Uniform Definability in Propositional Dependence Logic

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Dependence Logic: Theory and Applications
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1. Propositional dependence logic
2. Defining downwards closed collections of teams
3. $n$-context and uniform definability
4. $\lor$ and $\rightarrow$ are not uniformly definable in $\text{PD}$
This work is inspired by P. Galliani, *Epistemic Operators and Uniform Definability in Dependence Logic*, Studia Logica, to appear.
Propositional dependence logic
First-order dependence logic and its variants

[Väänänen 2007]:

Well-formed formulas of first-order dependence logic ($\mathbb{D}$) are given by the following grammar

$$\phi ::= \alpha \mid = (t_1, \ldots, t_n) \mid \phi \land \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi$$

where $\alpha$ is a first-order literal and $t_1, \ldots, t_n$ are first-order terms.
First-order dependence logic and its variants

[Väänänen 2007]:

- Formulas of first-order dependence logic ($D$):
  \[ \phi ::= \alpha \mid = (t_1, \ldots, t_n) \mid \phi \land \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi \]

[Abramsky, Väänänen 2009]:

- Formulas of first-order intuitionistic dependence logic ($ID$):
  \[ \phi ::= \alpha \mid \bot \mid = (t) \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \rightarrow \psi \mid \forall x \phi \mid \exists x \phi \]
Expressive power of logics

On sentence level

\[ \Sigma_1 \quad \text{FO} \quad \text{FO (team)} \]

Y. Väänänen

Enderton, Walkoe
Propositional dependence logic and its variants

- Formulas of propositional dependence logic ($\text{PD}$):

$$\phi ::= p \mid \neg p \mid =(p_1, \ldots, p_n) \mid \phi \land \phi \mid \phi \otimes \phi$$

where $p, p_1, \ldots, p_n$ are propositional variables.

- Formulas of propositional intuitionistic dependence logic ($\text{PID}$):

$$\phi ::= p \mid \bot \mid =(p) \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \rightarrow \psi$$

- $\text{PD}^\lor$ is the logic extended from $\text{PD}$ by adding the intuitionistic disjunction $\lor$. 


A team is a set of valuations, i.e. a set of functions $s : \text{Prop}_n \rightarrow \{0, 1\}$.

In particular, the emptyset $\emptyset$ is a team.
A *team* is a set of valuations, i.e. a set of functions $s : \text{Prop}_n \rightarrow \{0, 1\}$.

In particular, the emptyset $\emptyset$ is a team.
Let $X$ be a team.

- $X \models p$ iff $s(p) = 1$ for all $s \in X$
- $X \models \neg p$ iff $s(p) = 0$ for all $s \in X$

- $X \models (p_1, \ldots, p_n, q)$ iff for any $s_1, s_2 \in X$ such that $s_1(p_1) = s_2(p_1), \ldots, s_1(p_n) = s_2(p_n)$, we have that $s_1(q) = s_2(q)$

- $X \models \varphi \land \psi$ iff $X \models \varphi$ and $X \models \psi$

- $X \models \varphi \otimes \psi$ iff there exist $Y, Z \subseteq X$ s.t. $X = Y \cup Z$, $Y \models \varphi$ and $Z \models \psi$

- $X \models \varphi \lor \psi$ iff $X \models \varphi$ or $X \models \psi$

- $X \models \varphi \rightarrow \psi$ iff for all $Y \subseteq X$, if $Y \models \varphi$ then $Y \models \psi$
The logics $\text{PD}$, $\text{PD}^\lor$, $\text{PID}$

- are downwards closed, namely

$$X \models \phi \text{ and } Y \subseteq X \implies Y \models \phi;$$

- and have the empty team property, namely

$$\emptyset \models \phi \text{ for all } \phi.$$
The logics $\text{PD}$, $\text{PD}^\lor$, $\text{PID}$

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Expressive power of logics

PD  PD\lor  PID
Expressive power of logics

\[ PD = PD^\vee \quad \text{PID} \]

Huuskonen, 2012
Expressive power of logics

\[ PD = PD^\lor = PID \]

Huuskonen, 2012
Expressive power of logics

\[ PD = \text{PD}^\lor = \text{PID} \equiv \text{Inquisitive Logic} \]

Huuskonen, 2012

\[ \text{[Ciardelli, Roelofsen, 2011]} \]

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1 I thank Prof. Dick de Jongh and Dr. Tadeusz Litak for pointing out this surprising connection to me.
Defining downwards closed collections of teams
A team is a set of valuations, i.e. a set of functions $s : \text{Prop}_n \rightarrow \{0, 1\}$.  

- A team with domain $\{p_1, \ldots, p_n\}$ is called an \textit{n-team}.
- A formula $\phi(p_1, \ldots, p_n)$ whose propositional variables are among $p_1, \ldots, p_n$ is called an \textit{n-formula}.  

![Diagram of a team with indices and valuations](image-url)
A team is a set of valuations, i.e. a set of functions $s : \text{Prop}_n \rightarrow \{0, 1\}$.

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Recall: Team

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**Fact:** Fix \( n \), there are \( 2^{2^n} \) many \( n \)-teams.
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**Fact:** Fix \( n \), there are \( 2^{2^n} \) many \( n\text{-teams} \).
Let $\phi$ be a $n$-formula of PD. Define

$$\boxtimes[\phi] = \{ X \subseteq 2^n \mid X \models \phi \}.$$

Let $\nabla_n$ be the family of all downwards closed collections of $n$-teams, i.e.,

$$\nabla_n = \{ \mathcal{K} \subseteq 2^{2^n} \mid X \in \mathcal{K}, \ Y \subseteq X \text{ imply } Y \in \mathcal{K} \}.$$

Clearly, $\boxtimes[\phi] \in \nabla_n$ for every $n$-formula $\phi$ of PD.
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Clearly, $[\phi] \in \nabla_n$ for every $n$-formula $\phi$ of $\mathbf{PD}$. 
Lemma (Huuskonen)

For any $n$-team $Y$, there exists an $n$-formula $\Theta_Y$ of $\text{PD}$ such that for any $n$-team $X$,

$$X \models \Theta_Y \iff Y \not\in X.$$ 

Theorem (Huuskonen)

Every downwards closed collection of $n$-teams is definable in $\text{PD}$.

Proof. (sketch) For any $\mathcal{K} \in \nabla_n$,

$$\mathcal{K} = \bigwedge_{i \in I} \Theta_{Y_i},$$

where $2^{2^n} \setminus \mathcal{K} = \{ Y_i \mid i \in I \}$, since for any $n$-team $X$,

$$X \models \bigwedge_{i \in I} \Theta_{Y_i} \iff Y_i \not\in X \text{ for all } i \in I \iff X \in \mathcal{K}.$$ 

Corollary: For every $\text{PD}$ $n$-formula $\phi$, we have that $\phi \equiv \bigwedge_{i \in I} \Theta_{Y_i}$. 

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Since $\textbf{PD} = \textbf{PD}^\lor = \textbf{PID}$, both $\lor$ and $\rightarrow$ are definable in $\textbf{PD}$.

