$ for all $t\in T$ is only able to characterize the empty model class. Some new axioms are called for to guarantee non-trivial expressive power.

Definition 3. The logic $\mathscr{L}$ is *non-trivial* if for each $n\in\omega$ the class

$$\{\langle A,R,a_1,\dots,a_n\rangle | R\subset A^n,\ \langle a_1,\dots,a_n\rangle\in R\}$$

and the class

$$\{\langle A,a,b\rangle | a=b\}$$

are elementary classes of $\mathscr{L}$.
Non-triviality is not yet enough to give all the elementary classes of $\mathscr{L}_{\omega\omega}$; for that we would need quantifiers. It proves useful to introduce the following general framework: Recall that a model class of type t is called a *generalized quantifier of type t*, or just a *quantifier of type t* [2].

Definition 4. Suppose Q is a quantifier of type $t\in\omega^n$. Let $k=\max\{t(i)|i<n\}$, let t' be t concatenated with k zeros, and let t^i be t concatenated with $t(i)$ zeros. A logic $\mathscr{L}$ is *closed under the quantifier Q* if for all classes $\mathscr{K}_0,\dots,\mathscr{K}_{n-1}\in\mathscr{L}_{t'}$ such that $\mathscr{K}_i=\mathscr{M}_i^{t'}$ for some $\mathscr{M}_i\in\mathscr{L}_{t^i}$, the class

$$\{\mathfrak{A}\in\mathrm{Mod}_{t'}|\langle|\mathfrak{A}|,R_0^{\mathscr{K}_0},\dots,R_{n-1}^{\mathscr{K}_{n-1}}\rangle\in Q\},$$

where

$$R_i^{\mathscr{K}_i}=\{\langle a_0,\dots,a_{t(i)-1}\rangle|\langle\mathfrak{A},a_0,\dots,a_{t(i)-1}\rangle\in\mathscr{M}_i\},$$

is in $\mathscr{L}_{t'}$.
The most relevant quantifier is, of course, the *existential quantifier* $\{\langle A,B\rangle|B\subseteq A,\ B\neq\emptyset\}$. It is easily seen that there is a smallest non-trivial abstract logic closed under the existential quantifier, and the elementary classes of this logic are exactly the elementary classes of $L_{\omega\omega}$.
It would seem like a very desirable property of any abstract logic that it be closed under every quantifier which is its own elementary class. This is a strong form of a *substitution axiom.* Our axiom (3) legitimates only substitution of predicate symbols to predict symbols, but this stronger axiom would legitimate substitution of any formulae (under certain coherency conditions) to predicate symbols. There is a wealth of different substitution axioms of varying strength, and the one we are now going to define is among the weakest. This version is related to coding of predicates into one, something one does in the well-known quantifier manipulation

$$\forall x\,\exists R\phi(R(\cdot),x)\ \leftrightarrow\ \exists R'\,\forall x\phi(R'(x,\cdot),x).$$

Suppose $\mathscr{K}$ is a model class of type $t \in \omega^n$. Suppose $i_0, \ldots, i_k \in n$ such that $t(i_j) \neq 0$ for $j = 0, \ldots, k$. We define t' as follows:

$$t'(i) = \begin{cases} t(i) & \text{if } \; i \in n - \{i_0, \ldots, i_k\}, \\ t(i)+1 & \text{if } \; i \in \{i_0, \ldots, i_k\}, \\ 0 & \text{if } \; i = n. \end{cases}$$

A model class $\mathscr{K}_{i_0 \ldots i_k}$ of type t' is defined by

$$\mathscr{K}_{i_0 \ldots i_k} = \{\langle A, R_0, \ldots, R_{n-1}, a\rangle \mid \langle A, R_0^{(a)}, \ldots, R_{n-1}^{(a)}\rangle \in \mathscr{K}\},$$

where

$$R_i^{(a)} = \begin{cases} \{\langle a_0, \ldots, a_{t(i)}\rangle \mid \langle a_0, \ldots, a_{t(i)}, a\rangle \in R_i\} & \text{if } \; i \in \{i_0, \ldots, i_k\}, \\ R_i, & \text{otherwise}. \end{cases}$$

Definition 5. A logic $\mathscr{L}$ is AIR (admits indization of relations) if $\mathscr{K}_{i_0 \ldots i_k} \in \mathscr{L}$ whenever $\mathscr{K} \in \mathscr{L}$ and $i_0 \ldots i_k \in t(\mathscr{K})$ such that $t(\mathscr{K}) \neq 0$ for $j = 0, \ldots, k$.
For logics closed under the existential quantifier, AIR implies the property of being "strong" in the sense of [3].

Definition 6. An abstract logic is *normal* if it is non-trivial, AIR and closed under the existential quantifier. $\boldsymbol{NL}$ denotes the collection of all normal logics. The collection of all normal class-like logics is denoted by $\boldsymbol{NL}_c$.
We feel that normality is a stable and interesting property of an abstract logic, and cannot be weakened (apart from omitting closure under negation) without an essential decline in the ability to carry out model theoretic constructions. The definition of $\boldsymbol{NL}$ does not take place in class theory itself, but on a metalevel: normality is a definable property of classes.

2. The Ordering of Normal Logics

In this chapter we shall discuss the usual sublogic relation between normal logics. This relation turns out to provide $\boldsymbol{NL}$ with a non-modular lattice-ordering into which every distributive lattice can be embedded.

Definition 7. Let $\mathscr{L}$ and $\mathscr{L}'$ be normal logics. We say that $\mathscr{L}$ is *weaker than* $\mathscr{L}'$, in symbols $\mathscr{L} \leqq \mathscr{L}'$, if for all $t \in T$, $\mathscr{L}_t \subseteqq_{\text{ext}} \mathscr{L}'_t$. The logics $\mathscr{L}$ and $\mathscr{L}'$ are *equivalent*, $\mathscr{L} \equiv \mathscr{L}'$, if $\mathscr{L} \leqq \mathscr{L}'$ and $\mathscr{L}' \leqq \mathscr{L}$.
Note that $\leqq$ is a definable property of classes, thus defined on the metalevel only. The meaning of the definition

$$\mathscr{N}\mathscr{L} = \langle \boldsymbol{NL}, \leqq \rangle$$

is to be understood in the same way. A word about identity is here in order. Two classes are identical if they have the same sets as elements. For classes which are quasilogics we have the weaker notion of equivalence, defined above. It does not seem to be plausible to distinguish two logics with the same model classes, just because they happen to have a different definition. This is in accordance with our general suppression of syntax. With this in mind we adopt the notion of equivalence as the notion of identity between logics, and consider $\mathcal{NL}$ as a structure with this identity-relation and the indicated binary relation $\leqq$. Having made this convention, it is eminently obvious that $\mathcal{NL}$ is a partially ordered structure with a least element. We shall proceed to proving the following stronger result:

