Lindström's Theorem

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Abstract

Lindström's Theorem is an important fact about first order logic and cannot be overlooked by anybody interested in the question, why is first order logic so useful and so widely used. Are there some deeper reasons for this or is it just a coincidence. I give an overview of Lindström's theorem and a sketch of its proof in modern notation.

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Lindström's Theorem is a model-theoretic characterization of first order logic. It says in effect that any formal language that goes beyond first order logic has to distinguish between some infinite cardinalities in the sense that some sentence has a model of some infinite cardinality but not of all infinite cardinalities. Loosely speaking Lindström's Theorem tells us that any proper extension of first order logic has to detect something non-trivial about the set-theoretic universe. The equivalent original formulation says that first

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order logic is a maximal logic which satisfies the Downward Löwenheim-Skolem Property and the Countable Compactness Property. I will give an introduction to this result. The background and history of the result is told by Lindström himself in [13].

When we characterize first order logic we have to fix the domain where it is characterized. To this end we introduce the concept of abstract logic. We refer the reader to [2, Chapter II] for complete definitions. What we present here is merely a sketch. This concept was introduced by Lindström [9] and Mostowski [15] independently of each other. The concept was further developed by Barwise [1].

As a first approximation, an *abstract logic* is a pair $L = (S, T)$, where S is a set and T is a relation between arbitrary structures and elements of the set S. Intuitively, S is the set of sentences of the abstract logic L and T is the truth predicate. If τ is a vocabulary, let $\mathrm{Str}(\tau)$ denote the class of structures of vocabulary τ. Obviously we make some assumptions about S and T. They are listed below after we introduce some new concepts:

For $\varphi \in S$ we write $\mathrm{Mod}_{L,\tau}(\varphi) = \{\mathfrak{M} \in \mathrm{Str}(\tau) : T(\mathfrak{M}, \varphi)\}$. An abstract logic L is said to be *closed under negation*, if for all vocabularies τ and all $\varphi \in S$ there is $\neg\varphi \in S$ such that $\mathrm{Mod}_{L,\tau}(\neg\varphi) = \mathrm{Str}(\tau) \setminus \mathrm{Mod}_{L,\tau}(\varphi)$. We say L is *closed under conjunction* if for all vocabularies τ and all $\varphi, \psi \in S$ there is $\varphi \wedge \psi \in S$ such that $\mathrm{Mod}_{L,\tau}(\varphi \wedge \psi) = \mathrm{Mod}_{L,\tau}(\varphi) \cap \mathrm{Mod}_{L,\tau}(\psi)$. We say L is *closed under existential quantification*, if for all vocabularies τ, for all constant symbols c in τ and for all $\varphi \in S$, there is $\varphi' \in S$ such that:

$$\mathrm{Mod}_{L,\tau\setminus\{c\}}(\varphi') = \{\mathfrak{M} : (\mathfrak{M}, c^{\mathfrak{M}}) \in \mathrm{Mod}_{L,\tau}(\varphi) \text{ for some } c^{\mathfrak{M}} \in M\}.$$

We say that L is *closed under renaming* if whenever $\pi : \tau \to \tau'$ is a permutation which respects arity, and we extend π in a canonical way to $\hat{\pi} : \mathrm{Str}(\tau) \to \mathrm{Str}(\tau')$, then for all and $\varphi \in S$, there is $\varphi' \in S$ such that $\{\hat{\pi}(\mathfrak{M}) : \mathfrak{M} \in \mathrm{Mod}_{L,\tau}(\varphi)\} = \mathrm{Mod}_{L,\tau'}(\varphi')$. We say that L is *closed under free expansions* if whenever $\tau \subseteq \tau'$ and $\varphi \in S$, there is $\varphi' \in S$ such that $\mathrm{Mod}_{L,\tau}(\varphi) = \mathrm{Mod}_{L,\tau'}(\varphi')$. Finally, we say that L is *closed under isomorphisms*, if whenever $\varphi \in S$, $\mathfrak{M} \in \mathrm{Mod}_{L,\tau}(\varphi)$ and $f : \mathfrak{M} \cong \mathfrak{N}$, then also $\mathfrak{N} \in \mathrm{Mod}_{L,\tau}(\varphi)$.

