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Pseudo-finite model theory *

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Abstract

We consider the restriction of first–order logic to models, called pseudo–finite, with the property that every first–order sentence true in the model is true in a finite model. We argue that this is a good framework for studying first-order logic on finite structures. We prove a Lindström Theorem for extensions of first order logic on pseudo-finite structures.

1 Introduction

We introduce an "infinitary" approach to finite model theory, based on what we call pseudo–finite structures. We then review results and methods in ordinary model theory and adapt them to this new context. In this approach we do not restrict ourselves to finite structures but allow pseudo–finite structures, i.e. structures with the property that any true first order sentence has a finite model. Equivalently, the pseudo–finite structures are those structures that are elementarily equivalent with ultraproducts of finite structures. Thus a first order sentence is valid in finite models if and only it is valid in pseudo–finite models, i.e. allowing infinite pseudo–finite structures does not take us too far afield from finite model theory.

The main result of the paper is in the last section where Lindström's Theorem, a corner-stone of elementary first order model theory, is proved. A model theoretic characterization of first-order logic or any other logic on finite models is a well-known open problem. Therefore it is interesting to see that in the context of pseudo-finite models the effectiveness of standard model-theoretic methods extends even to this result.

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First order model theory on infinite structures is a well–developed field of abstract mathematics with powerful methods and results. In contrast, finite model theory has rather few general methods and also rather few general results. We propose an approach where we allow certain infinite models but not all. Some infinite models have infinite cardinality but no first-order properties which depend on this infinity. We distinguish these infinite models from those which actually satisfy some first-order axioms which do not have finite models at all. In the latter case the infinite cardinality of the model is not an accident but a necessity. We call a model **pseudo-finite** if every first-order sentence true in the model is also true in a finite model. We propose the study of first-order model theory restricted to pseudo-finite models as an alternative approach to finite model theory.

Let us think of the following problem: We have two first-order sentences ϕ and ψ and we want to know if ψ is true in all finite models of ϕ. If $\phi \to \psi$ is provable in predicate logic, ψ is true in all models of ϕ, finite or infinite, so we have solved our problem. But suppose we suspect that $\phi \to \psi$ is *not* provable in predicate logic. We may try to demonstrate this by exhibiting a model of ϕ where ψ is false. Such a model-construction may require elaborate techniques as we know from model theory and axiomatic set theory. We argue below that a substantial part of the machinery of infinite model theory is available in pseudo-finite model theory. If we succeed in using these methods to construct a pseudo-finite model of $\phi \wedge \neg\psi$, we have solved the problem: The sentence ψ cannot be true in all finite models of ϕ.

Pseudo-finiteness can be defined in several equivalent ways. The Lemma below exhibits three. If L is a vocabulary, we use Γ_L to denote the first-order theory of all finite L-structures.

Lemma 1 *Suppose L is a vocabulary and $\mathbf{A}$ is an L–structure. Then the following conditions are equivalent*

(1) $\mathbf{A} \models \Gamma_L$.

(2) Every first–order L–sentence which is true in $\mathbf{A}$ is true in a finite model.

(3) There is a set $\{\mathbf{A}_i : i \in I\}$ of finite L–structures and an ultrafilter F on I such that

$$\mathbf{A} \equiv \prod_{i \in I} \mathbf{A}_i/_F.$$

Proof. (2) $\to$ (3): Let T be the set of first–order L–sentences true in $\mathbf{A}$. Let I be the set of finite subsets of T. For $i = \{\varphi_1, \ldots, \varphi_n\} \in I$, let $\mathbf{A}_i$ be a finite L–structure such that $\mathbf{A}_i \models \varphi_1 \wedge \ldots \wedge \varphi_n$, and then X_i the set of $j \in I$ for which $\mathbf{A}_j \models \varphi_1 \wedge \ldots \wedge \varphi_n$. The family $\{X_i : i \in I\}$ can be extended to an ultrafilter F on I. If now $\mathbf{A} \models \varphi$, then $\{j \in I : \mathbf{A}_j \models \varphi\}$ contains X_i for $i = \{\varphi\}$, hence $\prod_{i \in I} \mathbf{A}_i/_F \models \varphi$. The other directions are trivial. $\square$

