Strong Logics

Jouko Väänänen

UH and UvA

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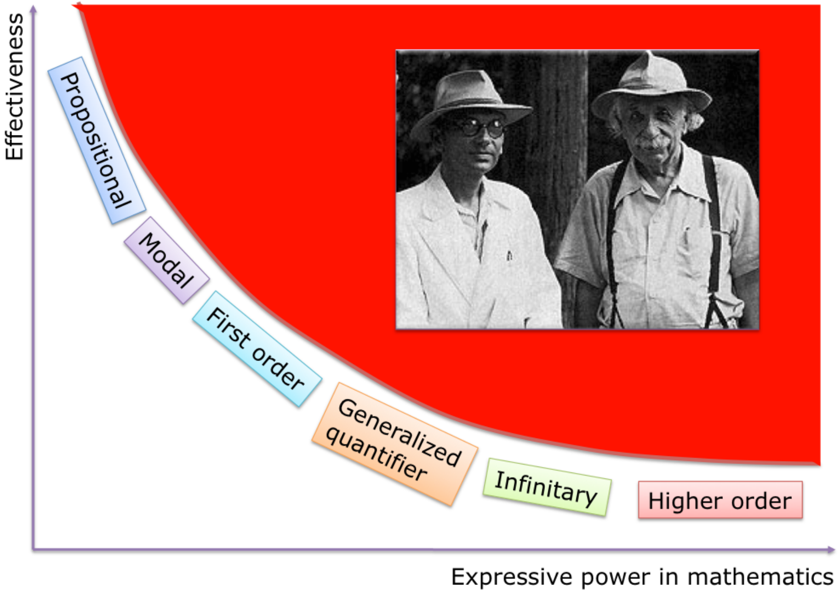
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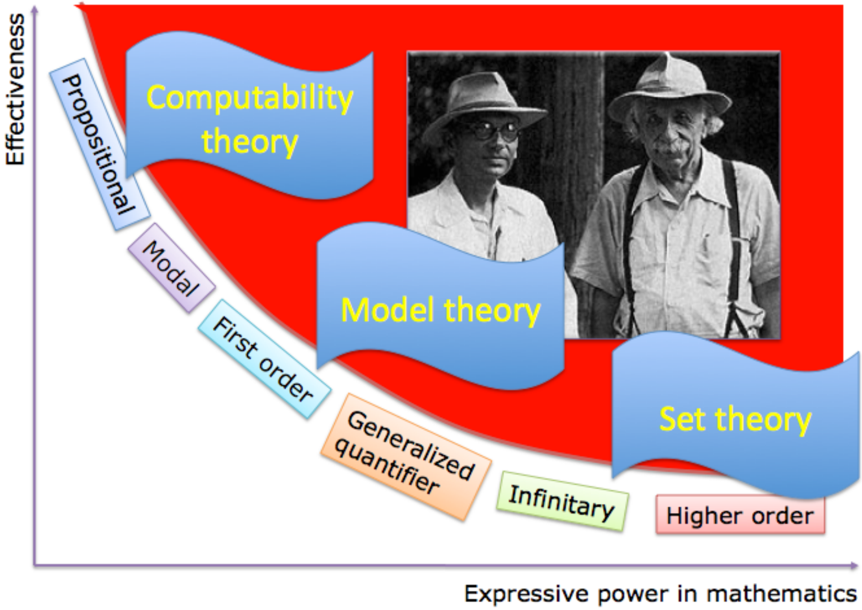
Propositional
Modal
First order
Generalized quantifier
Infinitary
Higher order
Expressive power in mathematics