Theorem 1. *$\mathcal{NL}$ is a complete lattice with a least element.*

Proof. Suppose $\{\mathcal{L}^i|i\in I\}$ is a family of logics and I is a set (such that $I\cap\omega=\emptyset$). We can define a new quasilogic $\mathcal{L}$ by letting for each $t\in T$:

$$\mathcal{L}_t=\bigcap_{i\in I}\mathcal{L}_t^i.$$

It is readily verified that $\mathcal{L}$ is the infimum of $\{\mathcal{L}^i|i\in I\}$ in $\mathcal{NL}$. Thus we can concentrate on the problem of finding the supremum of $\{\mathcal{L}^i|i\in I\}$. An obvious candidate is the quasilogic $\mathcal{L}^*$ defined for $t\in T$ by

$$\mathcal{L}_t^*=\bigcup_{i\in I}\mathcal{L}_t^i.$$

$\mathcal{L}^*$ not necessarily being a normal logic, we extend it successively to one. For this end, we shall consider four natural operations on quasilogics. In the following $\mathcal{L}$ is an arbitrary quasilogic.

Let Bool($\mathcal{L}$) be the quasilogic $\mathcal{L}'$ defined as follows:

$$\mathcal{L}_t'=\left\{\bigcup_{j=0}^{m-1}\bigcap_{i=0}^{n-1}\mathcal{K}_{ij}^{e(i,j)}|e\in 2^{nm},\ \mathcal{K}_{ij}\in\mathcal{L}_t,\ \mathcal{K}_{ij}^0=\mathcal{K}_{ij},\ \mathcal{K}_{ij}^1=\bar{\mathcal{K}}_{ij},\ n,m<\omega\right\}.$$

Fact 2. Bool($\mathcal{L}$) *is the least quasilogic containing $\mathcal{L}$ and closed under Boolean operations.*

Let Ext($\mathcal{L}$) be the quasilogic $\mathcal{L}'$ defined as follows:

$$\mathcal{L}_t'=\{\mathcal{K}|\text{ there is an extension } t_1 \text{ of } t \text{ such that } \mathcal{K}=\mathcal{M}^t \text{ for some } \mathcal{M}\in\mathcal{L}_{t_1}\}.$$

Fact 3. Ext($\mathcal{L}$) *is the least quasilogic containing $\mathcal{L}$ and closed under expansions.*

Suppose $\mathcal{K}\in\mathcal{L}_t$ and for some $i_0<k=\mathrm{dom}(T)$, $t(i_0)=0$. Let $t'(j)=t(j)$ if $j<i_0$ and $t'(j)=t(j+1)$ if $i_0<j<k-1$. We let $\exists x\mathcal{K}$ be the class

$$\{\langle A,R_0,\dots,R_{i_0-1},R_{i_0+1},\dots,R_k\rangle|\text{ for some } a\in A\ \langle A,R_0,\dots,R_{i_0-1},a,\dots,R_k\rangle\in\mathcal{K}\}.$$

Let $\mathrm{Eq}(\mathscr{L})$ be the quasilogic defined by

$$\mathscr{L}'_t=\{\exists x_1 \ldots \exists x_n \mathscr{K} \mid \mathscr{K}\in\mathscr{L}_{t'} \text{ and } t(\exists x_1 \ldots \exists x_n \mathscr{K})=t\}.$$

Fact 4. Eq ($\mathscr{L}$) *is the least quasilogic containing* $\mathscr{L}$ *and closed under the existential quantifier.*

The quasilogic $\mathrm{Ind}(\mathscr{L})$ is defined as the $\mathscr{L}'$ such that

$$\begin{aligned}\mathscr{L}'_t=\{\mathscr{K}\mid t(\mathscr{K})=t &\text{ and for some } \mathscr{K}_1\in\mathscr{L} \text{ and } i_0 \ldots i_k\in\mathrm{dom}(t(\mathscr{K}^1))\\ &\text{such that } t(\mathscr{K}_1)(i_j)\neq 0 \text{ for } j\leqq k,\\ &\mathscr{K}=\mathscr{K}_{i_0\ldots i_k}\}\cup\mathscr{L}_t.\end{aligned}$$

Fact 5. $\mathrm{Ind}(\mathscr{L})$ *is the smallest* AIR *quasilogic containing* $\mathscr{L}$.

Putting the above operations together we obtain a useful operation Nor as follows. Let $\{\mathscr{L}^n \mid n<\omega\}$ be a family defined by

$$\mathscr{L}^0=\mathscr{L}$$
$$\mathscr{L}^{n+1}=\mathrm{Ind}(\mathrm{Eq}(\mathrm{Ext}(\mathrm{Bool}(\mathscr{L}^n)))).$$

Let

$$\mathrm{Nor}(\mathscr{L})$$

be the quasilogic $\mathscr{L}'$ defined by

$$\mathscr{L}'_t=\bigcup_{n<\omega}\mathscr{L}^n_t.$$

Fact 6. *If* $\mathscr{L}$ *is a non-trivial quasilogic, then* $\mathrm{Nor}(\mathscr{L})$ *is the smallest normal logic stronger than* $\mathscr{L}$.

Let us now return to the proof of Theorem 1. Let $\mathscr{L}=\mathrm{Nor}(\mathscr{L}^*)$. The abstract logic $\mathscr{L}$ is the desired normal supremum of $\{\mathscr{L}^i \mid i\in I\}$.
The above proof also shows that $\mathscr{NL}$ is a complete sublattice of $\mathscr{NL}_c$ (defined as $\langle NL_c, \leqq\rangle$) and that $\mathscr{NL}_c$ has the following stronger completeness property: every indexed family $\{\mathscr{L}^i \mid i\in\mathscr{I}\}$, $\mathscr{I}$ a class, has a supremum and an infimum in $\mathscr{NL}_c$.
In the sequel we use $\bigcup$ and $\bigcap$ to denote the lattice operations of $\mathscr{NL}$.
What kind of sublattices does $\mathscr{NL}$ have? We have the following general result concerning distributive lattices. Non-distributive lattices will be discussed later.