We call models isomorphic to $\prod_{i \in I} \mathbf{A}_i/_F$ for some set I, finite $\mathbf{A}_i$ and some ultrafilter F, **UP-finite**. By the Keisler–Shelah Isomorphism Theorem [11, 15] a model $\mathbf{A}$ is pseudo-finite if and only if it has an ultrapower which is UP-finite. This gives a purely algebraic definition of pseudo-finiteness. The random graph (see e.g. [6]) is a well-known example of a pseudo-finite model. More generally, suppose μ_n is a uniform probability measure on the set $S_n(L)$ of L–structures with universe $\{1, \ldots, n\}$ and let

$$\mu(\varphi) = \lim_{n \to \infty} \mu_n(\{\mathbf{A} \in S_n(L) : \mathbf{A} \models \varphi\})$$

for any L–sentence φ. Then if $\mathbf{A}$ is a model of the *almost sure theory* of μ

$$\{\varphi : \varphi \text{ first order } L\text{–sentence and } \mu(\varphi) = 1\},$$

then $\mathbf{A}$ is pseudo-finite. In this case the pseudo-finite structures actually satisfy the stronger condition: true sentences are true in *almost all* finite models. There is a deep result of model theory which can be utilized to get pseudo-finite models: Cherlin, Harrington and Lachlan [3] proved that all models of totally categorical (κ–categorical for each κ) theories are pseudo-finite. For a stability theoretic analysis of pseudo-finite structures, see [10] and [4].

We use common model theoretic notation, as e.g. in [9]. When we talk about finite structures we do not assume that the vocabulary is necessarily finite.

2 Examples

Pseudo-finite models can be and have been used to prove results about finite structures. Gurevich [7] proved by means of Ehrenfeucht–Fraisse games that **even cardinality** is not expressible in first–order logic on ordered structures. One can alternatively use UP-finite models to prove this result. Let $\mathbf{A}_n$ be the linear order $\langle \{1, \ldots, n\}, < \rangle$. It is easy to see that if

$$\mathbf{A} = \prod_{n<\omega} \mathbf{A}_{2n}/_F, \quad \mathbf{B} = \prod_{n<\omega} \mathbf{A}_{2n+1}/_F,$$

where F is a non–principal ultrafilter on ω, then $\mathbf{A} \equiv \mathbf{B}$ as both $\mathbf{A}$ and $\mathbf{B}$ have order–type of the form

$$\omega + \sum_{i \in L} \mathbf{Z} + \omega^*,$$

where L is an $\aleph_1$-saturated dense linear order without endpoints. If even cardinality were expressible by a first–order sentence φ on finite ordered structures, then we would have $\mathbf{A} \models \varphi$ and $\mathbf{B} \models \neg\varphi$ contradicting $\mathbf{A} \equiv \mathbf{B}$. It is easy to see that a linear order is pseudo-finite if and only if its type is finite or $\omega + \sum_{\alpha \in L} \mathbf{Z} + \omega^*$, where L is a linear order.

Hajek [8] proved that **connectedness** is not first–order definable on finite graphs. It was observed by Turán [18] that this can be proved using UP-finite models as follows: Let $\mathbf{A}_n$ be the cycle of $2n$ vertices and $\mathbf{B}_n$ the union of two cycles of n vertices. Let $\mathbf{A} = \prod_{n<\omega} \mathbf{A}_n/_F$ and $\mathbf{B} = \prod_{n<\omega} \mathbf{B}_n/_F$, where F is a non–principal ultrafilter on ω. It is easy to see that $\mathbf{A} \cong \mathbf{B}$ as both graphs are the union of continuum many infinite chains of type $\mathbf{Z}$. If there were a first–order sentence φ expressing connectedness, then $\mathbf{A} \models \varphi$ and $\mathbf{B} \models \neg\varphi$ contradicting $\mathbf{A} \cong \mathbf{B}$.