We get an abstract logic $L_{\omega\omega} = (S_0, T_0)$ satisfying the above closure properties by letting S_0 be the set of all first order sentences and T_0 the usual truth predicate of first order logic:

$$T_0(\mathfrak{M}, \varphi) \iff \mathfrak{M} \models \varphi.$$

For example, the closure under free expansions can be satisfied simply by choosing $\varphi' = \varphi$. Other abstract logics arise from infinitary languages, generalized quantifiers, higher order logic and combinations of such.

An abstract logic $L = (S,T)$ is a *sublogic* of another abstract logic $L' = (S',T')$, in symbols $L \leq L'$, if for all $\varphi \in S$ there is $\varphi' \in S'$ such that for all τ $\mathrm{Mod}_{L,\tau}(\varphi) = \mathrm{Mod}_{L',\tau}(\varphi')$. If $L \leq L'$ and $L' \leq L$, we say that L and L' are *equivalent*, $L \equiv L'$.

Now we are ready to give the real definition:

Definition *An* abstract logic[1] *is a pair $L = (S,T)$, where S is a set and T is a relation between structures and elements of S, such that L is closed under isomorphisms, renaming, free expansions, negation, conjunction, and existential quantification.*

An abstract logic $L = (S,T)$ satisfies the *(Countable) Compactness Property* if for any (countable) $\Sigma \subseteq S$, if $\bigcap\{\mathrm{Mod}_{M,\tau}(\varphi) : \varphi \in \Sigma\} = \emptyset$, then $\bigcap\{\mathrm{Mod}_{M,\tau}(\varphi) : \varphi \in \Sigma_0\} = \emptyset$ for some finite $\Sigma_0 \subseteq \Sigma$. An abstract logic $L = (S,T)$ satisfies the *Downward Löwenheim-Skolem Property* if for every countable τ every non-empty $\mathrm{Mod}_{L,\tau}(\varphi)$, $\varphi \in S$, contains a countable model. Now we are ready to state Lindström's Theorem:

Lindström's Theorem *Suppose L is an abstract logic such that $FO \leq L$. Then the following conditions are equivalent:*

(1) L has the Countable Compactness Property and the Downward Löwenheim-Skolem Property.

(2) $L \equiv L_{\omega\omega}$.

There are several other characterizations of first order logic, all due to Lindström. An abstract logic $L = (S,T)$ satisfies the *Upward Löwenheim-Skolem Property* if every $\mathrm{Mod}_{L,\tau}(\varphi)$, $\varphi \in S$, which contains an infinite model, contains an uncountable model. We can replace condition (1) above by

(1)′ L has the Upward and Downward Löwenheim-Skolem Properties[2].

In this form Lindström's characterization of first order logic among all abstract logics extends Mostowski's characterization [14] of $L_{\omega\omega}$ among logics obtained by adding simple unary generalized quantifiers to $L_{\omega\omega}$.

For another characterization, we fix some notation. We use $\mathrm{Th}_L(\mathfrak{M})$ to denote $\{\varphi \in S : T(\mathfrak{M}, \varphi)\}$. If $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathrm{Th}_L((\mathfrak{M}, a)_{a\in M}) = \mathrm{Th}_L((\mathfrak{N}, a)_{a\in N})$,

[1] Lindström called abstract logics *extended first order logics*.

[2] Lindström makes here an additional assumption that he calls "strongness". It suffices that L is closed under substitution of (first order) formulas into atomic formulas.

we write $\mathfrak{M} <_L \mathfrak{N}$. An abstract logic $L = (S, T)$ satisfies the *Tarski Union Property* if $\mathfrak{M}_0 <_L \mathfrak{M}_1 <_L \ldots$ implies $\mathfrak{M}_n <_L \bigcup_n \mathfrak{M}_n$. We can replace condition (1) above by

(1)″ L has the Compactness Property and the Tarski Union Property [10].

We can also replace (1) by a condition derived from the Omitting-Types Theorem of first order logic [12]. By assuming a little more effectiveness about the abstract logics, (1) can be replaced by a combination of the Downward Löwenheim-Skolem Property and either a property derived from the Beth Definability Theorem of first order logic [10], extending a result of Mostowski [15], or a property derived from the Completeness Theorem of first order logic [10]. Lindström has himself written a very readable survey of his results in [11].

Barwise [1] characterized $L_{\kappa\omega}$ for $\kappa = \beth_\kappa$. Lindström's Theorem was rediscovered later by H. Friedman.

