Some proof theoretical results on propositional logics of dependence
— applications of normal forms

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Outline

1. propositional logics of dependence and normal forms
2. interpolation
3. admissible rules and structural completeness
Propositional Dependence Logic

\[ \equiv (\bar{p}, q) \]

- Whether \(-x > 0\) depends completely on whether \(x < 0\) or not.

- Whether you end up in the town depends entirely on whether you turn here left or right.
Well-formed formulas of *propositional dependence logic* (PD) are given by the following grammar:

\[
\phi ::= p \mid \neg p \mid (\vec{p}, q) \mid \phi \land \phi \mid \phi \otimes \phi
\]

*propositional intuitionistic dependence logic* (PID):

\[
\phi ::= p \mid \bot \mid (\vec{p}, q) \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \rightarrow \phi
\]

*inquisitive logic* (InqL):

\[
\phi ::= p \mid \bot \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \rightarrow \phi
\]

\[
(\neg \phi := \phi \rightarrow \bot)
\]
Team Semantics

Let $X$ be a team, i.e., a set of valuations $\nu : \text{Prop} \rightarrow \{0, 1\}$.

- $X \models p$ iff for all $\nu \in X$, $\nu(p) = 1$;
- $X \models \neg p$ iff for all $\nu \in X$, $\nu(p) = 0$;
- $X \models \bot$ iff $X = \emptyset$;
- $X \models = (\bar{p}, q)$ iff for all $\nu, \nu' \in X$: $\nu(\bar{p}) = \nu'(\bar{p}) \implies \nu(q) = \nu'(q)$;
- $X \models \phi \land \psi$ iff $X \models \phi$ and $X \models \psi$;
- $X \models \phi \otimes \psi$ iff there exist $Y, Z$ s.t. $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$;
- $X \models \phi \lor \psi$ iff $X \models \phi$ or $X \models \psi$;
- $X \models \phi \rightarrow \psi$ iff for any team $Y \subseteq X$: $Y \models \phi \implies Y \models \psi$.

A formula $\phi$ is said to be flat if

$$X \models \phi \iff \forall \nu \in X : \{\nu\} \models \phi.$$ 

Example:

- Classical formulas (i.e., formulas without any occurrences of $= (\bar{p}, q)$ and $\lor$) are flat.
- $\neg \phi$ is flat for all $\phi$.  

Theorem (Ciardelli and Roelofsen, 2011)

\textbf{InqL} PID \textit{is complete w.r.t. the following Hilbert style deductive system:}

\textbf{Axioms:}
- all substitution instances of IPC axioms
- all substitution instances of
  \[(\text{Kreisel – Putnam}) \quad (\neg p \rightarrow (q \lor r)) \rightarrow ((\neg p \rightarrow q) \lor (\neg p \rightarrow r)).\]
- \(\neg\neg p \rightarrow p\) for all propositional variables \(p\)
- \(=(p_1, \cdots, p_n, q) \leftrightarrow (\bigwedge_{i=1}^{n}(p_i \lor \neg p_i) \rightarrow (q \lor \neg q))\)

\textbf{Rules:}
- Modus Ponens

\(\text{I.e., InqL} = \text{KP} \oplus \neg\neg p \rightarrow p.\)

Theorem ((Y., Väänänen, 2014), (Sano, Virtema, 2014))

\textbf{PD is sound and complete w.r.t. its deductive systems.}

For any D-formula \(\phi, \psi\), if they are flat and do not contain \(=\text{(\vec{p}, q)}\), then

\[\phi \vdash_{\text{CPC}} \psi \iff \phi \vdash_{\text{D}} \psi.\]
Lemma

For any team \( X \neq \emptyset \) on \( V = \{p_1, \ldots, p_n\} \), there is a formula \( \Theta_X \) of PD, PID and InqL such that for any team \( Y \) on \( V \), \( Y \models \Theta_X \iff Y \subseteq X \).

