THEORIA
THEORIA, 2010, **76**, 100–107
doi:10.1111/j.1755-2567.2010.01069.x

Obituary

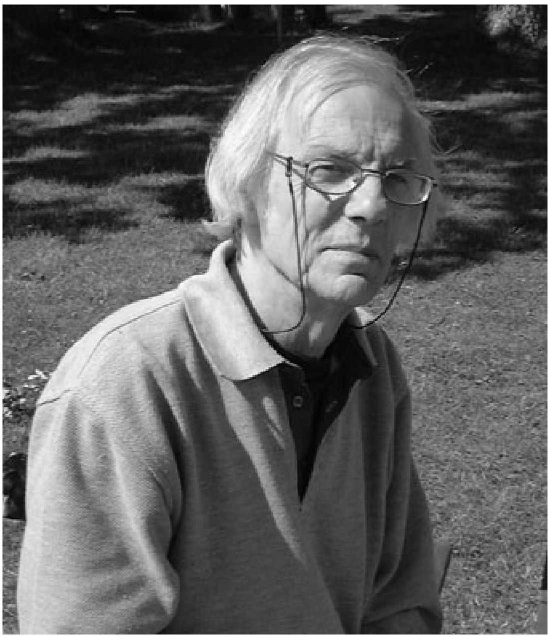
In Memoriam: Per Lindström

by

JOUKO VÄÄNÄNEN

and

DAG WESTERSTÅHL



WHEN PER LINDSTRÖM – or Pelle Lindström as he insisted on being called – died on 21 August 2009, Sweden lost one of its internationally most renowned mathematical logicians. Pelle Lindström was a rather withdrawn person, not a frequent conference participant, and he did not feel that his results, once published, needed further advertising: they should speak for themselves. So although he was well known to logicians all over the world, and to his colleagues in Sweden, his work was not as widely known outside this limited circle as perhaps it should have been. Indeed, the results for which he is famous – nowadays collected under the label *Lindström's Theorem* – are not only innovative and rich in mathematical content, but they also have philosophical implications that anyone interested in the role of logic will do well to be aware of.

Published by Blackwell Publishing Ltd

Pelle Lindström was born on 9 April 1936, and spent most of his academic life at the Department of Philosophy, University of Gothenburg, where he was employed first as a lecturer ("docent") and, from 1991, as a Professor of Logic, until his retirement in 2001. During his school years he showed little interest in the subjects taught, including mathematics. It was during his early university studies, in first practical and then theoretical philosophy,[1] that he became interested in logic, and began to develop his remarkable mathematical talent. Although he had contact with other Swedish logicians such as Stig Kanger in Uppsala and Sören Halldén in Lund, he had no teacher, but essentially taught himself mathematical logic, apparently by reading Kleene's *Introduction to Metamathematics* which appeared in 1952, and Tarski's, Vaught's, and Robinson's model-theoretic papers from the fifties.

Between 1964 and 1966 Pelle Lindström published four papers in *Theoria* which were to constitute his PhD dissertation. Each of these papers contained important results in model theory. "On Model-completeness" (1964) gives a criterion for a theory to be model-complete – a property Robinson had introduced for studying relations between model theory and algebra – which today is usually called Lindström's test for model-completeness. In "On Characterizability in $L_{\omega_1\omega}$" (1966a) he proved that the notion of well-order is not definable even if additional predicates are allowed and even in a logic that extends first-order logic (*FO*) by allowing countable conjunctions and disjunctions of sentences (a result proved independently and in more general form by Lopez-Escobar). The proof was an early example of the use of recursion theory to prove model-theoretic results. Further, in the early sixties Lindström had independently rediscovered what is now called the Ehrenfeucht-Fraïssé (EF) method of characterizing elementary equivalence between two models (i.e., that the same *FO* sentences are true in them), and in "On Relations between Structures" (1966b) he used that method to obtain a powerful interpolation/preservation theorem for *FO*. But the paper from his dissertation that was to have the most profound impact was "First Order Predicate Logic with Generalized Quantifiers" (1966c).