Let $\phi, \psi$ be $n$-formulas, where $\phi \equiv \bigwedge_{i \in I} \Theta x_i$ and $\psi \equiv \bigwedge_{j \in J} \Theta y_j$. Then we have that

$$\phi \lor \psi \equiv \bigwedge_{k \in K} \Theta z_k$$

for

$$\{z_k \mid k \in K\} = \{Z \subseteq 2^n \mid \exists i \in I, \exists j \in J, X_i \cup Y_j \subseteq Z\}.$$
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$n$-context and uniform definability
Consider the logic $\text{PD}^*$, whose well-formed formulas are built from the following grammar:

$$
\phi ::= \Xi_i \mid p \mid \neg p \mid = (p_{i_1}, \ldots, p_{i_k}) \mid (\phi \land \phi) \mid (\phi \otimes \phi),
$$

where $\Xi_i$ is a context atom, $p$ is a propositional variable. We call formulas of the forms $\Xi_i, p, \neg p$ or $=(p_{i_1}, \ldots, p_{i_k})$ atoms.

**Definition (n-context)**

An $n$-context of type $\langle n, n \rangle$ is an $n$-formula $\phi[\Xi_1, \Xi_2]$ of $\text{PD}^*$, whose context atoms are $\Xi_1, \Xi_2$.  

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\[ \]
Definition

Let $\mathcal{K}_1, \mathcal{K}_2 \in \nabla_n$, $\phi[\Xi_1, \Xi_2]$ an $n$-context of type $\langle n, n \rangle$, and $X$ an $n$-team.

We define the satisfaction relation $X \models \phi[\mathcal{K}_1, \mathcal{K}_2]$ recursively as follows:

- $X \models \Xi_i[\mathcal{K}_1, \mathcal{K}_2]$ iff $X \in \mathcal{K}_i$;
- $X \models \alpha[\mathcal{K}_1, \mathcal{K}_2]$ iff $X \models \alpha$ for $\alpha$ a non-context atom;
- $X \models (\psi \land \chi)[\mathcal{K}_1, \mathcal{K}_2]$ iff $X \models \psi[\mathcal{K}_1, \mathcal{K}_2]$ and $X \models \chi[\mathcal{K}_1, \mathcal{K}_2]$;
- $X \models (\psi \otimes \chi)[\mathcal{K}_1, \mathcal{K}_2]$ iff there exist $Y, Z$ such that $X = Y \cup Z$, $Y \models \psi[\mathcal{K}_1, \mathcal{K}_2]$ and $Z \models \chi[\mathcal{K}_1, \mathcal{K}_2]$. 
Define

$$
\llbracket \phi[k_1, k_2] \rrbracket := \{ X \subseteq 2^n \mid X \models \phi[k_1, k_2] \}.
$$

**Definition (Uniform definability)**

A binary operator $op : \nabla_2^n \rightarrow \nabla_n$ is said to be *uniformly definable* in $PD$ iff there exists an $n$-context $\phi[\Xi_1, \Xi_2]$ of type $\langle n, n \rangle$ such that

$$
\llbracket \phi[k_1, k_2] \rrbracket = op(k_1, k_2).
$$

In particular, a binary connective $*$ is *uniformly definable* in $PD$ iff there exists an $n$-context $\phi[\Xi_1, \Xi_2]$ such that for all $n$-formulas $\phi$ and $\psi$ of $PD$,

$$
\llbracket \phi[\llbracket \psi \rrbracket, [\chi] \rrbracket \rrbracket = ,
$$
Define

\[ \llbracket \phi[\kappa_1, \kappa_2] \rrbracket := \{ X \subseteq 2^n \mid X \models \phi[\kappa_1, \kappa_2] \}. \]

**Definition (Uniform definability)**

A binary operator \( op : \nabla_n^2 \to \nabla_n \) is said to be *uniformly definable* in \( \text{PD} \) iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) of type \( \langle n, n \rangle \) such that

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[[\phi[\mathcal{K}_1, \mathcal{K}_2]]] := \{X \subseteq 2^n \mid X \models \phi[\mathcal{K}_1, \mathcal{K}_2]\}.
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**Definition (Uniform definability)**

A binary operator \( \text{op} : \nabla_n^2 \to \nabla_n \) is said to be *uniformly definable* in \( \text{PD} \) iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) of type \( \langle n, n \rangle \) such that

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[[\phi[\mathcal{K}_1, \mathcal{K}_2]]] = \text{op}(\mathcal{K}_1, \mathcal{K}_2).
\]

In particular, a binary connective \( \ast \) is *uniformly definable* in \( \text{PD} \) iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) such that for all \( n \)-formulas \( \phi \) and \( \psi \) of \( \text{PD} \),

\[
[[\phi[\psi, \chi]]] = [[\psi]] \ast [[\chi]],
\]
Define
\[
\llbracket \phi[\mathcal{K}_1, \mathcal{K}_2] \rrbracket := \{ X \subseteq 2^n \mid X \models \phi[\mathcal{K}_1, \mathcal{K}_2] \}.
\]

**Definition (Uniform definability)**

A binary operator \( op : \nabla^2_n \to \nabla_n \) is said to be *uniformly definable* in \( \text{PD} \) iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) of type \( \langle n, n \rangle \) such that
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\llbracket \phi[\mathcal{K}_1, \mathcal{K}_2] \rrbracket = op(\mathcal{K}_1, \mathcal{K}_2).
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In particular, a binary connective \( * \) is *uniformly definable* in \( \text{PD} \) iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) such that for all \( n \)-formulas \( \phi \) and \( \psi \) of \( \text{PD} \),
\[
\llbracket \phi \llbracket \llbracket \psi \rrbracket, \llbracket \chi \rrbracket \rrbracket = \llbracket \psi \ast \chi \rrbracket,
\]
Define

\[
\llbracket \phi[\mathcal{K}_1, \mathcal{K}_2] \rrbracket := \{ X \subseteq 2^n \mid X \models \phi[\mathcal{K}_1, \mathcal{K}_2] \}.
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**Definition (Uniform definability)**

A binary operator \( op : \nabla_2^n \rightarrow \nabla_n \) is said to be *uniformly definable* in PD iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) of type \( \langle n, n \rangle \) such that

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\llbracket \phi[\mathcal{K}_1, \mathcal{K}_2] \rrbracket = op(\mathcal{K}_1, \mathcal{K}_2).
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In particular, a binary connective \( * \) is *uniformly definable* in PD iff there exists an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) such that for all \( n \)-formulas \( \phi \) and \( \psi \) of PD,

\[
\llbracket \phi[\llbracket \psi \rrbracket, \llbracket \chi \rrbracket] \rrbracket = \llbracket \psi * \chi \rrbracket,
\]

namely, for all \( n \)-team \( X \),

\[
X \models \phi[\llbracket \psi \rrbracket, \llbracket \chi \rrbracket] \iff X \models \psi * \chi.
\]
**Definition (syntax tree)**

The *syntax tree* of an $n$-context $\phi(\Xi_1, \Xi_2)$ is a labelled binary tree $\mathcal{T}_\phi = (T, \preceq, r, f)$ such that