Fagin [5] proved the stronger result that connectedness is not even monadic Σ^1_1 on finite graphs. Also this result can be proved using UP-finite models. For this, suppose $\exists P_1 \ldots \exists P_k \varphi$ is a monadic Σ^1_1 sentence. Suppose C_n is a cycle of length $2^{(k+2)n}$. Each C_n is connected, so we have predicates $P_1, \ldots, P_k$ on C_n making φ true. We can think of these predicates as components of a coloring of the vertices of C_n. Let a_n and b_n be vertices of C_n so that the neighborhoods of a_n and b_n of width n have the same color–pattern. Let C'_n be the result of diverting the edge connecting a_n to its successor to an edge from a_n to the successor of b_n, and the edge connecting b_n to its successor to an edge from b_n to the successor of a_n. Let $\mathbf{A}_n$ be the graph C_n and $\mathbf{B}_n$ the graph C'_n, both endowed with the above coloring. Let $\mathbf{A} = \prod_{n<\omega} \mathbf{A}_n/_F$ and $\mathbf{B} = \prod_{n<\omega} \mathbf{B}_n/_F$, where F is a non–principal ultrafilter on ω. Then $\mathbf{A} \cong \mathbf{B}$, whence $\mathbf{B} \models \varphi$. Let $n < \omega$, with $\mathbf{B}_n \models \varphi$. We get a contradiction since C'_n is disconnected.

Koponen and Luosto [12] show that neither **simplicity nor nilpotency** is first–order definable on finite groups. They accomplish this by using pseudo-finite groups. For simplicity they prove that

$$\prod_{p\in P} \mathbf{Z}_p/_F \cong \prod_{p\in P} (\mathbf{Z}_p + \mathbf{Z}_p)/_F$$

where P is the set of primes and F is a non–principal ultrafilter on P. For nilpotency they use

$$\prod_{i<\omega} \mathbf{Z}_{2^i}/_F \cong \prod_{i<\omega} \mathbf{Z}_{2^i p(i)}/_F.$$

Starting with [1] algebraists and model theorists have studied **pseudo-finite fields**.

Familiar model constructions may not yield a pseudo-finite structure. For example, if $\mathbf{A}$ is pseudo-finite and P is a subset of A, there is no guarantee that the expansion $(\mathbf{A}, P)$ is pseudo-finite. However, we may always add constants. Reducts of pseudo-finite structures are pseudo-finite. Ultraproducts of pseudo-finite structures are pseudo-finite, by the Łoś Lemma. Naturally, any model elementarily equivalent to a pseudo-finite model is itself pseudo-finite.

UP-finite structures constitute an important special case of pseudo-finiteness. Note that ultraproducts of UP-finite structures are again UP-finite:

$$\prod_{i\in I} (\prod_{j\in J_i} \mathbf{A}_{i,j}/_{G_i})/_F \cong \prod_{(i,j)\in K} \mathbf{A}_{i,j}/_H,$$

where

$$K = \{(i,j) : i \in I, j \in I_j\}$$

and

$$X \in H \iff \{i : \{j \in J_i : (i,j) \in X\} \in G_i\} \in F.$$

The main difference between pseudo-finite and UP-finite structures is that the former may omit types while the latter are always $\aleph_1$–saturated (if infinite). Also, the former may be countably infinite, while the latter cannot: Shelah [14] showed that the cardinality of an infinite UP-finite $\prod_{i \in I} \mathbf{A}_i/_F$ is always $|I|^\omega$. If we assume the Continuum Hypothesis (CH), the UP-finite models of cardinality $\aleph_1$ become (by $\aleph_1$–saturation) categorical, i.e., CH implies

$$\mathbf{A} \equiv \mathbf{B} \Leftrightarrow \mathbf{A} \cong \mathbf{B},$$

if $\mathbf{A}$ and $\mathbf{B}$ are UP-finite and of size $\leq \aleph_1$. Shelah [16] constructs a model of set theory with a non-categorical UP-finite model of cardinality 2^ω.

3 Model theory of pseudo-finite structures

Model theory of pseudo-finite structures can be studied in many respects in the same manner as the model theory of arbitrary structures. The pseudo-finite models of a first-order theory T of a vocabulary L are exactly the models of the theory $T \cup \Gamma_L$. So we have the following properties: ($\models_{FIN} \phi$ means ϕ is true in all finite finite models. Equivalently, ϕ is true in all pseudo-finite models, i.e., $\Gamma_L \vdash \phi$.)