Mostowski (1957) had generalized the usual *FO* quantifiers $\forall$ and $\exists$ to *cardinality quantifiers*: just as $\exists xAx$ says that (the extension of) A is non-empty, and $\forall xAx$ that the complement of A with respect to the universe of the model, $M - A$, is empty, so one can take $QxAx$ to express any given condition on the cardinalities of A and $M - A$. For instance, we have the quantifiers Q_α, where $Q_\alpha xAx$ says that A has cardinality at least $\aleph_\alpha$. Adding such quantifiers to the expressive means of *FO* can considerably increase expressive power. For example, if one adds Q_0, "there are infinitely many", the standard model of arithmetic (consisting of the natural

1 Academic philosophy in Sweden is by tradition divided into practical and theoretical philosophy, the first dealing with moral philosophy, philosophy of action, political philosophy, etc., and the second with metaphysics, epistemology, logic, philosophy of mind, philosophy of language, etc.

Published by Blackwell Publishing Ltd

numbers with addition and multiplication) becomes characterizable (up to isomorphism) with a single sentence. In his 1957 paper Mostowski gave a semantic characterization of *FO* among all logics based on his quantifiers. The proof of this characterization was rather simple, mainly because Mostowski's concept of a quantifier was so limited.

Pelle Lindström had first assumed that Mostowski had a completely general concept of (generalized) quantifier, but then saw that this wasn't so. First, a quantifier should be able to apply to more than one formula, thus $Qx(\varphi_1, \ldots, \varphi_k)$ rather than $Qx\varphi$, and bind x in each φ_i. For example, he realized (inspired by a remark by Rescher) that a simple condition like "Most *A* are *B*" (meaning that the number of *A*s that are *B* is greater than the number of *A*s that are not *B*) is *not* expressible by means of Mostowski's quantifiers (in contrast with, say, "Infinitely many *A* are *B*", which is expressible by means of Q_0 and Boolean connectives). Second, a quantifier should be able to bind more than one variable in a formula. Thus, one may allow formulas like $Qx; xy(Ax, Rxy)$, expressing, say, that the binary relation R well-orders the set A, a condition not expressible with quantifiers binding just one variable. As he himself recalled, "These simple observations opened up a landscape of surprising richness and variety."[2] In fact, his notion of a generalized quantifier, nowadays often called *Lindström quantifier*, has become a standard tool not only in model theory, but also in theoretical computer science and in formal semantics for natural languages.

Furthermore, he also saw that the EF method can yield results about logics with generalized quantifiers. Already in the 1966 paper, he used that method to characterize *FO* as maximal with respect to certain properties. These characterization results were greatly improved and generalized in his "On Extensions of Elementary Logic" (1969). This 11-page paper, also published in *Theoria*, is without a doubt Pelle Lindström's most important single contribution to mathematical logic. In it, he uses a surprisingly general notion of a logic: an (abstract) logic L is simply a pair $(S_L, \models_L)$ of a set S_L whose elements are called *sentences*, and a relation $\models_L$ between models and sentences, where $\mathcal{M} \models \varphi$ reads "φ is true in $\mathcal{M}$". L should satisfy some structural properties, such as *isomorphism closure*: If $\mathcal{M} \models \varphi$ and $\mathcal{M}'$ is isomorphic to $\mathcal{M}$, then $\mathcal{M} \models \varphi$. Also, L should have negation and conjunction, so that, for example, for every L-sentence φ there is an L-sentence which is true in a model exactly when φ is not true in that model. Now, there is a natural way of comparing the strength of such logics: L' is *at least as strong as* L, $L \leq L'$, if every L-sentence is equivalent to (true in the same models as) some L'-sentence, and L and L' are *equivalent*, $L \equiv L'$, if $L \leq L'$ and $L' \leq L$. *Lindström's Theorem* has the form: If $FO \leq L$, and L has certain familiar properties, then in fact $FO \equiv L$.

2 Lindström (1995), p. 22. In this short note Lindström gives an interesting account of how he arrived at the notion of a generalized quantifier and the subsequent characterizations of first-order logic.

First-order logic, *FO*, has become the logical formalism *par préférence* for at least two reasons. First, it has sufficient power to express most of modern mathematics (via the first-order formalization of Zermelo-Fraenkel set theory). Second, it has several important properties, such as *completeness*: the set of its valid sentences (the sentences true in all models) can be effectively generated from certain simple axioms (the set is recursively enumerable). It has long been known that these properties hinge on *FO* not being *too* expressive. For example, it follows from Gödel's incompleteness theorem that a logic in which Q_0 is expressible, and thus the standard model of arithmetic is characterizable, cannot be complete. What Lindström's Theorem does is to provide an explanation of this state of affairs.