Theorem 7. *Every distributive lattice (a set) can be embedded as a sublattice into* $\mathscr{NL}$.

Proof. The proof, as many subsequent ones, depends on the following special family of normal logics: For any ordinal α let $\mathscr{K}^0_\alpha(\mathscr{K}^1_\alpha, \mathscr{K}^2_\alpha)$ denote the class of structures of the empty type and of cardinality ω_α (at most ω_α, at least ω_α, respectively). If $\boldsymbol{K}$ is any family of model classes, we let $\mathrm{Nor}(\boldsymbol{K})$ denote the logic $\mathrm{Nor}(\mathscr{L})$ where $\mathscr{L}$ is the smallest non-trivial quasilogic having every $\mathscr{K} \in \boldsymbol{K}$ as an elementary class. If $\boldsymbol{K} = \{\mathscr{K}_0, \ldots, \mathscr{K}_n\}$, we write $\mathrm{Nor}(\mathscr{K}_0, \ldots, \mathscr{K}_n)$ for $\mathrm{Nor}(\boldsymbol{K})$. For the rest of the proof, $\boldsymbol{K}$ denotes a subfamily of $\{\mathscr{K}_\alpha^d | d \in 3, \alpha \in \mathrm{On}\}$.

Fact 8. *Every elementary class of* $\mathrm{Nor}(\boldsymbol{K})$ *is of the form*

$$\bigcup_{\alpha \in I} (\mathrm{Mod}_t(\phi_\alpha) \cap \alpha(0) \cdot \mathscr{K}_{\xi_0}^{i_0, t} \cap \ldots \cap \alpha(n-1) \cdot \mathscr{K}_{\xi_{n-1}}^{i_{n-1}, t}),$$

where $I \subseteq 2^n$, $\mathscr{K}_{\xi_j}^{i_j} \in \boldsymbol{K}$, ϕ_α *is a sentence of* $\mathscr{L}_{\omega\omega}$ *and*

$$e \cdot \mathscr{K}_\xi^{i,t} = \begin{cases} \mathscr{K}_\xi^{i,t} & \text{if } e = 0, \\ \overline{\mathscr{K}_\xi^{i,t}} & \text{if } e = 1. \end{cases}$$

The fact is proved by observing the inductive definition of $\mathrm{Nor}(\boldsymbol{K})$ and by verifying that the quasilogic of model classes of the form displayed is closed under the constituents Bool, Ext, Exp, Ind of Nor.

An immediate consequence of the above fact is:

Fact 9. *If the family* $\boldsymbol{K}$ *contains model classes of the form* $\mathscr{K}_\alpha^0$ *only, then for any* ξ *we have*

$$\mathscr{K}_\xi^0 \in \mathrm{Nor}(\boldsymbol{K}) \leftrightarrow \mathscr{K}_\xi^0 \in \boldsymbol{K}.$$

Let us now return to the proof of Theorem 7. Suppose $\boldsymbol{D} = \langle D, \leqq \rangle$ is a distributive lattice. We may assume that $\boldsymbol{D}$ is a lattice of subsets of On. For $d \in D$, let $\boldsymbol{K}_d = \{\mathscr{K}_\xi^0 | \xi \in d\}$. We define a mapping $F: D \to \boldsymbol{NL}$ by letting

$$F(d) = \mathrm{Nor}(\boldsymbol{K}_d).$$

If $a \leqq b$ in $\boldsymbol{D}$, then $\boldsymbol{K}_a \subseteq \boldsymbol{K}_b$, whence $F(a) \leqq F(b)$. On the other hand, if not $a \leqq b$, and $\alpha \in a - b$, then $\mathscr{K}_\alpha^0$ is an elementary class of $F(a)$, but, by Fact 9, not of $F(b)$. Thus $a \leqq b$ in $\boldsymbol{D}$ is equivalent to $F(a) \leqq F(b)$ in $\mathscr{NL}$. Moreover, by Fact 8, $F(a) \cap F(b) = \mathrm{Nor}(\boldsymbol{K}_a \cap \boldsymbol{K}_b) = \mathrm{Nor}(\boldsymbol{K}_{a \cap b}) = F(a \cap b)$, and similarly $F(a) \cup F(b) = F(a \cup b)$. □

The proof of Theorem 7 actually establishes the following stronger conclusion:

Theorem 10. *The complete lattice of all sets of ordinals can be completely embedded into* $\mathscr{NL}$.

Corollary 11. *Every partially ordered class every initial segment of which is a set, can be embedded into* $\mathscr{NL}$.

For the notion $\mathscr{NL}_c$ stronger embedding properties can be proved:

Theorem 12. *The complete (in the same sense as $\mathcal{NL}_c$) lattice of all classes of ordinals can be completely embedded into $\mathcal{NL}_c$. Every partially ordered class can be embedded into $\mathcal{NL}_c$.*

Another way of improving Theorem 7 is the following: Suppose $\mathcal{L}$ is a normal logic. Let ℓ be the Löwenheim number of $\mathcal{L}$, that is,

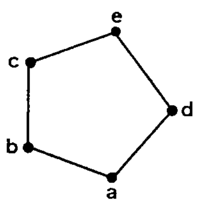
$$\ell = \sup\{\min\{|\mathfrak{A}|\,|\mathfrak{A}\in\mathcal{K}\}|\mathcal{K}\in\mathcal{L}\ \&\ \mathcal{K}\neq\emptyset\}.$$

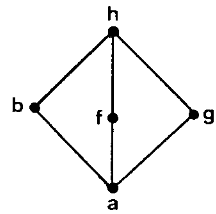
We can carry out the proof of Theorem 7 using classes $\mathcal{K}^0_\alpha$ for $\alpha > \ell$ only and adding them to $\mathcal{L}$ rather than to $\mathcal{L}_{\omega\omega}$. In this way we obtain the following result:

Theorem 13. *Let $\mathcal{L}$ be a normal logic. The complete lattice of all sets of ordinals can be completely embedded into $\mathcal{NL}$ above $\mathcal{L}$. Hence every distributive lattice and every partial ordering every initial segment of which is a set, can be embedded into $\mathcal{NL}$ above $\mathcal{L}$.*

Thus for example, every $\mathcal{L}\in\mathcal{NL}$ has a proper class of incomparable extensions and a proper linearly ordered class of extensions in $\mathcal{NL}$.
How about non-distributive lattices? Here the situation is very unclear. We do not even know if there is any lattice at all which cannot be embedded into $\mathcal{NL}$. However, the most common non-distributive lattices are embeddable as the following result indicates:

Proposition 14. *The lattices*





can be embedded as sublattices into $\mathcal{NL}$.