- **Finite Compactness Theorem** If T is a first–order theory every finite subset of which has a finite model, then T has a model (which w.l.o.g. is pseudo-finite).
- **Finite Löwenheim–Skolem Theorem** If T is a countable first–order theory and T has a pseudo-finite model, then T has a countable model (which w.l.o.g. is pseudo-finite).
- **Finite Omitting Types Theorem** If T is a first-order theory in a countable vocabulary L and p is an L-type, non-principal relative to $T \cup \Gamma_L$, then T has a pseudo-finite model which omits p.
- **Finite Lyndon Preservation Theorem** Suppose T is a first-order theory. A sentence is preserved by homomorphic images on pseudo-finite models of T if and only if it is equivalent in pseudo-finite models of T to a positive sentence.
- **Finite Łoś–Tarski Theorem** Suppose T is a first-order theory. A sentence is preserved by pseudo-finite submodels of pseudo-finite models of T if and only if it is equivalent in pseudo-finite models of T to a universal sentence.

The model theory of pseudo-finite structures is yet to be developed. Some more advanced results can be immediately derived from the infinite version. An example is the pseudo-finite version of **Morley's Theorem**: If a countable first-order theory has only one pseudo-finite model (up to isomorphism) in some uncountable cardinality, then it has only one pseudo-finite model (up to isomorphism) in every uncountable cardinality. However, for pseudo-finite model theory to make sense, it has to be developed systematically. One such systematic account is [4].

4 Interpolation

The usual forms of interpolation all fail in finite models. In their usual formulation they also fail in pseudo-finite models. However, in each case there is a weaker statement which is true. We start with a crucial example:

Example 2 *There are a finite vocabulary L_1, a finite vocabulary L_2, an L_1–sentence ϕ_1, an L_2–sentence ϕ_2, an L_1–model $\mathbf{A}_1$, and an L_2–model $\mathbf{A}_2$ such that $\models_{FIN} \phi_1 \to \phi_2$, $\mathbf{A}_1 \models \phi_1$, $\mathbf{A}_2 \models \neg\phi_2$, $\mathbf{A}_1 \restriction (L_0 \cap L_1) = \mathbf{A}_2 \restriction (L_0 \cap L_1)$ is pseudo-finite, and there is no $L_1 \cap L_2$–sentence θ such that $\models_{FIN} \phi_1 \to \theta$ and $\models_{FIN} \theta \to \phi_2$.*

Proof. Let $L_1 = \{R\}$ and $L_2 = \{P\}$ where R and P are distinct binary predicates. Let ϕ_1 say that R is an equivalence relation with all classes of size 2 and let ϕ_2 say $\neg$(P is an equivalence relation with all classes of size two except one of size one). Then $\models_{FIN} \phi_1 \to \phi_2$. If there were an identity sentence θ such that $\models_{FIN} \varphi \to \theta$ and $\models_{FIN} \theta \to \neg\psi$, then θ would characterize even cardinality in finite models. The models $\mathbf{A}_1$ and $\mathbf{A}_2$ as required are easy to construct. □

We can now immediately see that the usual form of Robinson's Theorem is false: Let T_1 be the theory of $\mathbf{A}_1$ and T_2 the theory of $\mathbf{A}_2$. The theories T_1 and T_2 both have a pseudo-finite model, and all pseudo-finite $L_1 \cap L_2$-models of $T_1 \cap T_2$ are elementary equivalent. Yet $T_1 \cup T_2$ has no pseudo-finite model. However, the following obtains easily from the classical Robinson's Theorem:

> **Finite Robinson Consistency Theorem** Suppose T_i $(i \in \{0,1\})$ is a first–order L_i–theory with a pseudo-finite model. Suppose also that pseudo-finite $L_0 \cap L_1$–models of $T_0 \cap T_1$ are elementary equivalent. Then there are pseudo-finite models $\mathbf{A}_0$ and $\mathbf{A}_1$ such that $\mathbf{A}_0 \models T_0$, $\mathbf{A}_1 \models T_1$ and $\mathbf{A}_0 \restriction (L_0 \cap L_1) = \mathbf{A}_1 \restriction (L_0 \cap L_1)$.