- $T := m + 1$, where $m$ is the number of all parentheses in $\phi$;
- $r := 0$;
- $\preceq := \{(0, k) | 0 < k \leq m\} \cup \{(k_1, k_2) | \text{the } k_2\text{-th parenthesis is inside the scope of the } k_1\text{-th parenthesis }\}$;
- $f$ is a function $f : T \rightarrow \text{Sub}(\phi)$ satisfying
  - $f(0) = \phi$;
  - $f(k) := \psi$, where $\psi$ is the subformula of $\phi$ bounded by the $k$-the parenthesis.
\[
\left( ( p_1 \land \Xi_1 ) \otimes ( = (p_2, p_3) \otimes ( \Xi_1 \land \Xi_2 ) ) \right)
\]
\[
(\ (p_1 \land \Xi_1 \ ) \otimes ( = (p_2, p_3) \otimes ( \Xi_1 \land \Xi_2 ) ) \ )
\]
\[
( ( p_1 \land \Xi_1 ) \otimes ( \Xi_1 \land \Xi_2 ) )
\]
\[(p_1 \land \Xi_1) \otimes \left(\left(= (p_2, p_3) \otimes (\Xi_1 \land \Xi_2)\right)\right)\]
Truth Function

\[
\left( (p_1 \land \Xi_1) \otimes \left( \equiv (p_2, p_3) \otimes (\Xi_1 \land \Xi_2) \right) \right)
\]
Truth Function

\[ X \models \left( (p_1 \land \Xi_1) \otimes \left( = (p_2, p_3) \otimes (\Xi_1 \land \Xi_2) \right) \right)[\mathcal{K}_1, \mathcal{K}_2] \]
Truth Function

\[
X \models \left( (p_1 \land \Xi_1) \otimes (= (p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) \right) [\mathcal{K}_1, \mathcal{K}_2]
\]

\[
\sigma : \exists \phi \rightarrow \wp(\mathcal{X})
\]
\[ X \models (p_1 \land \Xi_1) \otimes (=(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) [\mathcal{K}_1, \mathcal{K}_2] \]

\[ Y \models (p_1 \land \Xi_1)[\mathcal{K}_1, \mathcal{K}_2] \]

\[ Z \models (=(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) [\mathcal{K}_1, \mathcal{K}_2] \]

\[ \sigma : \Sigma_\phi \mapsto \wp(X) \]
Truth Function

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Truth Function

\[
X \models \left( (p_1 \land \Xi_1) \otimes (\vartheta(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) \right)[K_1, K_2]
\]

\[
Y \models (p_1 \land \Xi_1)[K_1, K_2]
\]

\[
Z \models (\vartheta(p_2, p_3) \otimes (\Xi_1 \land \Xi_2))[K_1, K_2]
\]

\[
W \models (p_2, p_3)[K_1, K_2]
\]

\[
U \models (\Xi_1 \land \Xi_2)[K_1, K_2]
\]

\[
\Sigma_\phi \rightarrow \vartheta(X)
\]
Truth Function

\[ X \models \left( (p_1 \land \Xi_1) \otimes (\phi(p_2, p_3) \land (\Xi_1 \land \Xi_2)) \right)[\mathcal{K}_1, \mathcal{K}_2] \]

\[ Y \models (p_1 \land \Xi_1)[\mathcal{K}_1, \mathcal{K}_2] \]

\[ Z \models (\phi(p_2, p_3) \land (\Xi_1 \land \Xi_2))[\mathcal{K}_1, \mathcal{K}_2] \]

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\[ Y \models \Xi_1[\mathcal{K}_1, \mathcal{K}_2] \]

\[ U \models \Xi_1[\mathcal{K}_1, \mathcal{K}_2] \]

\[ U \models \Xi_2[\mathcal{K}_1, \mathcal{K}_2] \]

\[ \sigma : \Sigma_\phi \rightarrow \wp(X) \]
Definition (truth function)

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ a $n$-team and $\mathcal{K}_1, \mathcal{K}_2 \in \nabla_n$. A function $\sigma : \mathcal{S}_\phi \rightarrow \wp(X)$ is called a truth function for $\phi[\mathcal{K}_1, \mathcal{K}_2]$ over $X$ iff

(i) $\sigma(0) = X$;

(ii) for all $k \in \mathcal{S}_\phi$, $\sigma(k) \models f(k)[\mathcal{K}_1, \mathcal{K}_2]$;

(iii) if $k$ is labeled with $\psi \land \chi$ and $k_0, k_1$ are two children of $k$, then $\sigma(k) = \sigma(k_0) = \sigma(k_1)$;

(iv) if $k$ is labeled with $\psi \otimes \chi$ and $k_0, k_1$ are two children of $k$, then $\sigma(k) = \sigma(k_0) \cup \sigma(k_1)$;
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ an $n$-team and $K_1, K_2 \in \nabla_n$. Then $X \models \phi[K_1, K_2]$ iff there exists a truth function $\sigma$ for $\phi[K_1, K_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$. □

\[
\left( (p_1 \land \Xi_1) \land \left( = (p_2, p_3) \land (\Xi_1 \land \Xi_2) \right) \right)
\]

\[
(p_1 \land \Xi_1)
\]

\[
(\Xi_1 \land \Xi_2)
\]

\[
\Xi_1
\]

\[
\Xi_2
\]

\[
(p_1 \land \Xi_1)
\]

\[
(\Xi_1 \land \Xi_2)
\]

\[
(p_1 \land \Xi_1)
\]

\[
(\Xi_1 \land \Xi_2)
\]

\[
(p_1 \land \Xi_1)
\]

\[
(\Xi_1 \land \Xi_2)
\]
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ an $n$-team and $\mathcal{K}_1, \mathcal{K}_2 \in \nabla_n$. Then $X \models \phi[\mathcal{K}_1, \mathcal{K}_2]$ iff there exists a truth function $\sigma$ for $\phi[\mathcal{K}_1, \mathcal{K}_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$. □
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ an $n$-team and $K_1, K_2 \in \bigtriangledown_n$. Then $X \models \phi[K_1, K_2]$ iff there exists a truth function $\sigma$ for $\phi[K_1, K_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$. □

\[
X \models (p_1 \land \Xi_1) \otimes (= (p_2, p_3) \otimes (\Xi_1 \land \Xi_2))^{[K_1, K_2]}
\]

\[
\sigma : \sum_\phi \rightarrow \wp(X)
\]
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ an $n$-team and $\mathcal{K}_1, \mathcal{K}_2 \in \nabla_n$. Then $X \models \phi[\mathcal{K}_1, \mathcal{K}_2]$ iff there exists a truth function $\sigma$ for $\phi[\mathcal{K}_1, \mathcal{K}_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$.