Proof. Let $a=\mathcal{L}_{\omega\omega}$, $b=\mathrm{Nor}(\mathcal{K}^1_1)$, $c=\mathrm{Nor}(\mathcal{K}^0_1, \mathcal{K}^1_1)$, $d=\mathrm{Nor}(\mathcal{K}^0_5, \mathcal{K}^0_1\cup\mathcal{K}^0_2)$, and $e=\mathrm{Nor}(\mathcal{K}^0_1, \mathcal{K}^1_1, \mathcal{K}^0_2, \mathcal{K}^0_5)$. By Fact 8 we may inter that $c\cap d=a$. It can be proved that $e=b\cup d$. This establishes the first claim. For the second lattice we take $f=\mathrm{Nor}(\mathcal{K}^0_1)$, $g=\mathrm{Nor}(\mathcal{K}^2_1)$, and $h=\mathrm{Nor}(\mathcal{K}^1_1, \mathcal{K}^2_1)$. The equations $b\leqq f\cup g$, $f\leqq b\cup g$, $g\leqq b\cup f$ follow easily. Trivially $b\cap f=f\cap g=g\cap b=a$. □

Corollary 15. *$\mathcal{NL}$ is not modular, and therefore not distributive.*

Thus the converse of Theorem 10 fails: $\mathcal{NL}$ cannot be embedded into the lattice of all sets of ordinals.

3. The Ordering Based on PC-Definability

In this chapter we shall consider a sublogic relation based on PC-classes. This ordering of NL is not as natural as $\leqq$, but it is in fact more frequent and more useful in practice. The main results concerning this new ordering follow the same pattern as the results of the previous chapter.

Let us start with some definitions. The reduct of a model $\mathfrak{A}$ to type t is denoted by $\mathfrak{A}|t$. The reduct of a model class $\mathscr{K}$ to type t is defined as $\{\mathfrak{A}|t \mid \mathfrak{A} \in \mathscr{K}\}$ and denoted by $\mathscr{K}|t$. Let $\mathscr{L}$ be a quasilogic. A PC-class of $\mathscr{L}$ is a model class which is the reduct of an elementary class of $\mathscr{L}$.

Definition 8. If $\mathscr{L}$ and $\mathscr{L}'$ are quasilogics, we say that $\mathscr{L}$ is PC-*weaker than* $\mathscr{L}'$, if every elementary class of $\mathscr{L}$ is a PC-class of $\mathscr{L}'$, in symbols: $\mathscr{L} \leqq_{PC} \mathscr{L}'$. We say that $\mathscr{L}$ and $\mathscr{L}'$ are PC-*equivalent*, if $\mathscr{L} \leqq_{PC} \mathscr{L}'$ and $\mathscr{L}' \leqq_{PC} \mathscr{L}$, in symbols: $\mathscr{L} =_{PC} \mathscr{L}'$.

We can now give NLa new structure by letting $=_{PC}$ be the identity and $\leqq_{PC}$ the ordering. Let us denote this new structure by $\mathscr{N}\mathscr{L}\mathscr{P}$. The structure $\mathscr{N}\mathscr{L}\mathscr{P}_r$ is defined similarly.

Theorem 16. *$\mathscr{N}\mathscr{L}\mathscr{P}$ is a complete lattice with a least element.*

Proof. Let I be a set $(I \cap \omega = \emptyset)$ and $\{\mathscr{L}^i | i \in I\}$ a family of normal logics. The straightforward intersection of the family does not necessarily yield the infimum, but let

$$\mathscr{L} = \mathrm{Nor}(\{\mathscr{K} | \mathscr{K} \text{ and } \bar{\mathscr{K}} \text{ are PC-classes of } \mathscr{L}^i \text{ for every } i \in I\}).$$

As it is clear that $\mathscr{L}' \leqq \mathscr{L}$ for any lower bound $\mathscr{L}'$ of $\{\mathscr{L}^i | i \in I\}$, we can concentrate on proving

$$\mathscr{L} \leqq_{PC} \mathscr{L}^i \quad \text{for all} \quad i \in I.$$

The definition of Nor gives a representation of $\mathscr{L}$ as a union $\bigcup_{n<\omega} \mathscr{L}^n$. Trivially $\mathscr{L}^0 \leqq_{PC} \mathscr{L}^i$ for all $i \in I$. The relevant induction step follows from the following eight equations:

Suppose $\mathscr{A}$ and $\mathscr{B}$ are model classes and $\mathscr{K} = \mathscr{A}|t$, $\mathscr{M} = \mathscr{B}|t$. Then:

(1) $\mathscr{K} \cup \mathscr{M} = (\mathscr{A}^{t'} \cup \mathscr{B}^{t'})|t$, where t' is some extension of $t(\mathscr{A})$ and $t(\mathscr{B})$,
(2) $\mathscr{K} \cap \mathscr{M} = (\mathscr{A}^{t'} \cap \mathscr{B}^{t'})|t$, where t' is as above,
(3) $\mathscr{K}^{t_1} = \mathscr{A}^{t,t'_1}|t_1$, where t' is some extension of t_1 and $t(\mathscr{A})$, and $t' \simeq t'_1$,
(4) $\overline{\mathscr{K}^{t_1}} = \overline{\mathscr{A}}^{t',t'_1}|t_1$, where t and t'_1 are as above,
(5) $\exists x \mathscr{K} = (\exists x A)|t'$, where $t'^\frown 0 = t$,
(6) $\forall x \mathscr{K} = \forall a \forall x (x \neq a \vee (\exists y_1 \ldots y_r \mathscr{A})_{n_1 \ldots n_1+p})|t'$, where $t'^\frown 0 = t$ and $n_1 = \mathrm{dom}(t)$,
(7) $\mathscr{K}_{i_0 \ldots i_k} = \mathscr{A}_{i_0 \ldots i_k}|t_1$, where $t_1 = t(\mathscr{K}_{i_0 \ldots i_k})$,
(8) $\overline{\mathscr{K}_{i_0 \ldots i_k}} = (\bar{\mathscr{K}})_{i_0 \ldots i_k}$.