Effectiveness
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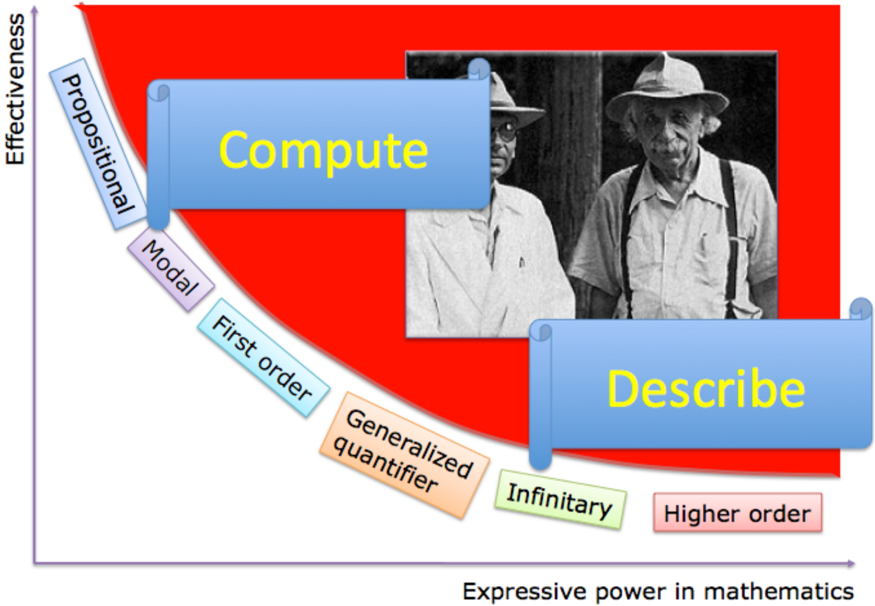
Model theory
Set theory
Expressive power in mathematics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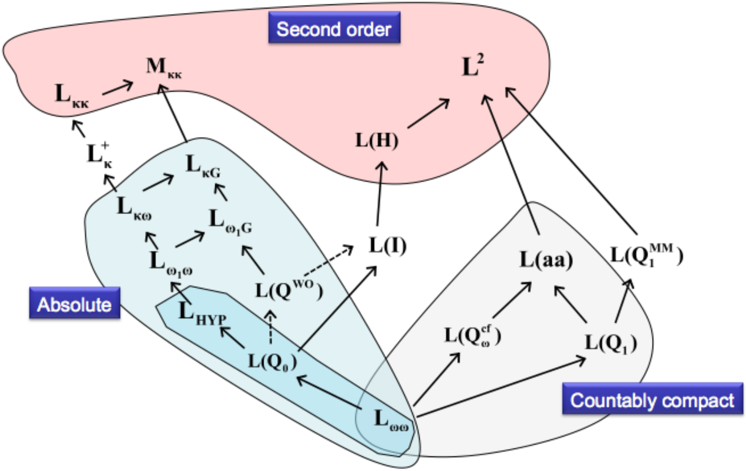
- Infinitely many, uncountably many, equally many, free group, uncountable homogeneous set, well-founded, stationarily many,...
- Some have compactness properties, some have absoluteness properties.

The two trails to higher order



Strong logics

Definitions

- A **logic** is a pair $(L, \models_L)$, where L is a class called the class of L-formulas and $\models_L$ is a relation between models and elements of L, respecting isomorphisms.
- A logic is **strong** if it is as strong as second order logic L^2, at least in some inner models.

Strong logics

Examples

- A logic is **strongly unbounded** if two sentences can be found in the logic such that these sentences both have arbitrarily large models but no models of the same cardinality.

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- $Ixy\phi(x)\psi(y) \iff |\phi(\cdot)| = |\psi(\cdot)|$.

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- $Ixy\phi(x)\psi(y) \iff |\phi(\cdot)| = |\psi(\cdot)|$.
- $Qx\phi(x) \iff |\phi(\cdot)|$ is a limit cardinal.

Questions

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Is it possible that there is a supercompact cardinal and $L(I)$ has the same strength as second order logic?