Here are some well-known properties of *FO*, that can be stated for an arbitrary logic *L*:

- The *Löwenheim property*: every sentence which has (is true in) an infinite model has a countable model. (Also, stronger versions like the *Löwenheim-Skolem property*: every *theory* (set of sentences) in a countable language with an infinite model has a countable model.)
- The *Tarski property*: every sentence with an infinite model has an uncountable model. (A stronger version is the *upward Löwenheim-Skolem-Tarski property*, which generalizes to theories, and says that there are models in every infinite cardinality.)
- (Countable) *compactness*: If every finite subset of a (countable) set of sentences has a model, then the whole set has a model.
- *Completeness*: The set of valid sentences is recursively enumerable.
- The *Craig interpolation property*: If $\varphi \models \psi$ (i.e., ψ is true in every model in which φ is true), there is an *interpolant*: a sentence θ containing only non-logical symbols common to both φ and ψ, such that $\varphi \models \theta$ and $\theta \models \psi$. This property is known to imply:
- The *Beth definability property*: If a sentence implicitly defines a predicate *P* occurring in it, then it also explicitly defines *P*.[3]

In the paper mentioned, Lindström proved the following results.[4]

Theorem: If *$FO \leq L$ and L has one of the following combinations of properties, then $FO \equiv L$:*

3 φ implicitly defines *P* if every model of φ has at most one interpretation of *P*. It explicitly defines *P* if there is a formula ψ in the same symbols but not containing *P*, such that in every model of φ, *P* and ψ have the same extension.

4 There are some slight differences as to what is required of *L* in (a)–(d). In particular, Lindström proved (d) for logics with generalized quantifiers (note that $L_{\omega_1\omega}$ has the Löwenheim property and the Beth definability property), but it holds in general if some extra computability assumptions are made on *L*. For exact formulations and proofs see, for example, Flum (1985).

(a) *The Löwenheim property and countable compactness.*
(b) *The Löwenheim and Tarski properties.*
(c) *The Löwenheim property and completeness.*
(d) *The Löwenheim property and the Beth definability property.*

Several further characterizations of *FO*, all due to Lindström, are known.[5]

Let us return to the question, why is it that first-order logic has become the logical formalism *par préférence*. Are there some deeper reasons for this or is it just a coincidence or a matter of convenience? What Lindström's Theorem does is nothing less than to sharply reveal such deeper reasons. If we extend first-order logic, we lose one or several of the very properties that make first-order logic so useful. It says in effect (think of (b)) that any formal language that goes beyond first-order logic has to distinguish between some infinite cardinalities in the sense that some sentence has a model of some infinite cardinality but not of all infinite cardinalities. Loosely speaking, Lindström's Theorem tells us that any proper extension of first-order logic has to detect something non-trivial about the set-theoretic universe. On the other hand, Lindström's Theorem reveals a deep robustness of first-order logic: it does not matter which way you define your syntax – as long as your semantics obeys certain basic principles (like (a)–(d) above): you will always get the same logic. It is remarkable that despite initial optimism very few – if indeed any – other logics have emerged with robustness of similar calibre.

From the mid-seventies and onwards, Lindström's research in logic focused on the area of *arithmetized metamathematics*. In particular, he was interested in the relation of *interpretability* between theories containing some arithmetic. This work is a continuation of the results and methods introduced by Gödel with his incompleteness theorems. Gödel had proved in 1931 not only that a consistent theory T containing some arithmetic (such as first-order Peano arithmetic, *PA*) is incomplete,[6] but also that (an arithmetized version of) the statement that T is inconsistent, Con_T, is itself an example of a sentence which is true but unprovable in T. These results were taken further by Feferman in the late fifties. For example, he strengthened Gödel's result to the statement that the theory obtained by adding Con_T as an axiom to T is not even *interpretable* in T.[7] Moreover, he showed that these facts

5 See Lindström (1973, 1974, 1978).

6 Gödel assumed T had the stronger property of *ω-consistency*; that consistency is enough was shown by Rosser in 1936.

7 Roughly, S is *interpretable* in T if there is a formula $\sigma(x)$ of T, defining the "universe" of the interpretation, and a translation from the language of S to the language of T such that the translation of every theorem of S, relativized to the universe given by $\sigma(x)$, is provable in T. The main use of interpretability is to show *relative consistency* results, since if S is interpretable in T and T is consistent, S must also be consistent. A classical example is the consistency of various non-euclidean geometries, obtained by showing that they can be interpreted in euclidean geometry. Tarski had applied the notion of interpretability to obtain numerous results about consistency and undecidability for arithmetical theories; see Tarski,

about Con_T hinge on the exact formulation of that sentence: there is a sentence Con_T^* in the language of arithmetic which *extensionally* expresses that T is consistent but which *is* provable in T. It turns out that such sentences are no mere curiosity but an essential tool in the theory of interpretability that was initiated by Feferman. Using them, he was able to prove that (roughly) a theory S is interpretable in T if and only if there is a sentence extensionally expressing that S is consistent which is provable in T. In other words, the notion of interpretability, for this type of theories, turns out to be essentially tied to the provability of consistency statements.