Proof of Lindström's Theorem[3]

Suppose there were an abstract logic $L = (S, T)$ that satisfies both the Downward Löwenheim Property and the Countable Compactness Property, but some $\varphi \in S$ is not first order definable, i.e. $\mathrm{Mod}_{L,\tau}(\varphi)$ is not of the form $\mathrm{Mod}_{L_{\omega\omega},\tau}(\psi)$ for any first order ψ. We assume w.l.o.g. that τ is finite and relational.

For every n there are only finitely many (logically non-equivalent) first order sentences $\psi_i^n, i = 1, \ldots, k_n$, of vocabulary τ and of quantifier rank at most n. Let us call two L-structures n-equivalent if they satisfy the same ψ_i^n. There are only $\leq 2^{k_n}$ different n-equivalence classes, and each class is first order definable. Since φ is not definable in first order logic, we can find for any n L-structures $\mathfrak{M}_n$ and $\mathfrak{N}_n$ such that:

$$\begin{array}{l} T(\mathfrak{M}_n, \varphi) \\ T(\mathfrak{N}_n, \neg\varphi) \\ \mathfrak{M}_n \text{ and } \mathfrak{N}_n \text{ are } n\text{-equivalent.} \end{array} \tag{1}$$

Lindström uses then a characterization of n-equivalence in terms of back-and-forth sequences. Ehrenfeucht [5] and Fraisse [6] showed that two models are n-equivalent if and only if there are relations $I_i, i < n$, such that

[3] This proof is from [9]. Lindström proved a very similar result for logics with generalized quantifiers already in [8].

- If $(a_1, ..., a_i)I_i(b_1, ..., b_i)$, then $a_1, ..., a_i \in M$ and $b_1, ..., b_i \in N$
- $()I_0()$
- If $(a_1, ..., a_i)I_i(b_1, ..., b_i)$ then for all $a_{i+1} \in M$ $(b_{i+1} \in N)$ there is $b_{i+1} \in N$ $(a_{i+1} \in M)$ such that $(a_1, ..., a_{i+1})I_{i+1}(b_1, ..., b_{i+1})$.
- If $(a_1, ..., a_{i-1})I_i(b_1, ..., b_{i-1})$, then for all atomic formulas $\varphi(v_1, ..., v_{i-1})$ we have $\mathfrak{M} \models \varphi(a_1, ..., a_{i-1})$ if and only if $\mathfrak{N} \models \varphi(b_1, ..., b_{i-1})$.

If there are such relations $I_i, i < \omega$, then we say that that $\mathfrak{M}$ and $\mathfrak{N}$ are ω-equivalent. Note:

$$\text{If } \mathfrak{M} \text{ and } \mathfrak{N} \text{ are countable and } \omega\text{-equivalent, then } \mathfrak{M} \cong \mathfrak{N}, \qquad (2)$$

as we can go "back-and-forth" between the countable models generating infinite sequences $(a_1, ..., a_i, ...)$ and $(b_1, ..., b_i, ...)$ such that for all i we have $(a_1, ..., a_i)I_i(b_1, ..., b_i)$, and moreover, $M = \{a_i : i < \omega\}$ and $N = \{b_i : i < \omega\}$.

Lindström writes (1), supplemented with a little bit of arithmetic, into a sentence $\psi(n)$ in S, using the above back-and-forth characterization of n-equivalence. By the Countable Compactness Property there is a model of $\psi(n)$ in which n is non-standard. Due to the coding used, this model yields two other models $\mathfrak{M}$ and $\mathfrak{N}$ such that $T(\mathfrak{M}, \varphi)$, $T(\mathfrak{N}, \neg\varphi)$ and $\mathfrak{M}$ and $\mathfrak{N}$ are ω-equivalent. By the Downward Löwenheim-Skolem Property, we may assume $\mathfrak{M}$ and $\mathfrak{N}$ are countable. But then they are isomorphic by (2). Thus L cannot be closed under isomorphisms, contrary to assumption. This ends the proof. □

The setup of Lindström's Theorem can be modified in many interesting ways. We can consider monadic structures [17], Banach spaces [7], topological structures [18], or modal logic [3] and prove similar characterizations. Also, the assumption $FO \leq L$ can be relaxed [4]. It is still open whether the Compactness Property and the Craig Interpolation Theorem together characterize first order logic. However, there is an extension of $L_{\omega\omega}$ with the Compactness Property that satisfies the Beth Definability Theorem [16].

An overview of model theoretic properties of various extensions of first order logic can be found in [2].

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