All these equations are entirely trivial, with the exception of (6), which can be verified as follows:

$$\begin{aligned}\mathscr{A}\in\forall x\mathscr{K} &\leftrightarrow \forall a\langle\mathfrak{A},a\rangle\in\mathscr{K} \leftrightarrow \forall a\exists R\exists b\langle\mathfrak{A},a,R,b\rangle\in\mathscr{A} \leftrightarrow \forall a\exists R\langle\mathfrak{A},a,R\rangle\in\exists y\mathscr{A}\\ &\leftrightarrow \exists R\forall a\langle\mathfrak{A},R,a\rangle\in\forall x(x\neq a\vee(\exists y\mathscr{A})_{n_1\ldots n_1+p})\\ &\leftrightarrow \exists R\langle\mathfrak{A},R\rangle\in\forall a\forall x(x\neq a\vee(\exists y\mathscr{A})_{n_1\ldots n_1+p})\\ &\leftrightarrow \mathfrak{A}\in\forall a\forall x(x\neq a\vee(\exists y\mathscr{A})_{n_1\ldots n_1+p})|t'.\end{aligned}$$

The Equations (1)–(8) give immediately the induction step for the proof of $\mathscr{L}^n\leqq_{\mathrm{PC}}\mathscr{L}^i$.

This ends the proof that $\mathscr{L}$ is the infimum of $\{\mathscr{L}^i|i\in I\}$. The proof that $\{\mathscr{L}^i|i\in I\}$ has a supremum goes along the same lines: Let

$$\mathscr{L}=\mathrm{Nor}(\{\mathscr{L}^i|i\in I\}).$$

Using (1)–(8) one readily shows that $\mathscr{L}$ is the desired supremum.

Theorem 17. *Every distributive lattice (a set) can be embedded as a sublattice into $\mathscr{NLP}$. Every partially ordered class can be embedded into $\mathscr{NLP}$. Both embeddings can be done above any given $\mathscr{L}\in\mathscr{NLP}$.*

Proof. There are no essential differences to the proof of Theorem 7. Note that if $\boldsymbol{K}$ is as in Fact 9, then $\mathscr{K}_\alpha^0$ is a PC-class of Nor($\boldsymbol{K}$) iff $\mathscr{K}_\alpha^0\in\boldsymbol{K}$.

Also the proof of Proposition 14 carries over:

Proposition 18. *$\mathscr{NLP}$ and $\mathscr{NLP}_c$ are non-modular.*

Closely related to $\leqq_{\mathrm{PC}}$ is the so called Δ-operation. We say that an abstract logic $\mathscr{L}$ has the Souslin-Kleene-interpolation property if $\mathscr{K}\in\mathscr{L}$ whenever $\mathscr{K}$ and $\bar{\mathscr{K}}$ are PC-classes of $\mathscr{L}$. Let *NLI* denote the collection of all normal logics with the Souslin-Kleene-interpolation property.

As a property of an abstract logic, Souslin-Kleene interpolation is clearly desirable, but unfortunately fairly rare. The most famous examples of logics with this property are $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega_1\omega}$. However, its importance is largely due to the following notion. For any quasilogic $\mathscr{L}$ let

$$\Delta(\mathscr{L})=\mathrm{Nor}(\{\mathscr{K}|\mathscr{K}\text{ and }\bar{\mathscr{K}}\text{ are PC-classes of }\mathscr{L}\}).$$

Clearly, $\Delta(\mathscr{L})$ is the smallest extension of $\mathscr{L}$ to a normal logic with the Souslin-Kleene-interpolation property. For details concerning this operation the reader is referred to [4], where also a proof of the following lemma can be found:

Lemma 19.

(1) *If $\mathscr{K}\in\Delta(\mathscr{L})$, then $\mathscr{K}$ is a* PC-*class of $\mathscr{L}$,*

(2) *$\mathscr{L}$ has* SKI *if and only if $\mathscr{L}=\Delta(\mathscr{L})$,*

(3) $\mathscr{L}=_{\mathrm{PC}}\Delta(\mathscr{L})$,

(4) *$\mathscr{L}=_{\mathrm{PC}}\mathscr{L}'$ if and only if $\Delta(\mathscr{L})=\Delta(\mathscr{L}')$.*

Corollary 20. *The structures $\langle NLI, \leqq \rangle$ and $\langle NLI, \leqq_{PC} \rangle$ coincide and are isomorphic to $\langle NL, \leqq_{PC} \rangle$.*

It follows from this corollary that $\langle NLI, \leqq \rangle$ is a complete lattice. It is not a sublattice or $\langle NL, \leqq \rangle$, though, and we only have the result:

Corollary 21. *$\langle NL, \leqq_{PC} \rangle$ can be embedded as a subordering into $\langle NL, \leqq \rangle$.*

As a final observation on $\mathcal{NLP}$ we have the following application of Lindström's theorem [3]:

Proposition 22. *Let $\mathscr{L}(Q_0)$ be the logic with the quantifier "there exists infinitely many". The logic $\Delta(\mathscr{L}(Q_0))$ is an atom of $\mathcal{NLP}$.*

Proof. Suppose $\mathscr{L}_{\omega\omega} <_{PC} \mathscr{L} \leqq_{PC} \Delta(\mathscr{L}(Q_0))$. Then every non-empty $\phi \in \mathscr{L}$ contains a countable model. Thus by Theorem 3.1 of [3], the class of models $\{\langle A, < \rangle | \langle A, < \rangle \simeq \langle \omega, < \rangle\}$ is PC-definable in $\mathscr{L}$. Hence indeed $\mathscr{L} =_{PC} \mathscr{L}(Q_0)$. □

4. The Ordering of Finitely Generated Logics

In this chapter we shall consider logics which are generated by a finite number of generalized quantifiers. These logics constitute a rather special but highly interesting subfamily of NL. The main result is, that every countable partial ordering can be embedded into the sublogic relation of these logics.