$X \models \left( (p_1 \land \Xi_1) \otimes ((p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) \right)[\mathcal{K}_1, \mathcal{K}_2]$

$Y \models (p_1 \land \Xi_1)[\mathcal{K}_1, \mathcal{K}_2]$

$Z \models ((p_2, p_3) \otimes (\Xi_1 \land \Xi_2))[\mathcal{K}_1, \mathcal{K}_2]$

$\sigma : \Sigma_\phi \rightarrow \wp(X)$
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ an $n$-team and $\mathcal{K}_1, \mathcal{K}_2 \in \nabla_n$. Then $X \models \phi[\mathcal{K}_1, \mathcal{K}_2]$ iff there exists a truth function $\sigma$ for $\phi[\mathcal{K}_1, \mathcal{K}_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$.

\[
X \models (p_1 \land \Xi_1) \otimes ((p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) [\mathcal{K}_1, \mathcal{K}_2]
\]

\[
\sigma : \Xi_\phi \rightarrow \wp(X)
\]
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an n-context of type $\langle n, n \rangle$, $X$ an n-team and $K_1, K_2 \in \nabla_n$. Then $X \models \phi[K_1, K_2]$ iff there exists a truth function $\sigma$ for $\phi[K_1, K_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$. $\square$

$$X \models ((p_1 \wedge \Xi_1) \otimes (=(p_2, p_3) \otimes (\Xi_1 \wedge \Xi_2)))[K_1, K_2]$$

$$Y \models (p_1 \wedge \Xi_1)[K_1, K_2]$$

$$Z \models (=(p_2, p_3) \otimes (\Xi_1 \wedge \Xi_2))[K_1, K_2]$$

$$W \models =(p_2, p_3)[K_1, K_2]$$

$$U \models (\Xi_1 \wedge \Xi_2)[K_1, K_2]$$

$$\sigma : \Sigma_{\phi} \rightarrow \wp(X)$$
Theorem

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$, $X$ an $n$-team and $\mathcal{K}_1, \mathcal{K}_2 \in \nabla_n$. Then $X \models \phi[\mathcal{K}_1, \mathcal{K}_2]$ iff there exists a truth function $\sigma$ for $\phi[\mathcal{K}_1, \mathcal{K}_2]$ over $X$.

Proof. Easy, by induction on $\phi[\Xi_1, \Xi_2]$.

\[
X \models (p_1 \land \Xi_1) \otimes (=(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) [\mathcal{K}_1, \mathcal{K}_2]
\]
Recall:

A function $\sigma : \mathcal{T}_\phi \rightarrow \wp(X)$ is called a truth function for $\phi[K_1, K_2]$ over $X$ iff

(i) $\sigma(0) = X$;

(ii) for all $k \in \mathcal{T}_\phi$, $\sigma(k) \models f(k)[K_1, K_2]$;

(iii) if $k$ is labeled with $(\psi \land \chi)$ and $k_0, k_1$ are two child nodes of $k$, then $\sigma(k) = \sigma(k_0) = \sigma(k_1)$;

(iv) if $k$ is labeled with $(\psi \otimes \chi)$ and $k_0, k_1$ are two child nodes of $k$, then $\sigma(k) = \sigma(k_0) \cup \sigma(k_1)$;

Lemma (I)

If $\sigma : \mathcal{T}_\phi \rightarrow \wp(X)$ is a function which satisfies conditions (i), (iii), (iv) and condition (ii) with respect to $K_1, K_2$ for all $k$ labeled with atoms, then $\sigma$ is a truth function for $\phi[K_1, K_2]$ over $X$. 
Recall:

A function \( \sigma : \mathcal{I}_\phi \rightarrow \wp(X) \) is called a truth function for \( \phi[K_1, K_2] \) over \( X \) iff

(i) \( \sigma(0) = X \);

(ii) for all \( k \in \mathcal{I}_\phi \), \( \sigma(k) \models f(k)[K_1, K_2] \);

(iii) if \( k \) is labeled with \( (\psi \land \chi) \) and \( k_0, k_1 \) are two child nodes of \( k \), then \( \sigma(k) = \sigma(k_0) = \sigma(k_1) \);

(iv) if \( k \) is labeled with \( (\psi \otimes \chi) \) and \( k_0, k_1 \) are two child nodes of \( k \), then \( \sigma(k) = \sigma(k_0) \cup \sigma(k_1) \);

Lemma (I)

If \( \sigma : \mathcal{I}_\phi \rightarrow \wp(X) \) is a function which satisfies conditions (i),(iii),(iv) and condition (ii) with respect to \( K_1, K_2 \) for all \( k \) labeled with atoms, then \( \sigma \) is a truth function for \( \phi[K_1, K_2] \) over \( X \).
Lemma (I)

If $\sigma : \mathcal{F}_\phi \rightarrow \mathcal{F}(X)$ is a function which satisfies conditions (i),(iii),(iv) and condition (ii) with respect to $K_1, K_2$ for all $k$ labeled with atoms, then $\sigma$ is a truth function for $\phi[K_1, K_2]$ over $X$.

Proof.

$$ \left( (p_1 \land \Xi_1) \otimes (=(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) \right) $$
Lemma (I)

If $\sigma : \mathfrak{S}_{\phi} \rightarrow \wp(X)$ is a function which satisfies conditions (i),(iii),(iv) and condition (ii) with respect to $\mathcal{K}_1, \mathcal{K}_2$ for all $k$ labeled with atoms, then $\sigma$ is a truth function for $\phi[\mathcal{K}_1, \mathcal{K}_2]$ over $X$.

Proof.

$$X ?? \models \left( (p_1 \land \Xi_1) \otimes (=(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) \right)[\mathcal{K}_1, \mathcal{K}_2]$$
Lemma (I)

If $\sigma : \mathcal{Z}_\phi \rightarrow \wp(X)$ is a function which satisfies conditions (i),(iii),(iv) and condition (ii) with respect to $K_1, K_2$ for all k labeled with atoms, then $\sigma$ is a truth function for $\phi[K_1, K_2]$ over $X$.

Proof.

$$X \models \left( (p_1 \land \Xi_1) \otimes \left( = (p_2, p_3) \otimes (\Xi_1 \land \Xi_2) \right) \right)[K_1, K_2]$$

$$Y \models (p_1 \land \Xi_1)[K_1, K_2]$$
$$Y \models (p_2, p_3) \otimes (\Xi_1 \land \Xi_2)$$
Lemma (I)

If \( \sigma : \mathcal{T} \phi \rightarrow \mathcal{V}(X) \) is a function which satisfies conditions (i),(iii),(iv) and condition (ii) with respect to \( K_1, K_2 \) for all \( k \) labeled with atoms, then \( \sigma \) is a truth function for \( \phi[K_1, K_2]\) over \( X \).

Proof.

\[
X \models (p_1 \land \Xi_1) \otimes (p_2, p_3) \otimes (\Xi_1 \land \Xi_2) \Big) \Big)[K_1, K_2]
\]

\[
Y \models (p_1 \land \Xi_1)[K_1, K_2]
\]

\[
W \cup U \models (p_2, p_3) \otimes (\Xi_1 \land \Xi_2)[K_1, K_2]
\]

\[
W \models (p_2, p_3)[K_1, K_2]
\]

\[
U \models (\Xi_1 \land \Xi_2)[K_1, K_2]
\]

\[
Y \models p_1[K_1, K_2]
\]

\[
Y \models \Xi_1[K_1, K_2]
\]

\[
U \models \Xi_1[K_1, K_2]
\]

\[
U \models \Xi_2[K_1, K_2]
\]
Lemma (I)

If \( \sigma : \mathcal{T}_\phi \to \wp(\phi(X)) \) is a function which satisfies conditions (i), (iii), (iv) and condition (ii) with respect to \( K_1, K_2 \) for all \( k \) labeled with atoms, then \( \sigma \) is a truth function for \( \phi[K_1, K_2] \) over \( X \).