The answer is yes for $L(I, Q_{\text{exp}})$, where

$$Q_{\text{exp}} xy\phi(x)\psi(y) \iff |\phi(\cdot)| = 2^{|\psi(\cdot)|}.$$

Not a strong (enough) logic

Well-order quantifier:

$$Wxy\phi(x,y) \iff \phi(\cdot,\cdot) \text{ is a well-order.}$$

Game-quantifier:

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \bigwedge_n \phi_n(x_0, y_0, \ldots, x_n, y_n).$$

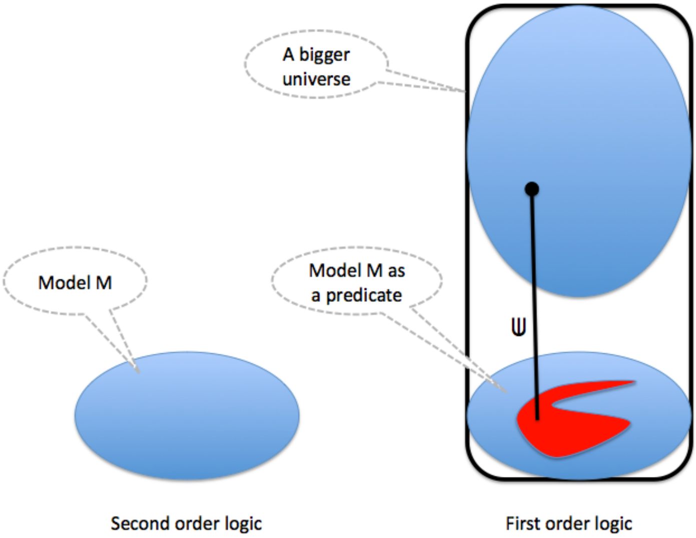
Infinitary languages $L_{\kappa\lambda}$, even when $\lambda > \omega$.

Above second order logic — third order logic?

Fact (Hintikka 1955, Montague 1965)

Higher order logic can be reduced to second order logic.

Basic superstructure $[N; M]$



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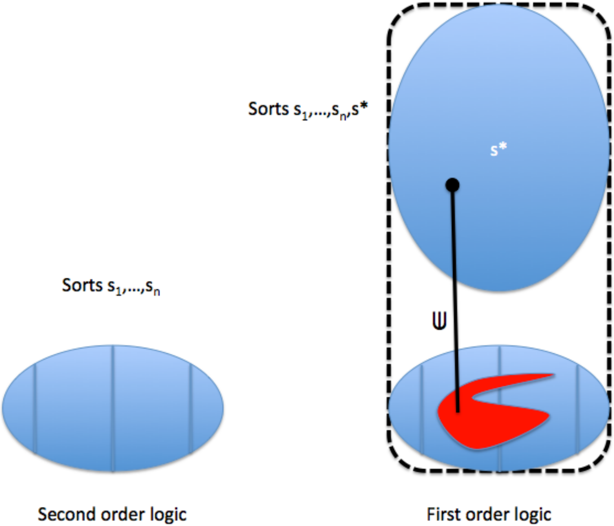
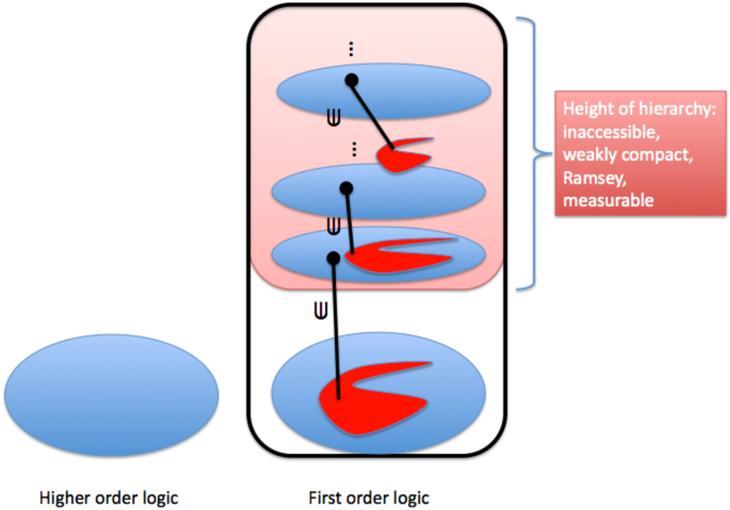