Proof.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( p )</th>
<th>( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_1 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \nu_2 )</td>
<td>1</td>
<td>0</td>
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<tr>
<td>( \nu_3 )</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Let \( \Theta_X := \bigotimes_{v \in X} (p_1^{v(p_1)} \land \cdots \land p_n^{v(p_n)}) \), for PD;
\( \neg\neg \bigvee_{v \in X} (p_1^{v(p_1)} \land \cdots \land p_n^{v(p_n)}) \), for PID, InqL.

Then \( Y \models \Theta_X \iff Y \subseteq X \), for any team \( Y \) on \( N \).

Corollary

\( \phi \equiv \bigvee_{X \in [[\phi]]} \Theta_X \), where \([\phi] = \{ X \subseteq \{0, 1\}^V \mid X \models \phi \} \), for any consistent formula \( \phi \) of PD, PID and InqL.

- A consistent formula \( \phi \) is flat iff \( \phi \equiv \Theta_X \) for some \( X \). In particular, for PID and InqL, a formula \( \phi \) is flat iff \( \phi \equiv \neg\neg\phi \).
- PD \( \equiv \) PID \( \equiv \) InqL (Ciardelli, Huuskonen, Y.).
Applications of the normal form:
1. Interpolation
Lemma

For any flat formula $\phi$ and any set $\{\psi_j \mid j \in J\}$ of flat formulas,

$$\phi \models \bigvee_{j \in J} \psi_j \implies \phi \models \psi_j \text{ for some } j \in J.$$  

Proof. Easy. For the case of PID or InqL, this corresponds to KP axiom.

Theorem

PD, PID and InqL have the Craig interpolation property, i.e., if $\phi(\vec{p}, \vec{q}) \models \psi(\vec{q}, \vec{r})$, then there exists a formula $\delta(\vec{q})$ such that $\phi \models \delta$ and $\delta \models \psi$.

Proof. 

\[
\bigvee_{i \in I} \Theta x_i(\vec{p}, \vec{q}) \models \bigvee_{j \in J} \Theta x_j(\vec{q}, \vec{r})
\]

\[
\implies \text{ for each } i \in I, \quad \Theta x_i(\vec{p}, \vec{q}) \models \bigvee_{j \in J} \Theta x_j(\vec{q}, \vec{r})
\]

\[
\implies \text{ for each } i \in I, \quad \text{there exists } j \in J \text{ s.t. } \Theta x_i(\vec{p}, \vec{q}) \vdash \text{CPC } \Theta x_j(\vec{q}, \vec{r})
\]

Now, by the interpolation theorem of CPC, for each $i \in I$, there exists a $\delta_i(\vec{q})$ such that $\Theta x_i(\vec{p}, \vec{q}) \vdash \text{CPC } \delta_i(\vec{q})$ and $\delta_i(\vec{q}) \vdash \text{CPC } \Theta x_j(\vec{q}, \vec{r})$. It follows that $\Theta x_i(\vec{p}, \vec{q}) \vdash \delta_i'(\vec{q})$ and $\delta_i'(\vec{q}) \vdash \Theta x_j(\vec{q}, \vec{r})$.

Hence $\bigvee_{i \in I} \Theta x_i(\vec{p}, \vec{q}) \vdash \bigvee_{i \in I} \delta_i'(\vec{q})$ and $\bigvee_{i \in I} \delta_i'(\vec{q}) \vdash \bigvee_{j \in J} \Theta x_j(\vec{q}, \vec{r})$. \qed
Theorem. Every locally finite logic that has interpolation has the uniform interpolation.