This line of research, which had lain dormant (with a few exceptions) since Feferman left it in 1960, was taken up by Pelle Lindström, and at about the same time, but independently, by a group of logicians around Petr Hájek in Prague. Lindström contributed several memorable results in this and related areas. His approach to interpretability was abstract: he studied various lattices of *degrees* of interpretability (classes of theories mutually interpretable in each other), such as the lattice generated by a fixed extension of *PA*, or the lattices of Σ_n- and Π_n-sentences over *PA*, for fixed n. The results are technical and many have extremely clever proofs. For example, there is the Lindström fixed-point construction, a far-reaching generalization of Gödel's diagonal method; such constructions continue to be crucial in this area of research, ever since Gödel used it to construct a sentence "saying" of itself that it is not provable. Other results include the Lindström-Solovay theorem that the interpretability relation between sentences over *PA* is Π^0_2-complete, and the characterization of faithful interpretability over *PA* as a combination of Π_1- and Σ_1-conservativity. He also contributed to provability logic and interpretability logic, in which the provability predicate (or the interpretability relation) is treated as a sentential operator in the style of modal logic. Most of the results on interpretability are presented in detail in his book *Aspects of Incompleteness* (1997a; second edition in 2002 published by the Association of Symbolic Logic). Very readable are also the two survey papers he published in *Theoria*: "Provability Logic – a Short Introduction" (1996), and "Interpretability in Reflexive Theories – a Survey" (1997b).[8]

Throughout his life Pelle Lindström also took an active interest in philosophical issues. In fact, many of those who knew him have vivid memories of heated

---

Mostowski, and Robinson (1953). Feferman was the first to study interpretability for its own sake. His classical paper is "The Arithmetization of Metamathematics in a General Setting" (Feferman, 1960). For an interesting recollection of how these results emerged, with a reference to Lindström's work, see Feferman (1997).

8 Thus, he remained faithful to *Theoria*; we have now mentioned each of the eight (!) papers he published there. Some have complained that Lindström's Theorem would have become quicker known among logicians – it took a couple of years before that happened – if he had published his results in a journal more familiar to mathematical logicians. Not being one for advertising himself, we think he simply felt it a duty to his academic background to publish in a Swedish journal.

discussions (in seminars, in cafeterias, or in his home) on logic, philosophy, literature, music, politics, and other topics. Intellectual discussion and argument was his passion. For example, he had strong views on the philosophy of mathematics, but was for a long time reluctant to publish them, feeling he did not have enough to say. Eventually he did, however (Lindström, 2000), expounding, among other things, his "quasi-realist" view that the *visualizable* parts of mathematics were beyond doubt and that classical logic holds for them.[9] To the visualizable parts he counted not only the ω-sequence of natural numbers, but also the set of arbitrary sets of natural numbers, since this set can be visualized by means of branches in the infinite binary tree, whereas nothing similar can be said for, e.g., sets of sets of numbers. But he also published on other philosophical topics: he contributed over the years numerous short papers and notices to the Swedish popular philosophy journal *Filosofisk Tidskrift*, on issues as diverse as the freedom of the will, the mind-body problem, utilitarianism, and counterfactuals.

Pelle Lindström remained active in logic and philosophy until the end. As late as 2006 he published in the *Journal of Philosophical Logic* a paper on various ways to prove the de Jongh-Sambin fixed point theorem in provability logic, including his own simplified proof(s) (2006a). And he had two papers in the same journal (2001, 2006b), on Roger Penrose's attempts to revive the argument that Gödel's incompleteness theorem shows that the mind cannot be mechanical.[10] Furthermore, a paper with V. Shavrukov, "The $\forall\exists$ theory of Peano Σ_1 sentences", will appear in the *Journal of Mathematical Logic*, another contribution to *Filosofisk Tidskrift* is about to be published, and a book manuscript, *First Order Logic*, where he presents first-order logic and its properties in just the way he thought it should be presented, is also under publication.