Definition 9. Suppose Q is a quantifier of type $t \in \omega^n$. Let t' and t^i be as in Definition 3. We define a logic $\mathscr{L}(Q)$ as follows: $\mathscr{L}(Q) = \bigcup_{n \in \omega} \mathscr{L}_n$, where $\mathscr{L}_0 = \mathrm{Nor}(Q)$,

$$\mathscr{L}_{n+1} = \mathrm{Nor}(\{M(\mathscr{K}_0, \ldots, \mathscr{K}_{n-1}) | \mathscr{K}_0, \ldots, \mathscr{K}_{n-1} \in \mathscr{L}_n, \mathscr{K}_i = \mathscr{M}_i^{t'}, \mathscr{M}_i \in (\mathscr{L}_n)_{t^i}\}),$$
$$M(\mathscr{K}_0, \ldots, \mathscr{K}_{n-1}) = \{\mathfrak{A} \in \mathrm{Mod}_{t'} | \langle |\mathfrak{A}|, R_0^{\mathscr{K}_1}, \ldots, R_{n-1}^{\mathscr{K}_{n-1}} \rangle \in Q\},$$

and

$$R_i^{\mathscr{K}_i} = \{\langle a_0, \ldots, a_{t(i)-1} \rangle | \langle \mathfrak{A}, a_0, \ldots, a_{t(i)-1} \rangle \in \mathscr{M}_i\}.$$

If $Q_1, \ldots, Q_n$ are quantifiers, a logic $\mathscr{L}(Q_1, \ldots, Q_n)$ can be defined in an analogous way, closing $\mathscr{L}_{\omega\omega}$ inductively under each Q_i.
Note that $\mathscr{L}(Q_1, \ldots, Q_n)$ is the smallest normal logic which is closed under quantifiers $Q_1, \ldots, Q_n$. Logics of this kind were introduced in [1].

Definition 10. A logic $\mathscr{L}$ is *finitely generated*, if there are quantifiers $Q_1, \ldots, Q_n$ such that $\mathscr{L} = \mathscr{L}(Q_1, \ldots, Q_n)$.

If $\mathscr{L}$ is finitely generated, one single quantifier Q can be found such that $\mathscr{L} = \mathscr{L}(Q)$. In a simple case, suppose $\mathscr{L} = \mathscr{L}(Q_1, Q_2, Q_3)$ where Q_1, Q_2, and Q_3 are arbitrary quantifiers of type $\langle 2 \rangle$. Then $\mathscr{L} = \mathscr{L}(Q)$, where

$$\begin{aligned} QxyzA(x)B(y,z) \leftrightarrow \; & (|A(\cdot)| = 1 \;\&\; Q_1 yzB(y,z)) \vee \\ & (|A(\cdot)| = 2 \;\&\; Q_2 yzB(y,z)) \vee \\ & (|A(\cdot)| = 3 \;\&\; Q_3 yzB(y,z)). \end{aligned}$$

Thus it is natural to take logics of the form $\mathscr{L}(Q)$ as canonical representatives of finitely generated logics. It is not true that every logic is finitely generated; for example, second order logic and the Δ-closure of weak second order logic are not.

The problem whether a given logic is finitely generated seems interesting and has certainly wider relevance, but it will not be treated here.

Let us use $\boldsymbol{NLQ}$ to denote the family of all (necessarily normal) logics of the form $\mathscr{L}(Q)$, and $\mathscr{N}\mathscr{L}\mathscr{Q}$ to denote the structure $\langle \boldsymbol{NLQ}, \leqq \rangle$.

By virtue of what has been said above, it is obvious that:

Proposition 23. *$\mathscr{N}\mathscr{L}\mathscr{Q}$ is an upper semilattice.*

Problem 24. *Is $\mathscr{N}\mathscr{L}\mathscr{Q}$ a lattice?*

The structure of $\mathscr{N}\mathscr{L}\mathscr{Q}$ seems much harder to reveal than that of $\mathscr{N}\mathscr{L}$ and many open problems immediately suggest themselves. Our main result concerning $\mathscr{N}\mathscr{L}\mathscr{Q}$ is the following universality property:

Theorem 25. *Every countable distributive lattice can be embedded as an upper subsemilattice into $\mathscr{N}\mathscr{L}\mathscr{Q}$.*

Proof. Suppose $\boldsymbol{D} = \langle D, \leqq \rangle$ is a countable distributive lattice. We may assume that $\boldsymbol{D}$ is a lattice of non-empty subsets of $\omega - \{0\}$. Let

$$D = \{d_n | n < \omega\}.$$

For $d_n \in D$, let

$$Q_n = \{\langle M, A, B \rangle | \, |A| = m \in d_n \text{ and } |B| = \omega_m\}$$

and

$$Q = \{\langle M, A, B \rangle | \, |A| = n \text{ and } |B| \in d_n\}.$$

We obtain the desired embedding by letting

$$\mathscr{L}_n = \mathscr{L}(Q_n, Q).$$

Suppose at first $d_n \leqq d_m$. Now $Q_n xyA(x)B(y)$ is equivalent to

$$Q_m xyA(x)B(y) \wedge \exists x_1 \ldots x_n \Big(\bigwedge_{1 \leqq i < j \leqq n} x_i \neq x_j \wedge Qxy \Big(\bigvee_{i \leqq n} x = x_i \Big) A(y) \Big),$$

whence $\mathscr{L}_n \leqq \mathscr{L}_m$. Note also that if $d_n \vee d_m = d_k$, then $\mathscr{L}_n \vee \mathscr{L}_m = \mathscr{L}_k$ because on the one hand

$$Q_k xyA(x)B(y) \leftrightarrow Q_n xyA(x)B(y) \vee Q_m xyA(x)B(y)$$

and on the other hand $\mathscr{L}_n \leqq \mathscr{L}_k$, $\mathscr{L}_m \leqq \mathscr{L}_k$. Thus to end the proof, we just have to prove that $\mathscr{L}_n \leqq \mathscr{L}_m$ implies $d_n \leqq d_m$. For this end, suppose not $d_n \leqq d_m$, and $i \in d_n - d_m$. Let $\mathfrak{A} = \langle \omega \rangle$ and $\mathfrak{B} = \langle \omega_i \rangle$ (note that $i \neq 0$).