Propositional logics of dependence are locally finite, i.e., given $n$ propositional variables, there are only finitely many (at most $2^{2^n}$) non-equivalent formulas.
Applications of the normal form:
2. admissible rules and structural completeness

Joint work with Rosalie Iemhoff (Utrecht University)
Substitution $\sigma : \text{Prop} \rightarrow \text{Form}$ is not well-defined in PD and PID, since, e.g., $=(\phi, \psi), \neg \phi$ are not always well-formed formulas in the logics. One can expand the languages of PD and PID such that for all flat formulas $\phi$ and $\psi$, strings of the form $=(\phi, \psi), \neg \phi$ are well-formed formulas. There are sound and complete deductive systems for the extended logics PD and PID.

None of the logics is closed under uniform substitution. E.g., for PID and InqL: $\vdash \neg \neg p \rightarrow p, \not \vdash \neg \neg (p \lor \neg p) \rightarrow (p \lor \neg p)$; for PD: $\vdash p \otimes \neg p, \not \vdash = (p) \otimes \neg = (p)$.

**Lemma**

PD, PID and InqL are closed under flat substitutions, i.e., substitutions $\sigma$ such that $\sigma(p)$ is flat for all $p \in \text{Prop}$.
• $\Gamma \vdash \phi$: a consequence relation on $\wp(\text{Form}) \times \text{Form}$.
• A logic $L$ is a set of theorems, i.e., $L = \{ \phi : \emptyset \vdash_L \phi \}$.

A rule $\phi/\psi$ is said to be **admissible** in $L$, in symbols $\phi \vdash_L \psi$, if $\vdash_L \sigma(\phi) \implies \vdash_L \sigma(\psi)$ for all substitutions $\sigma$.

Alternatively, a rule $R$ is admissible in $L$ iff
\[ \{ \phi : \emptyset \vdash_L \phi \} = \{ \phi : \emptyset \vdash^R_L \phi \}. \]

[Friedman, Citkin, Rybakov, Ghilardi, etc.]

A rule $\phi/\psi$ is said to be **derivable** in $L$ if $\phi \vdash_L \psi$, or $\vdash_L \phi \to \psi$.

$\phi \vdash_L \psi \implies \phi \vdash_L \psi$

**Definition**

A logic $L$ is said to be **structurally complete** if every admissible rule is derivable in $L$, i.e., $\phi \vdash_L \psi \iff \phi \vdash_L \psi$.

• **CPC** is structurally complete.

• KP rule $\neg p \to q \lor r/(\neg p \to q) \lor (\neg p \to r)$ is admissible in all intermediate logics, but KP rule is not derivable in **IPC**.

• **KP** is not structurally complete.
A logic \( L \) is a set of theorems, i.e., \( L = \{ \phi : \emptyset \vdash_L \phi \} \).

A rule \( \phi/\psi \) of \( L \) is said to be \textit{admissible}, in symbols \( \phi \mathrel{\vdash_L} \psi \), if \( \vdash_L \sigma(\phi) \implies \vdash_L \sigma(\psi) \) for all substitutions \( \sigma \).

Let \( S \) be a set of substitutions under which \( \vdash_L \) is closed. A rule \( \phi/\psi \) of \( L \) is said to be \textit{\( S \)-admissible}, in symbols \( \phi \mathrel{\vdash^S_L} \psi \), if \( \vdash_L \sigma(\phi) \implies \vdash_L \sigma(\psi) \) for all substitutions \( \sigma \in S \).

Alternatively, a rule \( R \) is \( S \)-admissible in \( L \) iff
\[
\{ \phi : \emptyset \vdash_L \phi \} = \{ \phi : \emptyset \vdash^R_L \phi \}.
\]

A rule \( \phi/\psi \) of \( L \) is said to be \textit{derivable} if \( \phi \vdash_L \psi \) or \( \vdash_L \phi \to \psi \).

\[ \phi \vdash_L \psi \implies \phi \mathrel{\vdash^S_L} \psi \]

\textit{Pf.} For any \( \sigma \in S \),
\[
\vdash_L \sigma(\phi) \implies \vdash_L \sigma(\psi).
\]

by assumption: \( \sigma(\phi) \vdash_L \sigma(\psi) \)

(since \( \vdash_L \) is closed under \( \sigma \))

[\text{Friedman, Citkin, Rybakov, Ghilardi, etc.}]
**Definition**

A logic $L$ is said to be *$S$-structurally complete* if every $S$-admissible rule is derivable in $L$, i.e., $\phi \vdash^S L \psi \iff \phi \vdash_L \psi$.