Enhanced basic superstructure $[N; M]$



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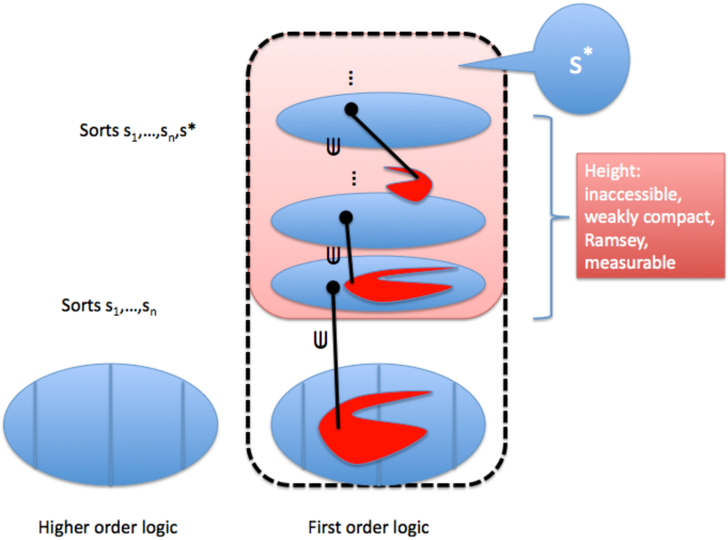
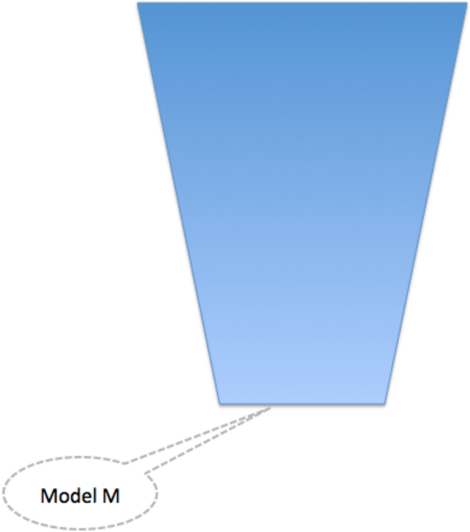


Set theory with urelements



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- Translate higher order ϕ to first order ϕ^*.
- Formulate a (Π^1_1) second order Θ which states the correctness of the superstructure.
- The following are equivalent:
  1. $M \models \phi$
  2. $[N; M] \models \phi^*$ for all $[N; M] \models \Theta$.
  3. $[N; M] \models \phi^*$ for some $[N; M] \models \Theta$.
- We did not define ϕ explicitly in L^2, only implicitly, but this is enough for the invariants that we consider.

Invariants of logics

Definition

- The *decision problem* of a logic L is the set

$$Val(L) = \{\ulcorner\phi\urcorner : \phi \in L,\ \phi \text{ valid}\}.$$

- The *spectrum* of a formula ϕ is the class of cardinalities of models of ϕ. A class is a *spectrum* of L is it is the spectrum of some sentence in L.
- The *Löwenheim-number* of a logic L is

$$\ell(L) = \sup\{\min S : S \text{ is a spectrum of } L \text{ and } S \neq \emptyset\}.$$

- The *Hanf number* of a logic L is

$$h(L) = \sup\{\sup S : S \text{ is a spectrum of } L \text{ and bounded}\}.$$

Fact (Hintikka 1955)

- *$Val(L^2)$ is not Π^m_n for any m, n.*

Fact (Magidor 1971)

- *The first MC $< \ell(L^2) <$ the first succ $< h(L^2) <$ the first extendible.*

A cardinal κ is λ-**strong** if there exists an elementary embedding $j : V \to M$ with critical point κ such that $V_\lambda \subseteq M$. A cardinal κ is **extendible** if for all $\alpha > \kappa$ there exists β and an elementary embedding from $V(\alpha)$ into $V(\beta)$ with critical point κ.