Example 2 shows that the usual formulation of Craig's Interpolation Theorem does not hold on pseudo-finite models. However, the following formulation follows from the classical Craig Interpolation Theorem:

> **Finite Craig Interpolation Theorem** Suppose L_0 and L_1 are vocabularies, φ_i is an L_i–sentence $(i \in \{0,1\})$ and for all pseudo-finite $L_0 \cap L_1$–models $\mathbf{A}$ and all pseudo-finite expansions $\mathbf{A}_i$ of $\mathbf{A}$ to an L_i–model satisfy
>
> $$\mathbf{A}_0 \models \varphi_0 \Rightarrow \mathbf{A}_1 \models \varphi_1.$$

Then there is an $L_0 \cap L_1$–sentence θ such that

$$\models_{FIN} \varphi_0 \to \theta \ \text{ and } \ \models_{FIN} \theta \to \varphi_1.$$

To see more clearly how this follows, we may write it in an equivalent form: Suppose L_0 and L_1 are vocabularies, φ_i is an L_i–sentence ($i \in \{0,1\}$) and

$$\Gamma_{L_0} \cup \Gamma_{L_1} \models \varphi_0 \to \varphi_1.$$

Then there is an $L_0 \cap L_1$–sentence θ such that $\models_{FIN} \varphi_0 \to \theta$ and $\models_{FIN} \theta \to \varphi_1$. Example 2 shows that the assumption $\Gamma_{L_0 \cup L_1} \models \varphi_0 \to \varphi_1$ i.e. $\models_{FIN} \varphi_0 \to \varphi_1$ would not be enough (as is well-known). The situation is the same with Beth Definability Theorem. The above Finite Craig Interpolation Theorem gives:

> **Finite Beth Definability Theorem** Suppose L is a vocabulary, P a new predicate symbol, and $\varphi(P)$ a first-order $L \cup \{P\}$–sentence such that for all pseudo-finite $L \cup \{P\}$–models $\mathbf{A}$ and $\mathbf{A}'$ of $\varphi(P)$ we have
>
> $$\mathbf{A} \restriction L = \mathbf{A}' \restriction L \Rightarrow P^{\mathbf{A}} = P^{\mathbf{A}'}.$$
>
> Then there is an L–formula $\theta(\vec{x})$ such that
>
> $$\varphi(P) \models_{FIN} \forall \vec{x}(\theta(\vec{x}) \leftrightarrow P(\vec{x})).$$

Equivalently: If

$$\Gamma_{L \cup \{P\}} \cup \Gamma_{L \cup \{P'\}} \cup \{\varphi(P), \varphi(P')\} \vdash P = P',$$

then there is an L–formula $\theta(\vec{x})$ such that $\varphi(P) \models_{FIN} \forall \vec{x}(\theta(\vec{x}) \leftrightarrow P(\vec{x}))$. Again, it is easy to see that the weaker assumption

$$\Gamma_{L \cup \{P, P'\}} \cup \{\varphi(P), \varphi(P')\} \vdash P = P'$$

would not be enough, indeed, it is well-known that $\phi(P) \wedge \phi(P') \models_{FIN} P = P'$ is not enough.

Let us call K a **pseudo-finite PC–class** if there is a vocabulary L and a first–order φ such that

$$K = \{\mathbf{A} \restriction L : \mathbf{A} \text{ pseudo-finite and } \mathbf{A} \models \varphi\},$$

and a **pseudo-finite Σ^1_1-class** if there is a vocabulary L and a first–order φ such that

$$K = \{\mathbf{A} : \mathbf{A} \text{ pseudo-finite and } \exists \mathbf{A}'(\mathbf{A} = \mathbf{A}' \restriction L, \mathbf{A}' \models \varphi)\}.$$

Note the difference between pseudo-finite PC and pseudo-finite Σ^1_1: the latter quantifies over more relations than the former. Every non-empty pseudo-finite PC-class contains a finite model, but a pseudo-finite Σ^1_1-class may contain only infinite models. Hence PC and Σ^1_1 are not the same concept on pseudo-finite models, but they coincide both on finite models and on all models. On finite models they coincide with NP-describability.

A consequence of the Finite Craig Interpolation Theorem is the **Finite Souslin–Kleene Interpolation Theorem for PC**: If K is pseudo-finite PC and the complement $\{\mathbf{A} \notin K : \mathbf{A} \text{ pseudo-finite }\}$ is pseudo-finite PC, too, then K is first–order definable of pseudo-finite models. Thus first–order logic is $\triangle$–*closed* on pseudo-finite structures. To get a similar result for Σ^1_1 we first modify the Interpolation Theorem:

> **Alternative Finite Craig Interpolation Theorem** Suppose L_0 and L_1 are vocabularies, φ_i is an L_i–sentence ($i \in \{0,1\}$) and for all pseudo-finite $L_0 \cap L_1$–models $\mathbf{A}$ and all expansions $\mathbf{A}_i$ of $\mathbf{A}$ to an L_i–model
>
> $$\mathbf{A}_0 \models \varphi_0 \Rightarrow \mathbf{A}_1 \models \varphi_1.$$
>
> Then there is an $L_0 \cap L_1$–sentence θ such that for all pseudo-finite $L_0 \cap L_1$–models $\mathbf{A}$ and all expansions $\mathbf{A}_i$ of $\mathbf{A}$ to an L_i–model
>
> $$\mathbf{A}_0 \models \varphi_0 \rightarrow \theta \quad \text{and} \quad \mathbf{A}_1 \models \theta \rightarrow \varphi_1.$$

The **Finite Souslin–Kleene Interpolation for** Σ^1_1 now follows: If K is pseudo-finite Σ^1_1 and $\{\mathbf{A} \notin K : \mathbf{A} \text{ pseudo-finite }\}$ is pseudo-finite Σ^1_1, then K is first–order definable on pseudo-finite structures. On finite models the Souslin-Kleene Theorem for PC (and Σ^1_1) is false, but the analogous question, whether NP∩co-NP=PTIME is a famous open problem. On all models the Souslin-Kleene Theorem for PC (and Σ^1_1) holds.

5 A Lindström Theorem

On all structures first order logic has a nice characterization, the Lindström Theorem [13]: first-order logic is the maximal logic with the Compactness and the Löwenheim-Skolem Theorems. Although first order logic seems very robust on finite structures, too, no analogous characterization in terms of model theoretic properties is known. Indeed, no model theoretic properties comparable to Compactness and the Löwenheim-Skolem Theorems are known for first order logic on finite structures. As we have seen above, this is not so in pseudo-finite structures. In this section we investigate to what extent the Lindström Theorem holds on pseudo-finite structures.

For basic definitions concerning abstract logics we refer to [2]. We denote first order logic by $L_{\omega\omega}$. The most commonly studied extensions of first-order logic on finite structures are the **fixed point logic** FP and its extension $\mathcal{L}^{\omega}_{\infty\omega}$. These logics make perfect sense on pseudo-finite structures, too. However, pseudo-finite structures need not have the finite model property relative to these logics. Let us call a model $\mathbf{A}$ L^***-pseudo-finite**, if every L^*-sentence true in $\mathbf{A}$ is true in a finite structure. We may, and it would make sense, restrict to abstract logics L^* where satisfaction is defined for L^*-pseudo-finite structures. The pseudo-finite linear order $\omega + \mathbf{Z} + \omega^*$ satisfies a sentence of FP with no finite models, so it is not FP-pseudo-finite. On the other hand, the random graph is a pseudo-finite structure which is even $\mathcal{L}^{\omega}_{\infty\omega}$-pseudo-finite. An extreme case is the logic L^* with the quantifier "There exists infinitely many". Here only finite models are L^*-pseudo-finite.

If L^* and L^{**} are abstract logics, then $L^* \leq L^{**}$ **on finite models** means that for every $\phi \in L^*$ there is $\phi^* \in L^{**}$ such that

$$\mathbf{A} \models \phi \iff \mathbf{A} \models \phi^*$$

for all finite $\mathbf{A}$. If $\mathbf{A}$ is a relational structure in a language L and $U \in L$, then $\mathbf{A}^{(U)}$ is the relativization of $\mathbf{A}$ to the set $U^{\mathbf{A}}$. The abstract logic L^* **relativizes** if for all $\phi \in L^*$ and all unary predicates U there is $\phi^{(U)} \in L^*$ such that

$$\mathbf{A} \models \phi^{(U)} \iff \mathbf{A}^{(U^{\mathbf{A}})} \models \phi$$

for all L^*-pseudo-finite models $\mathbf{A}$. We assume our abstract logics all relativize.

An abstract logic L^* satisfies the **Finite Compactness Theorem** if whenever T is a theory in L^*, every finite subset of which has a finite model, then T has a model. W.l.o.g. we may take this model to be L^*-pseudo-finite. L^* satisfies the **Finite Löwenheim–Skolem Theorem** if whenever T is a countable L^*-theory and T has an L^*-pseudo-finite model, then T has a countable model. W.l.o.g. we may take this model to be L^*-pseudo-finite.