Lemma 26. *For any formula $\phi(x_1, \ldots, x_k)$ of $\mathscr{L}_m$ and any $a_1, \ldots, a_k \in \omega$,*

$$\mathfrak{A} \models \phi(a_1, \ldots, a_k) \quad \text{if and only if} \quad \mathfrak{B} \models \phi(a_1, \ldots, a_k).$$

Proof. For any a and b in ω_i there is an automorphism of $\mathfrak{A}$ which interchanges a and b, but leaves the other elements fixed. The structure $\mathfrak{B}$ has the same property. Hence the Boolean algebra of subsets of $|\mathfrak{A}|(|\mathfrak{B}|)$ which are $\mathscr{L}(Q_m)$-definable using a given sequence $a_1, \ldots, a_n$ of parameters, is generated by the elements $\{a_1\}, \ldots, \{a_n\}$. Using this fact, the claim is easily proved by induction on the length of $\phi(x_1, \ldots, x_n)$. □

Let us return to the proof of Theorem 25. The structures $\mathfrak{A}$ and $\mathfrak{B}$ can be separated by the following $\mathscr{L}_n$-sentence

$$\exists x_1 \ldots \exists x_n \Big(\bigwedge_{1 \leqq k < l \leqq i} x_k \neq x_l \wedge Q_{X_n} xy \Big(\bigvee_{k \leqq i} x = x_i \Big)(y = y) \Big).$$

On the other hand, Lemma 26 implies that these models cannot be separated by any sentence of $\mathscr{L}_m$. Therefore $\mathscr{L}_n \not\leqq \mathscr{L}_m$ and the proof is complete. □

Corollary 27. *Every countable partial ordering can be embedded into $\mathscr{N}\mathscr{L}\mathscr{Q}$.*

The embedding of Theorem 25 can also be done above any given $L \in NLQ$ – one just has to choose the models $\mathfrak{A}$ and $\mathfrak{B}$ more carefully. The result cannot be generalized to uncountable lattices because $\mathscr{N}\mathscr{L}\mathscr{Q}$ is ω_1-like, that is, every $\mathscr{L} \in NLQ$ has but countably many predecessors. Whether every ω_1-like partial ordering can be embedded into $\mathscr{N}\mathscr{L}\mathscr{Q}$ remains open.
Let us then turn into non-distributive lattices. Again the situation is even more unclear than with $\mathscr{N}\mathscr{L}$. It is obvious that the five-element lattices of Proposition 14 can be embedded into $\mathscr{N}\mathscr{L}\mathscr{Q}$ such that sups are preserved. But to preserve infs more powerful methods seem to be needed:
For any classes $\mathscr{C}$ of cardinals let

$Q^{dc}_{\mathscr{C}} = \{\langle A, B \rangle | B$ is a linear ordering of its field with a dedekind cut (B_1, B_2) such that the cofinality of B_1 and the cofinality of the reverse ordering of B_2 are in $\mathscr{C}\}$.

Theorem 28 (S. Shelah [6]). *If there is a weakly compact cardinal, or a certain combinatorial principle (true in L) holds, then there is a class $\mathscr{C}$ of regular cardinals including ω such that the logic $\mathscr{L}(Q^{dc}_{\mathscr{C}})$ is fully compact.*

Corollary 29. *Under the hypothesis of Theorem 28, the pentagon lattice (as in Proposition 14) can be embedded into $\mathscr{N}\mathscr{L}\mathscr{Q}$ such that sups, infs, and least elements are preserved.*

Proof. Using the notation of Proposition 14, let $a=\mathscr{L}_{\omega\omega}$, $b=\mathscr{L}(Q_0)$, $d=\mathscr{L}(Q^{dc}_{\mathscr{C}})$, $e=\mathscr{L}(Q_0,Q^{dc}_{\mathscr{C}})$, and $c=\mathscr{L}(Q_0,Q)$, where

$$\begin{aligned} Q=\{\langle A,B,F\rangle \mid & \text{There are an } a\in A \text{ and an } \alpha\in\text{On such that} \\ & \langle A,B,a\rangle \simeq \langle \alpha,<,\omega\rangle \text{ and } F \text{ has the properties} \\ & (1)\ \forall xyzu(F(x,y,z)\ \&\ F(x,u,z) \to y=u) \\ & (2)\ \forall x\exists y\forall z\exists u(B(z,x) \to B(u,a)\ \&\ F(y,z,u)). \end{aligned}$$

A simple Fuhrken-type reduction-argument shows that c satisfies the Löwenheim-Skolem theorem down to ω. As d is compact, we obtain from Lindström's theorem the result that $d\wedge c$ exists and equals a. Thus it suffices to prove that $c\leqq b\vee d$. But consider the conjunction $\phi(B,F)$ of the following sentences of e:

(1), (2) as in the definition of Q,
(3) "B is a strict linear ordering with a least element such that every non-maximal element b has an immediate successor $b+1$",
(4) $\exists x\forall y(B(y,x) \to -Q_0 zB(z,y))$,
(5) $-Q^{dc}_{\mathscr{C}}xyB(x,y)$.

If an infinite model of $\phi(B,F)$ contains a sequence $(b_n)_{n<\omega}$ such that $B(b_{n+1},b_n)$ for all $n<\omega$, then the sets

$$A_1=\{x|\forall n<\omega: B(x,b_n)\}, \qquad A_2=\{x|\exists n<\omega: B(b_n,x)\}$$

constitute a Dedekind cut. By (1), (2), and (4) the model is countable. As $\omega\in C$, the cut (A_1,A_2) contradicts (5). □

We have observed already that $\mathscr{N}\mathscr{L}\mathscr{Q}$ contains countable chains only. But the following trivial result shows that there is no bound on the size of anti-chains:

Proposition 30. *Let $Q_\alpha=\{\langle A,B\rangle|B\subsetneq A,\ |B|\geqq\omega_\alpha\}$. The logics $\mathscr{L}(Q_\alpha)$, $\alpha\in$On, are incompatible in $\mathscr{N}\mathscr{L}\mathscr{Q}$ (even in $\mathscr{N}\mathscr{L}$).*

Proof. Suppose $\mathscr{L}\leqq\mathscr{L}(Q_\alpha)$ and $\mathscr{L}\leqq\mathscr{L}(Q_\beta)$ where $\alpha<\beta$. Let $\phi\in\mathscr{L}$. As $\phi\in\mathscr{L}(Q_\beta)$, we can find a $\psi\in\mathscr{L}_{\omega\omega}$ such that ϕ and ψ have the same models of power $\leqq\omega_\alpha$. If $\mathfrak{A}$ is any model, there is an elementary substructure $\mathfrak{B}$ of $\mathfrak{A}$ of power $\leqq\omega_\alpha$ with respect to the logic $\mathscr{L}(Q_\alpha)$. Now we have

$$\mathfrak{A}\models\phi \leftrightarrow \mathfrak{B}\models\phi \leftrightarrow \mathfrak{B}\models\psi \leftrightarrow \mathfrak{A}\models\psi$$

whence ϕ and ψ have the same models. This shows that $\mathscr{L}=\mathscr{L}_{\omega\omega}$. □

It is also easily proved that any $\mathscr{L} \in \mathscr{N}\mathscr{L}\mathscr{Q}$ has a proper class of incomparable extensions in $\mathscr{N}\mathscr{L}\mathscr{Q}$. The following new notion leads us to a study of the *local* structure of $\mathscr{N}\mathscr{L}\mathscr{Q}$.