Fact (V. 1977, 1981)

- *$Val(L)$ can be Δ^1_3 (but not less) for strongly unbounded L.*
- *$\ell(L)$ can be $< 2^\omega$ for a strongly unbounded logic L. Can it be less than the first weakly inaccessible?*
- *$h(L)$ can be less than the first weakly compact for a strongly unbounded L.*

Theorem ([V. 2001])

The set $Val(L^2)$ is the complete Π_2-set of natural numbers.

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Proof.

$Val(L^2)$ is Π_2: Suppose $\phi \in L^2$. The following are equivalent:

1. $M \models \phi$
2. $\exists N([N; M] \models \phi^* \wedge \Theta)$

Thus (1) is Σ_2 (in M) and hence $Val(L^2)$ is Π_2.

□

Proof (contd.)

Fact (Levy Reflection Principle)

If $\kappa > \omega$, then $H(\kappa) \prec_1 V$. If $\kappa = \beth_\kappa$, then $V_\kappa \prec_1 V$.

Proof (contd.)

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$Val(L^2)$ is Π_2-hard: Suppose $\exists x \forall y \phi(x, y, n)$ is Σ_2. The following are equivalent:

1. $\exists x \forall y \phi(x, y, n)$
2. There is (M, E) such that $(M, E) \models (\exists x \forall y \phi(x, y, \underline{n}) \wedge \bigwedge ZFC_m)$, and $(M, E) \cong (V_\alpha, \in)$ for some $\alpha = \beth_\alpha$.

Thus (1) can be reduced to model existence in L^2, and hence $Val(L^2)$ is Π_2-hard.

Projections

Definition

Suppose K is a class of L-structures and $L' \subseteq L$. The **projection** of K to L' is the model class

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Examples

- Leaving out order from a class of ordered groups $(G, +, <)$.
- Leaving out field structure from a class of vector spaces.
- Leaving out "superstructure".

PC in a logic

Definition (Tarski)

A model class K is

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- **co-PC** in a logic L, $K \in \Pi(L)$, if K is the complement of a class of projections of a model class definable in L.

Like analytic, or Σ^1_1, sets.

Projective model classes

Definition

The Δ-extension of a logic L is the family $\Delta(L)$ of all model classes that are projective and co-projective in L. We think of $\Delta(L)$ as a logic.

Analogous to Δ^1_1 or Δ^2_1.

Some computations

Examples

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- $Val(L)$, $\ell(L)$ and (in most cases) $h(L)$ are invariant under the Δ-operation. [V. 1983].
- Reduction via superstructure: "All" higher order logics have the same Δ-extension, hence the same decision problem, as well as Löwenheim and Hanf numbers.

Symbiosis of set theory and model theory

Definition

If P is a predicate of set theory, let $K[P]$ be the class of structures isomorphic to $(M, \in, a_1, ..., a_n)$ for some transitive M and some $a_1, ..., a_n \in M$ such that $P(a_1, ..., a_n)$.

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Definition ([V. 1979])

A predicate P of set theory and a logic L are **symbiotic** if

- $K[P]$ is $\Delta(L)$-definable,
- If $\phi \in L$, then $Mod(\phi)$ is $\Delta_1(P)$ in ϕ.

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- $L(I)$ and Cd are symbiotic.
- $L(W)$ and On are symbiotic.

Set theory and logic

Theorem

Suppose a predicate P and a logic L are symbiotic. Then the following conditions are equivalent for any model class K:

1. *K is $\Sigma(L)$-definable (in logic).*
2. *K is $\Sigma_1(P)$-definable (in set theory).*

Levy-hierarchy

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Respectively, Π_n-definable or Δ_n-definable.

A hierarchy of definable ordinals

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Lemma

For $n > 1$: $\pi_n = \delta_n < \tau_n < \sigma_n = \pi_{n+1}$

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Example

For $n > 1$:
1st MC $< \pi_2 < \tau_2 <$ 1st succ $< \sigma_2 = \pi_3 < \tau_3 <$ 1st extendible $< \ldots$

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Fact

$L^2 < \Delta_2 < \Delta_3 < \ldots$

Theorem ([V. 1979])

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Corollary

$h(\Delta_n) = \ell(\Delta_{n+1})$.