Proof.

\[
X = Y \cup W \cup U \models \left((p_1 \land \Xi_1) \otimes (\equiv(p_2, p_3) \otimes (\Xi_1 \land \Xi_2))\right)[K_1, K_2]
\]

\[
Y \models (p_1 \land \Xi_1)[K_1, K_2]
\]

\[
W \cup U \models (\equiv(p_2, p_3) \otimes (\Xi_1 \land \Xi_2))[K_1, K_2]
\]

\[
W \models \equiv(p_2, p_3)[K_1, K_2]
\]

\[
Y \models \Xi_1[K_1, K_2]
\]

\[
U \models \Xi_1 \land \Xi_2[K_1, K_2]
\]

\[
U \models \Xi_2[K_1, K_2]
\]
Lemma (I)

If $\sigma : \mathcal{S}_\phi \rightarrow \mathcal{F}(X)$ is a function which satisfies conditions (i),(iii),(iv) and condition (ii) with respect to $\mathcal{K}_1, \mathcal{K}_2$ for all $k$ labeled with atoms, then $\sigma$ is a truth function for $\mathcal{F}[\mathcal{K}_1, \mathcal{K}_2]$ over $X$.

Proof.

$$X = Y \cup W \cup U \models \left( (p_1 \land \Xi_1) \otimes (=(p_2, p_3) \otimes (\Xi_1 \land \Xi_2)) \right)[\mathcal{K}_1, \mathcal{K}_2]$$
∧ and → are not uniformly definable in PD
Fact: Let $\sigma$ be a truth function for $\phi[K_1, K_2]$ over $X$.

In the syntax tree $\mathcal{T}_{\phi}$, if a node $k$ has no ancestor node with a label of the form $\psi \otimes \chi$, then $\sigma(k) = X$. 

\[
X \models ( (p_1 \land \Xi_1) \land ( = (p_2, p_3) \otimes (\Xi_1 \land \Xi_2) ) ) [K_1, K_2]
\]

\[
X \models ( p_1 \land \Xi_1 ) [K_1, K_2]
\]

\[
X \models ( = (p_2, p_3) \otimes (\Xi_1 \land \Xi_2) ) [K_1, K_2]
\]

\[
W \models = (p_2, p_3) [K_1, K_2]
\]

\[
U \models (\Xi_1 \land \Xi_2) [K_1, K_2]
\]

\[
U \models \Xi_1 [K_1, K_2]
\]

\[
U \models \Xi_2 [K_1, K_2]
\]
Observation

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Sigma_\phi$ there is a node $k$ labeled with $\Xi_1$, which has no ancestor node with a label of the form $\psi \otimes \chi$. Then $\phi[\Xi_1, \Xi_2]$ does not define the intuitionistic disjunction $\lor$, namely

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]]$$

fails for some PD $n$-formulas $\psi, \chi$.

Proof. Let $\psi := \bot$ and $\chi := \top$. Then we have that for the full $n$-team $2^n$,

$$2^n \models \top, \text{ so } 2^n \in [[\bot \lor \top]].$$

Suppose $2^n \in [[\phi[[\bot], [\top]]]]$, i.e., $2^n \models \phi[[\bot], [\top]]$, and $\sigma$ is a truth function for $\phi[[\bot], [\top]]$ over $2^n$. By Fact, we must have that $\sigma(k) = 2^n$, which by the definition of a truth function means that $2^n \models \bot$; a contradiction.
Observation

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\mathcal{T}_\phi$ there is a node $k$ labeled with $\Xi_1$, which has no ancestor node with a label of the form $\psi \otimes \chi$. Then $\phi[\Xi_1, \Xi_2]$ does not define the intuitionistic disjunction $\lor$, namely

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]]$$

tails for some $\text{PD} \ n$-formulas $\psi, \chi$.

Proof. Let $\psi := \bot$ and $\chi := \top$. Then we have that for the full $n$-team $2^n$,

$$2^n \models \top, \text{ so } 2^n \in [[\bot \lor \top]].$$

Suppose $2^n \in [[\phi[[\bot], [\top]]]]$, i.e., $2^n \models \phi[[\bot], [\top]]$, and $\sigma$ is a truth function for $\phi[[\bot], [\top]]$ over $2^n$. By Fact, we must have that $\sigma(k) = 2^n$, which by the definition of a truth function means that $2^n \models \bot$; a contradiction.
Observation

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Sigma_\phi$ there is a node $k$ labeled with $\Xi_1$, which has no ancestor node with a label of the form $\psi \otimes \chi$. Then $\phi[\Xi_1, \Xi_2]$ does not define the intuitionistic disjunction $\vee$, namely

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]]$$

fails for some PD $n$-formulas $\psi, \chi$.

Proof. Let $\psi := \bot$ and $\chi := \top$. Then we have that for the full $n$-team $2^n$,

$$2^n \models \top, \text{ so } 2^n \in [[\bot \lor \top]].$$

Suppose $2^n \in [[\phi[[\bot], [\top]]]]$, i.e., $2^n \models \phi[[\bot], [\top]]$, and $\sigma$ is a truth function for $\phi[[\bot], [\top]]$ over $2^n$. By Fact, we must have that $\sigma(k) = 2^n$, which by the definition of a truth function means that $2^n \models \bot$; a contradiction.
**Observation**

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Sigma_\phi$ there is a node $k$ labeled with $\Xi_1$, which has no ancestor node with a label of the form $\psi \otimes \chi$. Then $\phi[\Xi_1, \Xi_2]$ does not define the intuitionistic disjunction $\lor$, namely

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]]$$

fails for some PD $n$-formulas $\psi, \chi$.

**Proof.** Let $\psi := \bot$ and $\chi := \top$. Then we have that for the full $n$-team $2^n$,

$$2^n \models \top, \text{ so } 2^n \in [\bot \lor \top].$$

Suppose $2^n \in [\phi[[\bot], [\top]]]$, i.e., $2^n \models \phi[[\bot], [\top]]$, and $\sigma$ is a truth function for $\phi[[\bot], [\top]]$ over $2^n$. By Fact, we must have that $\sigma(k) = 2^n$, which by the definition of a truth function means that $2^n \models \bot$; a contradiction.
Observation

Let $\phi[\Xi_1, \Xi_2]$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Sigma_\phi$ there is a node $k$ labeled with $\Xi_1$, which has no ancestor node with a label of the form $\psi \otimes \chi$. Then $\phi[\Xi_1, \Xi_2]$ does not define the intuitionistic disjunction $\lor$, namely

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]]$$

fails for some PD $n$-formulas $\psi, \chi$.