**Theorem**

PD, PID and $\text{Inq}_L$ are $F$-structurally complete, where $F$ is the class of all flat substitutions, i.e., substitutions $\sigma$ such that $\sigma(p)$ is flat for all $p \in \text{Prop}$.

**Proof.** (sketch) “$\implies$”: [Use the normal form!] Assume $\bigvee_{i \in I} \Theta X_i \vdash^F \psi$, where $\phi \equiv \bigvee_{i \in I} \Theta X_i$ and $\phi$ is consistent. Each $\Theta X_i$ is an $F$-projective formula, therefore $\Theta X_i \vdash \psi$ for all $i \in I$. Hence $\phi \vdash \psi$. □
Definition (Projective formula)

Let $S$ be a set of substitutions under which $\vdash_L$ is closed. A formula $\phi$ is said to be $S$-projective in $L$ if there exists $\sigma \in S$ such that

1. $\vdash_L \sigma(\phi)$
2. $\phi, \sigma(\psi) \vdash_L \psi$ and $\phi, \psi \vdash_L \sigma(\psi)$ for all formulas $\psi$.

Such $\sigma$ is called a $S$-projective unifier of $\phi$ in $L$.

- Every consistent classical formula is projective in $\text{CPC}$.
- Every consistent negated formula (i.e. $\neg \phi$) is projective in every intermediate logic.
- For $D \in \{\text{PD, PID, InqL}\}$, the formula

$$\Theta_X = \begin{cases} \bigotimes_{v \in X} (p_{v(p_1)}^v \land \cdots \land p_{v(p_n)}^v), & \text{for } \text{PD}; \\ \neg\neg \bigvee_{v \in X} (p_{v(p_1)}^v \land \cdots \land p_{v(p_n)}^v), & \text{for } \text{PID, InqL}. \end{cases}$$

is projective in $D$. 
Theorem

PD, PID and InqL are $\mathcal{F}$-structurally complete, where $\mathcal{F}$ is the class of all flat substitutions, i.e., substitutions $\sigma$ such that $\sigma(p)$ is flat for all $p \in \text{Prop}$.

Proof. (sketch) “$\implies$”: [Use the normal form!]

Assume $\bigvee_{i \in I} \Theta_{X_i} \models_{\mathcal{L}} \psi$, where $\phi \equiv \bigvee_{i \in I} \Theta_{X_i}$ and $\phi$ is consistent. Each $\Theta_{X_i}$ is an $\mathcal{F}$-projective formula, therefore $\Theta_{X_i} \vdash \psi$ for all $i \in I$. Hence $\phi \vdash \psi$. □
Since $\text{InqL} = KP \oplus \neg\neg p \rightarrow p = KP \neg = ND \neg = ML \neg$, we have: (Ciardelli, Roelofsen, 2011) & (Miglioli, et al, 1989)

**Corollary**

$\text{ND} \neg$, $\text{KP} \neg$ and $\text{ML} \neg$ are *ST-hereditarily structurally complete*, where $\text{ST}$ is the class of all stable substitutions, i.e., substitutions $\sigma$ s.t. $\vdash \neg\neg\sigma(p) \leftrightarrow \sigma(p)$.

- **ML** is structurally complete but not hereditarily structurally complete.
- **ML$\neg$** is $\text{ST}$-structurally complete. [ (Miglioli, Moscato, Ornaghi, Quazza, Usberti, 1989), proved using disjunction property]

[Recall: $\text{ND} \subseteq \text{KP} \subseteq \text{ML}$]
Applications of the normal form: 3., 4., ... n. ?

Thank you!