Pelle Lindström was a true logician. He put his energy only into the deepest questions of logic, such as the semantic character of logic, the extent of incompleteness in number theory, and the fundamental questions in philosophy of mathematics. He did not publish vigorously and he travelled sparingly; but when met in person, he was a vivid lecturer and despite (or perhaps because of) his intimidating sharpness, a wonderful person to talk to. Like Skolem, he stands out as a Scandinavian logician whose name will always remain a household name in logic.

---

9 The view is quasi-realist in that he agreed with Kreisel's dictum that it is not the existence of mathematical objects which is at stake, but the objectivity of mathematical truths.

10 Penrose presented such an argument first in *The Emperor's New Mind* (1989), then a different and more detailed argument in *Shadows of the Mind* (1994), and finally a third version in "Beyond the Doubting of a Shadow. A Reply to Commentaries on Shadows of the Mind" (1995). In "Penrose's New Argument" (2001) and "Remarks on Penrose's 'New Argument' " (2006b), Lindström drew on his metamathematical expertise to argue that Penrose's reasoning was far from conclusive.

References

Feferman, S. (1960) "The Arithmetization of Metamathematics in a General Setting." *Fundamenta Mathematicae*, 49: 35–92.

Feferman, S. (1997) "My Route to Arithmetization." *Theoria*, 63: 168–181.

Flum, J. (1985) "Characterizing Logics." In J. Barwise and S. Feferman (eds), *Model-theoretic Logics*, pp. 77–120. New York: Springer.

Lindström, P. (1964) "On Model-completeness." *Theoria*, 30: 183–196.

Lindström, P. (1966a) "On Characterizability in $L_{\omega_1\omega}$." *Theoria*, 32: 186–95.

Lindström, P. (1966b) "On Relations between Structures." *Theoria*, 32: 172–85.

Lindström, P. (1966c) "First Order Predicate Logic with Generalized Quantifiers." *Theoria*, 32: 186–95.

Lindström, P. (1969) "On Extensions of Elementary Logic." *Theoria*, 35: 1–11.

Lindström, P. (1973) "A Characterization of Elementary Logic." In *Modality, Morality, and Other Problems of Sense and Nonsense*, pp. 189–191. Lund: Gleerups Bokforlag.

Lindström, P. (1974) "On Characterizing Elementary Logic." In S. Stenlund (ed.), *Logical Theory and Semantic Analysis*, Synthese Library vol. 63, pp. 129–146. Dordrecht: D. Reidel.

Lindström, P. (1978) "Omitting Uncountable Types and Extensions of Elementary Logic." *Theoria*, 44: 152–156.

Lindström, P. (1995) "Prologue." In M. Krynicki, M. Mostowski, and L. Szczerba (eds), *Quantifiers: Logics, Models, and Computation*, pp. 21–24. Dordrecht: D. Reidel.

Lindström, P. (1996) "Provability Logic – a Short Introduction." *Theoria*, 62: 19–61.

Lindström, P. (1997a) *Aspects of Incompleteness*. Lecture Notes in Logic, Springer.

Lindström, P. (1997b) "Interpretability in Reflexive Theories – a Survey." *Theoria*, 63: 182–209.

Lindström, P. (2000) "Quasi-realism in Mathematics." *The Monist*, 83: 122–149.

Lindström, P. (2001) "Penrose's New Argument." *Journal of Philosophical Logic*, 30: 241–250.

Lindström, P. (2006a) "Note on Some Fixed Point Constructions in Provability Logic." *Journal of Philosophical Logic*, 35: 225–230.

Lindström, P. (2006b) "Remarks on Penrose's 'New Argument'." *Journal of Philosophical Logic*, 35: 231–237.

Mostowski, A. (1957) "On a Generalization of Quantifiers." *Fundamenta Mathematicae*, 44: 12–36.

Penrose, R. (1989) *The Emperor's New Mind*. Oxford: Oxford University Press.

Penrose, R. (1994) *Shadows of the Mind*. Oxford: Oxford University Press.

Penrose, R. (1995) "Beyond the Doubting of a Shadow. A Reply to Commentaries on Shadows of the Mind." *Psyche*, 2: 89–129.

Tarski, A., Mostowski, M. and Robinson, R. (1953) *Undecidable Theories*. Amsterdam: North-Holland.