Proof. Let $\psi := \bot$ and $\chi := \top$. Then we have that for the full $n$-team $2^n$,

$$2^n \models \top, \text{ so } 2^n \in [[\bot \lor \top]].$$

Suppose $2^n \in [[\phi[[\bot], [\top]]]]$, i.e., $2^n \models \phi[[\bot], [\top]]$, and $\sigma$ is a truth function for $\phi[[\bot], [\top]]$ over $2^n$. By Fact, we must have that $\sigma(k) = 2^n$, which by the definition of a truth function means that $2^n \models \bot$; a contradiction.
Lemma (II)

Let \( \phi[\Xi_1, \Xi_2] \neq \bot \) be an \( n \)-context of type \( \langle n, n \rangle \) such that in the syntax tree \( \Xi_\phi \) every node labeled with \( \Xi_i \) has an ancestor node labeled with a formula of the form \( \psi \otimes \chi \).

If \( 2^n \models \phi[[\top]], [[\top]] \), then there exists a truth function \( \sigma \) for \( \phi[[\top]], [[\top]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_i \) in \( \Xi_\phi \), \( \sigma(k) \not\subseteq 2^n \).

Proof.

![Diagram of the syntax tree \( \Xi_\phi \) with labels and subformulas.](image)
Lemma (II)

Let $\phi[\Xi_1, \Xi_2] \not\equiv \perp$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Xi_\phi$ every node labeled with $\Xi_i$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

If $2^n \models \phi[[T], [T]]$, then there exists a truth function $\sigma$ for $\phi[[T], [T]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_i$ in $\Xi_\phi$, $\sigma(k) \not\subseteq 2^n$.

Proof.

$2^n \models \left( (p_1 \otimes \Xi_1) \land (=(p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right)[[T], [T]]$

Diagram:

```
0
 / \  \\
1   8
 / \   /
2 3 4 7
 / \\  /   \
5 6 8 7
```

$\Xi_1$
Lemma (II)

Let \( \phi[\Xi_1, \Xi_2] \not\equiv \bot \) be an n-context of type \( \langle n, n \rangle \) such that in the syntax tree \( T_\phi \) every node labeled with \( \Xi_i \) has an ancestor node labeled with a formula of the form \( \psi \otimes \chi \).

If \( 2^n \models \phi[[T], [T]] \), then there exists a truth function \( \sigma \) for \( \phi[[T], [T]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_i \) in \( T_\phi \), \( \sigma(k) \not\subseteq 2^n \).

Proof.

\[
2^n \models \left( (p_1 \otimes \Xi_1) \land (= (p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right)[[T], [T]]
\]
Lemma (II)

Let $\phi[\Xi_1, \Xi_2] \not\equiv \bot$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Xi_\phi$ every node labeled with $\Xi_i$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

If $2^n \models \phi[[T], [T]]$, then there exists a truth function $\sigma$ for $\phi[[T], [T]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_i$ in $\Xi_\phi$, $\sigma(k) \subset 2^n$.

Proof.

$2^n \models \left( (p_1 \otimes \Xi_1) \land ( = (p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right) [[T], [T]]$
Lemma (II)

Let \( \varphi[\Xi_1, \Xi_2] \neq \bot \) be an \( n \)-context of type \( \langle n, n \rangle \) such that in the syntax tree \( \mathcal{T}_\varphi \) every node labeled with \( \Xi_i \) has an ancestor node labeled with a formula of the form \( \psi \otimes \chi \).

If \( 2^n \models \varphi[[T], [T]] \), then there exists a truth function \( \sigma \) for \( \varphi[[T], [T]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_i \) in \( \mathcal{T}_\varphi \), \( \sigma(k) \not\subseteq 2^n \).

Proof.

\[
2^n \models \left( (p_1 \otimes \Xi_1) \land (=(p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right) [[T], [T]]
\]
Lemma (II)

Let $\phi[\Xi_1, \Xi_2] \not\equiv \bot$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Xi_\phi$ every node labeled with $\Xi_i$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

If $2^n \models \phi[[T], [T]]$, then there exists a truth function $\sigma$ for $\phi[[T], [T]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_i$ in $\Xi_\phi$, $\sigma(k) \subsetneq 2^n$.

Proof.

$$2^n \models \left( (p_1 \otimes \Xi_1) \land (\equiv (p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right)[[T], [T]]$$

![Diagram](image.png)
Lemma (II)

Let $\phi[\Xi_1, \Xi_2] \not\equiv \bot$ be an $n$-context of type $\langle n, n \rangle$ such that in the syntax tree $\Xi_\phi$ every node labeled with $\Xi_i$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

If $2^n \models \phi[[T], [T]]$, then there exists a truth function $\sigma$ for $\phi[[T], [T]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_i$ in $\Xi_\phi$, $\sigma(k) \subsetneq 2^n$.

Proof.

$$2^n \models \left( (p_1 \otimes \Xi_1) \land ( = (p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right)[[T], [T]]$$

$$= (p_1 \otimes \Xi_1)$$

$$= (p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)$$

$$Y \models = (p_2, p_3)[[T], [T]]$$

$$Z \models (\Xi_1 \otimes \Xi_2)[[T], [T]]$$

$$W \models \Xi_1[[T], [T]]$$

$$U \models \Xi_2[[T], [T]]$$
Lemma (II)

Let \( \phi[\Xi_1, \Xi_2] \neq \bot \) be an \( n \)-context of type \( \langle n, n \rangle \) such that in the syntax tree \( \mathcal{Z}_\phi \) every node labeled with \( \Xi_i \) has an ancestor node labeled with a formula of the form \( \psi \otimes \chi \).

If \( 2^n \models \phi[[T], [T]] \), then there exists a truth function \( \sigma \) for \( \phi[[T], [T]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_i \) in \( \mathcal{Z}_\phi \), \( \sigma(k) \subseteq 2^n \).

Proof.

\[
2^n \models \left( (p_1 \otimes \Xi_1) \land (= (p_2, p_3) \otimes (\Xi_1 \otimes \Xi_2)) \right) [[T], [T]]
\]
Theorem

Intuitionistic disjunction $\lor$ is not uniformly definable in PD.

Proof. Suppose $\lor$ was uniformly definable in PD. Then there would exist an $n$-context $\phi[\Xi_1, \Xi_2]$ of type $\langle n, n \rangle$ such that for all PD $n$-formulas $\psi, \chi$,

$$[\phi[[\psi], [\chi]]] = [\psi \lor \chi],$$

namely, for all $n$-team $X$,

$$X \models \phi[[\psi], [\chi]] \iff X \models \psi \text{ or } X \models \chi.$$ (*)

Clearly, $\phi[\Xi_1, \Xi_2] \not=} \bot$ and by Observation, in the syntax tree $T_\phi$ of $\phi[\Xi_1, \Xi_2]$, every node labeled with $\Xi_1$ or $\Xi_2$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

Clearly, $2^n \models \top$, hence by (*) we have that

$$2^n \models \phi[[\top], [\top]],$$

thus by Lemma (II), there exists a truth function $\sigma$ for $\phi[[\top], [\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $T_\phi$, $\sigma(k) \subseteq 2^n$. 
Theorem

*Intuitionistic disjunction* \( \lor \) *is not uniformly definable in PD.*

**Proof.** Suppose \( \lor \) was uniformly definable in PD. Then there would exist an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) of type \( \langle n, n \rangle \) such that for all PD \( n \)-formulas \( \psi, \chi \),

\[
[\phi[[\psi], [\chi]]] = [\psi \lor \chi],
\]

namely, for all \( n \)-team \( X \),

\[
X \models \phi[[\psi], [\chi]] \iff X \models \psi \text{ or } X \models \chi.
\]  

\((*)\)

Clearly, \( \phi[\Xi_1, \Xi_2] \neq \bot \) and by Observation, in the syntax tree \( T_\phi \) of \( \phi[\Xi_1, \Xi_2] \), every node labeled with \( \Xi_1 \) or \( \Xi_2 \) has an ancestor node labeled with a formula of the form \( \psi \otimes \chi \).