First-order logic satisfies both Finite Compactness Theorem and Finite Löwenheim–Skolem Theorem. Any abstract logic which satisfies the Compactness Theorem on all structures, satisfies also, a fortiori, the Finite Compactness Theorem, and some infinite models are L^*-pseudo-finite. Likewise, any abstract logic satisfying the Löwenheim–Skolem Theorem (every countable theory with a model has a countable model) satisfies the Finite Löwenheim–Skolem Theorem. This inference shows that the fixed point logic FP satisfies the Finite Löwenheim–Skolem Theorem. The generalized quantifier

$$Q_{BA}uxy\phi(x, y, \vec{z})$$

which says that $\phi(x, y, \vec{z})$ defines on the set of elements satisfying $\psi(u)$ a Boolean algebra with an even (or infinite) number of atoms, satisfies the Compactness Theorem on all structures, (assuming GCH) [17]. Hence it satisfies the Finite Compactness Theorem, too. The logic $L_{\omega\omega}(Q_{BA})$ is an example of a logic which satisfies the Compactness Theorem and which extends the first-order logic even on finite structures.

Theorem 3 *Suppose L^* is an abstract logic such that $L_{\omega\omega} \leq L^*$ on finite models. If L^* satisfies the Finite Compactness Theorem and the Finite Löwenheim-Skolem Theorem, then $L^* \leq L_{\omega\omega}$ on finite models.*

Proof. The proof is an adaption of the original proof of Lindström's Theorem [13]. Suppose $\theta \in L^*$ is not first order definable on finite models. Because L^*

satisfies the Finite Compatcness Theorem, we may assume θ has finite vocabulary τ (see [2, page 79]). Let

$$\phi_n^{m,k}(x_1, ..., x_m), n < q(m,k)$$

be the list of all (up to equivalence) first order formulas of quantifier rank $\leq k$ with m free variables. Let

$$\phi^k = \bigwedge \{\phi_n^{0,k} : n < q(0,k), \models_{FIN} \theta \to \phi_n^{0,k}\}.$$

Clearly, $\models_{FIN} \theta \to \phi^k$. Thus we may choose a finite $\mathbf{A}_k \models \phi^k \wedge \neg\theta$. Let

$$\psi^k = \bigwedge \{\phi_n^{0,k} : n < q(0,k), \mathbf{A}_k \models \phi_n^{0,k}\}.$$

If $\models_{FIN} \theta \to \neg\psi^k$, then $\models_{FIN} \phi^k \to \neg\psi^k$, contrary to $\mathbf{A}_k \models \phi^k \wedge \psi^k$. Therefore we can choose a finite $\mathbf{B}_k \models \theta \wedge \psi^k$. Thus $\mathbf{A}_k$ and $\mathbf{B}_k$ satisfy the same first order sentences of quantifiers rank $\leq k$, i.e. $\mathbf{A}_k \equiv_k \mathbf{B}_k$.

Let U_0 and U_1 be new unary predicate symbols. By assumption, the relativisation $\theta^{(U_0)}$ of θ to U_0 and the relativisation $(\neg\theta)^{(U_1)}$ of $\neg\theta$ to U_1 are in L^*. Let P be a new binary predicate and F a new ternary predicate. Let T be the L^*-theory consisting of the following sentences:

1. $\theta^{(U_0)}$
2. $(\neg\theta)^{(U_1)}$
3. "P is a linear order"
4. "F codes a P-ranked back-and-forth system[1] in the language L between U_0 and U_1."
5. $\exists x_1 \exists x_0 P(x_1, x_0)$
6. $\forall x_n ... \forall x_0 [(P(x_n, x_{n-1}) \wedge ... \wedge P(x_1, x_0)) \to \exists x_{n+1} P(x_{n+1}, x_n)]$ for all $n < \omega$.
7. $\neg\phi$ for each L^*-sentence in the vocabulary τ.

If $T_0 \subseteq T$ is finite, we can construct a model $\mathbf{C}$ for T_0 from $\mathbf{A}_k$ and $\mathbf{B}_k$ for a sufficiently large k. The model $\mathbf{C}$ is finite, hence satisfies also the sentences in condition 7. By the Finite Compactness Theorem there is a model $\mathbf{C}$ of the whole T. Because of the conditions 7, $\mathbf{C}$ is L^*-pseudo-finite. By the Finite Löwenheim.Skolem Theorem, we can choose $\mathbf{C}$ to be countable. Let $\mathbf{C}_0$ be the restriction of $\mathbf{C}$ to $U_0^{\mathbf{C}}$. Respectively, let $\mathbf{C}_1$ be the restriction of $\mathbf{C}$ to $U_1^{\mathbf{C}}$.