Definition 11. Let κ be a cardinal. A generalized quantifier Q is *bounded by* κ, if every structure in Q has cardinality $\leqq \kappa$. A logic $\mathscr{L}$ is *bounded by* κ, if $\mathscr{L} = \mathscr{L}(Q)$ for some Q which is bounded by κ.
The most obvious example is the quantifier "there exists at most ω_α many", which is bounded by ω_α. Trivially, there are at most 2^{2^κ} non-equivalent logics bounded by κ.

Theorem 31. *Let κ be a cardinal $> \omega$. There are exactly 2^{2^κ} incomparable finitely generated logics bounded by κ.*

Proof. For any $x \subsetneqq \kappa$, let $\mathfrak{A}_x$ be the structure $\langle \kappa, <_x \rangle$ where the order type of $<_x$ results from that of κ by replacing every $\alpha \in x$ by a copy of $\eta(\kappa^* + \kappa)$ (η is the order type of the rationals), and every $\alpha \notin x$ by a copy of $\eta(\kappa^* + \kappa + 1)$. Thus $\mathfrak{A}_x \not\simeq \mathfrak{A}_y$ whenever $x \neq y$.
For $\mathscr{F} \subset \mathscr{P}(\kappa)$, let

$$Q_{\mathscr{F}} = \{\mathfrak{A} | \mathfrak{A} \simeq \mathfrak{A}_x \text{ for some } x \in \mathscr{F}\}.$$

Let $\mathscr{F}$ and $\mathscr{G}$ be different subsets of $\mathscr{P}(\kappa) - \{\kappa\}$. We claim that $\mathscr{L}(Q_{\mathscr{F}})$ and $\mathscr{L}(Q_{\mathscr{G}})$ are incomparable. Indeed, suppose $Q_{\mathscr{F}}$ is definable by a sentence ϕ of $\mathscr{L}(Q_{\mathscr{G}})$. Now we need a simple lemma:

Lemma 32. *For every formula $\phi(x_1, \ldots, x_n)$ of $\mathscr{L}(Q_{\mathscr{G}})$ there is a quantifierfree formula $\psi(x_1, \ldots, x_n)$ such that for all $x \notin \mathscr{G}$,*

$$\mathfrak{A}_x \models \phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n).$$

Proof. We use induction on the length of $\phi(x_1, \ldots, x_n)$. As the theory of dense linear order admits elimination of quantifiers, we only have to consider the induction step for the quantifier Q. Suppose therefore

$$\phi(x_1, \ldots, x_n) = Q_{\mathscr{G}} uv \theta(u, v, x_1, \ldots, x_n),$$

where $\theta(u, v, x_1, \ldots, x_n)$ is quantifierfree. If $a_1, \ldots, a_n \in \kappa$ and $x \notin \mathscr{G}$, the order type of $\{\langle b, c \rangle | \mathfrak{A}_x \models \theta(b, c, a_1, \ldots, a_n)\}$ is never that of an $\mathfrak{A}_y$, $y \in \mathscr{G}$. Therefore, in such models $\theta(x_1, \ldots, x_n)$ is equivalent to $x_1 \neq x_1$. □

We are ready to continue the proof of the theorem. For the chosen ϕ we can find a quantifierfree ψ such that for $x \notin \mathscr{G}$,

$$\mathfrak{A}_x \models \phi \leftrightarrow \psi.$$

But now it follows that $\mathfrak{A}_\kappa \models \phi$, whence $\kappa \in \mathscr{F}$, a contradiction. □

The previous theorem was one of the very few places so far where we have made use of non-monadic (of type $t \notin {}^{\omega}2$) quantifiers. This suggests the study of the substructure of $\mathcal{N}\mathcal{L}\mathcal{Q}$ consisting of monadic quantifiers only.
Let MLQ be the collection of all $\mathcal{L}(Q)$ such that Q is a generalized quantifier of a monadic type $t \in {}^{n}2$ ($n \in \omega$). Summing up the results the proofs of which do not use non-monadic quantifiers, yields:

Theorem 33. a) *The structure* $\mathcal{M}\mathcal{L}\mathcal{Q} = \langle MLQ, \leqq \rangle$ *is an upper subsemilattice of* $\mathcal{N}\mathcal{L}\mathcal{Q}$.
b) *Every* $\mathcal{L} \in \mathcal{M}\mathcal{L}\mathcal{Q}$ *has a proper class of incomparable extensions.*
c) *Every countable distributive lattice can be embedded as an upper subsemilattice into* $\mathcal{M}\mathcal{L}\mathcal{Q}$.

As a complement to these similarities between $\mathcal{N}\mathcal{L}\mathcal{Q}$ and $\mathcal{M}\mathcal{L}\mathcal{Q}$ we have the following local difference:

Theorem 34. *Let* κ *be a cardinal* $\geqq \omega$ *and* $\lambda = |\{\mu < \kappa \mid \mu \text{ a cardinal}\}|$. *There are exactly* 2^{λ} *incomparable logics of the form* $\mathcal{L}(Q)$, *where* Q *is a monadic quantifier bounded by* κ.

Proof. Suppose at first $\lambda > \omega$. For any set $\mathcal{X}$ of infinite cardinals $\leqq \kappa$, let

$$Q_{\mathcal{X}} = \{\langle A, B\rangle \mid B \subseteq A \text{ and } |B| \in \mathcal{X}\}.$$

The number of different such sets $\mathcal{X}$ is 2^{λ} and different sets $\mathcal{X}$ give rise to incomparable logics $\mathcal{L}(Q_{\mathcal{X}})$ (this is seen as in the proof of Theorem 27). Suppose then $\lambda = \omega$. For $\mathcal{X} \subseteq \omega$, let $Q_{\mathcal{X}}$ be defined as above. Suppose $L(Q_{\mathcal{X}}) \leqq L(Q_{\mathcal{Y}})$. Then there is an identity sentence ϕ of $L(Q_{\mathcal{Y}})$ such that ϕ is equivalent to $Q_{\mathcal{X}} v\ (v = v)$. Thus

$$n \in \mathcal{X} \quad \text{if and only if} \quad \langle n \rangle \models \phi .$$

It follows that $\mathcal{X}$ is Turing reducible to $\mathcal{Y}$. Now the claim follows from the fact that there are 2^{ω} incomparable degrees of unsolvability. □

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