Clearly, \( 2^n \models \top \), hence by \((*)\) we have that

\[
2^n \models \phi[[\top], [\top]],
\]

thus by Lemma (II), there exists a truth function \( \sigma \) for \( \phi[[\top], [\top]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_1 \) or \( \Xi_2 \) in \( T_\phi \), \( \sigma(k) \subsetneq 2^n \).
Theorem

Intuitionistic disjunction $\vee$ is not uniformly definable in PD.

Proof. Suppose $\vee$ was uniformly definable in PD. Then there would exist an $n$-context $\phi[\Xi_1, \Xi_2]$ of type $\langle n, n \rangle$ such that for all PD $n$-formulas $\psi, \chi$,

$$[[\phi[[\psi],[\chi]]]] = [[\psi \vee \chi]],$$

namely, for all $n$-team $X$,

$$X \models \phi[[\psi],[\chi]] \iff X \models \psi \text{ or } X \models \chi. \quad (*)$$

Clearly, $\phi[\Xi_1, \Xi_2] \neq \bot$ and by Observation, in the syntax tree $\Sigma_\phi$ of $\phi[\Xi_1, \Xi_2]$, every node labeled with $\Xi_1$ or $\Xi_2$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

Clearly, $2^n \models \top$, hence by $(*)$ we have that

$$2^n \models \phi[[\top],[\top]],$$

thus by Lemma (II), there exists a truth function $\sigma$ for $\phi[[\top],[\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $\Sigma_\phi$, $\sigma(k) \subsetneq 2^n$. 
Theorem

*Intuitionistic disjunction $\lor$ is not uniformly definable in PD.*

**Proof.** Suppose $\lor$ was uniformly definable in PD. Then there would exist an $n$-context $\phi[\Xi_1, \Xi_2]$ of type $\langle n, n \rangle$ such that for all PD $n$-formulas $\psi, \chi$,

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]],$$

namely, for all $n$-team $X$,

$$X \models \phi[[\psi], [\chi]] \iff X \models \psi \text{ or } X \models \chi. \quad (*)$$

Clearly, $\phi[\Xi_1, \Xi_2] \neq \bot$ and by Observation, in the syntax tree $\mathcal{T}_\phi$ of $\phi[\Xi_1, \Xi_2]$, every node labeled with $\Xi_1$ or $\Xi_2$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

Clearly, $2^n \models \top$, hence by (*) we have that

$$2^n \models \phi[[\top], [\top]],$$

thus by Lemma (II), there exists a truth function $\sigma$ for $\phi[[\top], [\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $\mathcal{T}_\phi$, $\sigma(k) \subsetneq 2^n$. 

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Theorem

**Intuitionistic disjunction $\lor$ is not uniformly definable in PD.**

Proof. Suppose $\lor$ was uniformly definable in PD. Then there would exist an $n$-context $\phi[\Xi_1, \Xi_2]$ of type $\langle n, n \rangle$ such that for all PD $n$-formulas $\psi, \chi$,

$$[[\phi[[\psi], [\chi]]]] = [[\psi \lor \chi]],$$

namely, for all $n$-team $X$,

$$X \models \phi[[\psi], [\chi]] \iff X \models \psi \text{ or } X \models \chi. \quad (*)$$

Clearly, $\phi[\Xi_1, \Xi_2] \neq \bot$ and by Observation, in the syntax tree $T_\phi$ of $\phi[\Xi_1, \Xi_2]$, every node labeled with $\Xi_1$ or $\Xi_2$ has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

Clearly, $2^n \models \top$, hence by $(*)$ we have that

$$2^n \models \phi[[\top], [\top]],$$

thus by Lemma (II), there exists a truth function $\sigma$ for $\phi[[\top], [\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $T_\phi$, $\sigma(k) \subsetneq 2^n$. 

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Proof (ctd). ... ... $X \models \phi[[\psi], [\chi]] \iff X \models \psi$ or $X \models \chi$.  

... ... there exists a truth function $\sigma$ for $\phi[[\top], [\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $\Xi_\phi$, $\sigma(k) \subsetneq 2^n$. 

\[ \Xi_1 \Xi_2 \]

\[ k_1 \quad a \quad b \quad c \quad k_2 \quad k_3 \]

$\Xi_1$ $\Xi_1$ $\Xi_2$
Proof (ctd). ... ... $X \models \phi[[\psi], [\chi]] \iff X \models \psi$ or $X \models \chi$. (*)

... ... there exists a truth function $\sigma$ for $\phi[[\top], [\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $\mathcal{G}_\phi$, $\sigma(k) \subset 2^n$. 

![Diagram](34/36)
Proof (ctd). ... ... $X \models \phi[[\psi], [\chi]] \iff X \models \psi$ or $X \models \chi$. 

... ... there exists a truth function $\sigma$ for $\phi[[T], [T]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $\mathcal{T}_\phi$, $\sigma(k) \subsetneq 2^n$.

Now, as $2^n \notin \sigma(k)$, we have that [Recall: $X \models \Theta_{2^n}$ iff $2^n \notin X$]

$$
\sigma(k) \models \Theta_{2^n}, \text{ i.e. } \sigma(k) \models f(k)[[\Theta_{2^n}], [\Theta_{2^n}]].
$$
Proof (ctd). ... ... \( X \models \phi[[\psi], [\chi]] \iff X \models \psi \) or \( X \models \chi \). (*)

... ... there exists a truth function \( \sigma \) for \( \phi[[\top], [\top]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_1 \) or \( \Xi_2 \) in \( \mathcal{X}_\phi \), \( \sigma(k) \subset 2^n \).

Now, as \( 2^n \not\subset \sigma(k) \), we have that

[Recall: \( X \models \Theta_{2^n} \) iff \( 2^n \not\subset X \)]

\[
\sigma(k) \models \Theta_{2^n}, \text{ i.e. } \sigma(k) \models f(k)[[[\Theta_{2^n}],[\Theta_{2^n}]]].
\]
Proof (ctd). ... ... $X \models \phi[[\psi], [\chi]] \iff X \models \psi$ or $X \models \chi$. (*)

... ... there exists a truth function $\sigma$ for $\phi[[\top], [\top]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ or $\Xi_2$ in $\mathcal{T}_\phi$, $\sigma(k) \subsetneq 2^n$.