Sort logic

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Theorem

Every (definable) model class is definable in L^s. Every abstract logic is a sublogic of L^s.

Sort logic

Fact

In its many-sorted disguise sort logic permits a simple description. One adds to second order logic the quantifier "there exists a sort such that ..."

Stronger invariants

Definition ([Stavi-V. 2002])

The Löwenheim-Skolem-Tarski number $LST(L)$ of a logic L is the least κ such that for any model M there is $N \prec_L M$ of cardinality $\leq \kappa$.

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Definition

The Upward Löwenheim-Skolem-Tarski number $ULST(L)$ of a logic L is the least κ such that for any model M of cardinality $\geq \kappa$ there is N of arbitrary large cardinality $\geq \kappa$ such that $M \prec_L N$.

Fact

- *$LST(L) = ULST(L) = \omega$ characterizes first order logic (Per Lindström).*
- *$LST(L)$ can be $\leq 2^{\omega}$ for a strong logic.*
- *$\ell(L) \leq LST(L)$.*
- *$h(L) \leq ULST(L)$.*

Theorem (Magidor 1971)

- *The first supercompact* $= LST(L^2) < ULST(L^2) \leq$ *the first extendible.*

Recall: The first MC $< \ell(L^2) <$ the first succ $< h(L^2) <$ the first extendible.

Equicardinality quantifier

Theorem ([Magidor-V. 2010])

For the strongly unbounded $L = L(I)$:

- *If $LST(L)$ exists then SCH holds above $LST(L)$ and PD hold. In fact $\square_\lambda$ fails for all $\lambda \geq LST(L)$, and moreover $\square_{\lambda,\lambda}$ fails if $cof(\lambda) = \omega$.*

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- *It is consistent, relative to the consistency of a supercompact cardinal, that $LST(L)$ is the first supercompact cardinal.*
- *It is consistent, relative to the consistency of a supercompact cardinal, that $LST(L)$ is the first weakly inaccessible cardinal (and 2^ω is above it).*

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Theorem ([Magidor-V. 2010])

For the strongly unbounded $L = L(I)$:

- *If $LST(L)$ exists then SCH holds above $LST(L)$ and PD hold. In fact $\square_\lambda$ fails for all $\lambda \geq LST(L)$, and moreover $\square_{\lambda,\lambda}$ fails if $cof(\lambda) = \omega$.*
- *It is consistent, relative to the consistency of a supercompact cardinal, that $LST(L)$ is the first supercompact cardinal.*
- *It is consistent, relative to the consistency of a supercompact cardinal, that $LST(L)$ is the first weakly inaccessible cardinal (and 2^ω is above it).*

Corollary

$\ell(L)$ can be less than the first weakly inaccessible cardinal.

Vopenka's Principle

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- *Vopenka's Principle*: Every proper class of structures of the same vocabulary has two members one of which is isomorphic to an elementary substructure of the other.
- Equivalently: For every class A there is an A-supercompact cardinal.
- A cardinal κ is *A-supercompact* if for all $\eta > \kappa$ there is $\alpha < \kappa$ and an elementary embedding $j : (V_\alpha, \in, A \cap V_\alpha) \to (V_\eta, \in, A \cap V_\eta)$ with a critical point γ such that $j(\gamma) = \kappa$.

The ultimate LST-theorem

Theorem (Stavi, [Magidor-V. 2010])

The following are equivalent:

- *Vopenka's Principle holds.*
- *$LST(L)$ exists for every logic L.*

Summary

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- Large cardinals of set theory correspond to model theoretic properties of various fragments of sort logic (Löwenheim and Hanf numbers).
- Definability ($\Delta_1(P)$) in set theory corresponds to implicit (Δ)-definability in various fragment of sort logic (symbiosis).
- The strongest logic is set theory itself (more exactly, sort logic).

Jouko Väänänen.

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Thank you!