[1]This is as in the original proof by Lindström [13]. A P-ranked back-and-forth system in the language L between U_0 and U_1 is a sequence $I = \langle I_i : i \in P \rangle$ of non-empty sets of partial isomorphimsm relative to the language L. If $p \in I_i$ and $P(j,i)$, then for every $a \in U_0$ there is $b \in U_1$ such that $p \cup \{\langle a, b \rangle\} \in I_j$. Also, for every $b \in U_1$ there is $a \in U_0$ such that $p \cup \{\langle a, b \rangle\} \in I_j$.

Note that $\mathbf{C}_0$ and $\mathbf{C}_1$ are L^*-pseudo-finite. Now $\mathbf{C}_0$ and $\mathbf{C}_1$ are countable partially isomorphic models, hence isomorphic. But $\mathbf{C}_0 \models \theta$ and $\mathbf{C}_1 \models \neg\theta$, a contradiction. □

A corollary of the theorem is: $L^* \leq L_{\omega\omega}$ on pseudo-finite models if and only if L^* satisfies the Finite Compactness Theorem, the Finite Löwenheim-Skolem Theorem, and every pseudo-finite model is L^*-pseudo-finite.

The above theorem can be interpreted as saying that even though first-order logic seems very weak on finite structures, we cannot hope to find logic which is stronger on finite models and which would admit both compactness and existence of countably infinite models in the proximity of finite models. Something has to be given up. One possibility is to give up countably infinite models. Let us say that an abstract logic L^* satisfies the **Ultraproduct Compactness Theorem** if whenever T is a theory in L^*, every finite subset of which has a finite model, then T has a UP-finite model $\mathbf{A}$. W.l.o.g. $\mathbf{A}$ may be taken to be L^*-pseudo-finite. L^* satisfies the **Ultraproduct Löwenheim–Skolem Theorem** if whenever T is an L^*-theory of cardinality $\leq 2^{\aleph_0}$ and T has an L^*-pseudo-finite UP-finite model, then T has a UP-finite model of cardinality $\leq 2^{\aleph_0}$. W.l.o.g. this model can be chosen to be L^*-pseudo-finite. First-order logic satisfies the Ultraproduct Compactness Theorem and the Ultraproduct Löwenheim–Skolem Theorem.

Theorem 4 *Assume the Continuum Hypothesis. Suppose L^* is an abstract logic such that $FO \leq L^*$ on finite models. If L^* satisfies the Ultraproduct Compactness Theorem and the Ultraproduct Löwenheim-Skolem Theorem, then $L^* \leq L_{\omega\omega}$ on finite structures.*

Proof. The proof is like that of Theorem 3. At the end we note that $\mathbf{C}_0$ and $\mathbf{C}_1$ are $\aleph_1$-saturated elementary equivalent models of cardinality $\leq 2^\omega = \aleph_1$, hence isomorphic. □

We can eliminate the use of Continuum Hypothesis in the above theorem, if we assume instead **absoluteness for countably closed forcing**, i.e. truth of L^* is preserved by countably closed forcing. More exactly, if $\phi \in L^*$, $\mathbf{A}$ is a model and $\mathbf{P}$ is countably closed forcing, then $\Vdash_{\mathbf{P}} [\mathbf{A} \models \phi]$.

Theorem 5 *Suppose L^* is an abstract logic such that $L_{\omega\omega} \leq L^*$ on finite structures and L^* is absolute for countably closed forcing. If L^* satisfies the Ultraproduct Compactness Theorem, then $L^* \leq L_{\omega\omega}$ on finite models.*

Proof. The proof is again like that of Theorem 3. At the end we note that $\mathbf{C}_0$ and $\mathbf{C}_1$ are $\aleph_1$-saturated and elementary equivalent. If we collapse canonically their cardinalities to $\aleph_1$, they are still $\aleph_1$-saturated and elementary equivalent, hence isomorphic. This leads to a contradiction with the absoluteness assumption. □

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