Now, as $2^n \not\subseteq \sigma(k)$, we have that

[Recall: $X \models \Theta_{2^n}$ iff $2^n \not\subseteq X$]

$$\sigma(k) \models \Theta_{2^n}, \text{ i.e. } \sigma(k) \models f(k)[[\Theta_{2^n}], [\Theta_{2^n}]].$$

Hence, by Lemma (I), we obtain that $\sigma$ is also a truth function for $\phi[[\Theta_{2^n}], [\Theta_{2^n}]]$ over $2^n$.

\[ \sigma(k_1) \models \Xi_1[[\Theta_{2^n}], [\Theta_{2^n}]] \]

\[ \sigma(k_2) \models \Xi_1[[\Theta_{2^n}], [\Theta_{2^n}]] \]

\[ \sigma(k_3) \models \Xi_2[[\Theta_{2^n}], [\Theta_{2^n}]] \]
Proof (ctd). ... ... \( X \models \phi[[\psi], [\chi]] \iff X \models \psi \) or \( X \models \chi \). \((*)\)

... ... there exists a truth function \( \sigma \) for \( \phi[[T], [T]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_1 \) or \( \Xi_2 \) in \( \mathcal{T}_\phi \), \( \sigma(k) \subsetneq 2^n \).

Now, as \( 2^n \not\subseteq \sigma(k) \), we have that

\[
\sigma(k) \models \Theta_{2^n}, \text{ i.e. } \sigma(k) \models f(k)[[\Theta_{2^n}], [\Theta_{2^n}]].
\]

Hence, by Lemma (I), we obtain that \( \sigma \) is also a truth function for \( \phi[[\Theta_{2^n}], [\Theta_{2^n}]] \) over \( 2^n \). Therefore \( 2^n \models \phi[[\Theta_{2^n}], [\Theta_{2^n}]] \),

\[
\sigma(k_1) \models \Xi_1[[\Theta_{2^n}], [\Theta_{2^n}]] \quad \sigma(k_2) \models \Xi_1[[\Theta_{2^n}], [\Theta_{2^n}]] \quad \sigma(k_3) \models \Xi_2[[\Theta_{2^n}], [\Theta_{2^n}]]
\]
Proof (ctd). ... ... \( X \models \phi[\psi, \chi] \iff X \models \psi \) or \( X \models \chi \). \( (\ast) \)

... ... there exists a truth function \( \sigma \) for \( \phi[\top, \top] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_1 \) or \( \Xi_2 \) in \( \mathcal{F}_\phi \), \( \sigma(k) \subsetneq 2^n \).

Now, as \( 2^n \not\subseteq \sigma(k) \), we have that

\[
\sigma(k) \models \Theta_{2^n}, \text{ i.e. } \sigma(k) \models f(k)[[\Theta_{2^n}], [[\Theta_{2^n}]]].
\]

Hence, by Lemma (I), we obtain that \( \sigma \) is also a truth function for \( \phi[[\Theta_{2^n}], [[\Theta_{2^n}]]] \) over \( 2^n \). Therefore \( 2^n \models \phi[[\Theta_{2^n}], [[\Theta_{2^n}]]] \), thus by \((\ast)\), we must have that \( 2^n \models \Theta_{2^n} \), which is impossible.

\[\square\]
Observation (II)

Let \( \phi[\Xi_1, \Xi_2] \) be an \( n \)-context of type \( \langle n, n \rangle \) such that in the syntax tree \( T_\phi \) there is a node \( k \) labeled with \( \Xi_1 \), which has no ancestor node with a label of the form \( \psi \otimes \chi \). Then \( \phi[\Xi_1, \Xi_2] \) does not define the intuitionistic implication \( \to \), namely

\[
[\phi[[\psi],[\chi]]] = [\psi \to \chi]
\]

fails for some \( \textbf{PD} \ n \)-formulas \( \psi, \chi \).

Proof. Let \( \psi := \bot \) and \( \chi := \bot \). Then

\[
2^n \models \bot \to \bot, \text{ i.e., } 2^n \in [\bot \to \bot].
\]

Suppose \( 2^n \in [\phi[[\bot],[\bot]]] \) and \( \sigma \) is a truth function for \( \phi[[\bot],[\bot]] \) over \( 2^n \). By assumption, \( \sigma(k) = 2^n \), which means that \( 2^n \models \bot \); a contradiction.
Theorem

**Intuitionistic implication** $\rightarrow$ is not uniformly definable in PD.

Proof. Suppose $\rightarrow$ was uniformly definable in PD. Then there would exist an $n$-context $\phi[\Xi_1, \Xi_2]$ of type $n^2$ such that for all PD $n$-formulas $\psi, \chi$, all $n$-team $X$,

$$X \models \phi[[\psi], [\chi]] \iff X \models \psi \rightarrow \chi.$$  \hfill (**)  

Clearly, $2^n \models T \rightarrow T$, hence by (**) we have that

$$2^n \models \phi[[T], [T]],$$

thus by Observation (II) and Lemma (II), there exists a truth function $\sigma$ for $\phi[[T], [T]]$ over $2^n$ such that for all nodes $k$ labeled with $\Xi_1$ and all nodes $m$ labeled with $\Xi_2$ in $\Sigma\phi$, it holds that $\sigma(k) \subsetneq 2^n$. Since

$$\sigma(k) \models T, \text{ i.e. } \sigma(k) \models f[[T], [\Theta_{2^n}]],$$

$$\sigma(m) \models \Theta_{2^n}, \text{ i.e. } \sigma(m) \models f(k)[[T], [\Theta_{2^n}]].$$

$\sigma$ is also a truth function for $\phi[[T], [\Theta_{2^n}]]$ over $2^n$. Therefore

$2^n \models \phi[[T], [\Theta_{2^n}]]$, thus by (**), we must have that $2^n \models T \rightarrow \Theta_{2^n}$, which is impossible.
Theorem

Intuitionistic implication \( \to \) is not uniformly definable in PD.

Proof. Suppose \( \to \) was uniformly definable in PD. Then there would exist an \( n \)-context \( \phi[\Xi_1, \Xi_2] \) of type \( n^2 \) such that for all PD \( n \)-formulas \( \psi, \chi \), all \( n \)-team \( X \),

\[
X \models \phi[[\psi], [\chi]] \iff X \models \psi \to \chi. \tag{**}
\]

Clearly, \( 2^n \models \top \to \top \), hence by (**) we have that

\[
2^n \models \phi[[\top], [\top]],
\]

thus by Observation (II) and Lemma (II), there exists a truth function \( \sigma \) for \( \phi[[\top], [\top]] \) over \( 2^n \) such that for all nodes \( k \) labeled with \( \Xi_1 \) and all nodes \( m \) labeled with \( \Xi_2 \) in \( \Sigma \phi \), it holds that \( \sigma(k) \not\subseteq 2^n \). Since

\[
\sigma(k) \models \top, \text{ i.e. } \sigma(k) \models f[[\top], [\Theta_{2^n}]],
\]

\[
\sigma(m) \models \Theta_{2^n}, \text{ i.e. } \sigma(m) \models f(k)[[\top], [\Theta_{2^n}]].
\]

\( \sigma \) is also a truth function for \( \phi[[\top], [\Theta_{2^n}]] \) over \( 2^n \). Therefore \( 2^n \models \phi[[\top], [\Theta_{2^n}]] \), thus by (**), we must have that \( 2^n \models \top \to \Theta_{2^n} \), which is impossible.