# Regular Ultrafilters and Finite Square Principles

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**Abstract**

We show that many singular cardinals $\lambda$ above a strongly compact cardinal have regular ultrafilters $D$ that violate the finite square principle $\square^{fin}_{\lambda,D}$ introduced in [3]. For such ultrafilters $D$ and cardinals $\lambda$

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there are models of size $\lambda$ for which $M^\lambda/D$ is not $\lambda^{++}$-universal and elementarily equivalent models $M$ and $N$ of size $\lambda$ for which $M^\lambda/D$ and $N^\lambda/D$ are non-isomorphic. The question of the existence of such ultrafilters and models was raised in [1].

# 1 Introduction

In [3] and [4] the equivalence of a finite square principle $\Box^{\mathit{fin}}_{\lambda,D}$ with various model theoretic properties of structures of size $\lambda$ and regular ultrafilters was established.

The model theoretic properties were the following: Firstly, if $D$ is an ultrafilter, then $\Box^{\mathit{fin}}_{\lambda,D}$ is equivalent to $M^\lambda/D$ being $\lambda^{++}$-universal for each model $M$ in a vocabulary of size $\leq \lambda$. Secondly, if $2^\lambda = \lambda^+$, then $\Box^{\mathit{fin}}_{\lambda,D}$ is equivalent to $M^\lambda/D$ and $N^\lambda/D$ being isomorphic for any elementarily equivalent models $M$ and $N$ of size $\lambda$ in a vocabulary of size $\leq \lambda$. The existence of such ultrafilters and models is related to Open Problems 18 and 19 in [1].

The consistency of the failure of $\Box^{\mathit{fin}}_{\lambda,D}$ at a singular strong limit cardinal $\lambda$ was proved in [4] relative to the consistency of a supercompact cardinal. The drawback of the result in [4] was that only a regular *filter* $D$ with $\neg\Box^{\mathit{fin}}_{\lambda,D}$ was obtained, while the existence of such an *ultrafilter* $D$ would be relevant from the point of view of model theoretic consequences. When we have failure of $\Box^{\mathit{fin}}_{\lambda,D}$ for an ultrafilter, we get the failure of $\lambda^{++}$-universality of $M^\lambda/D$ for some $M$, as well as failure of isomorphism of some regular ultrapowers $M^\lambda/D$ and $N^\lambda/D$. In this paper we indeed construct regular *ultrafilters* with $\neg\Box^{\mathit{fin}}_{\lambda,D}$ for singular $\lambda$ above a strongly compact. And thus the present paper answers negatively problems 18 and 19 of [1] modulo large cardinal assumptions. The use of large cardinals is justified by [3], [4] and [6] as the failure of $\Box^{\mathit{fin}}_{\lambda,D}$ for singular strong limit $\lambda$ implies the failure of $\Box_\lambda$, which implies the consistency of large cardinals.

A filter $D$ on a set $I$ is said to be $\mu$*-regular* if there is a family $E \subseteq D$, such that $|E| = \mu$ and $\bigcap F = \emptyset$ for all infinite $F \subseteq E$. We then say that $E$ is a $\mu$*-regular* family in $D$. If $\mu = |I|$, it is omitted. Regular filters have given rise to hard set theoretical problems but at the same time they are very useful in model theory.

## 2 An equivalent condition for $\square^{fin}_{\lambda,D}$

The following finite square principle was introduced in [3]:

$\square^{fin}_{\lambda,D}$ : $D$ is a filter on a cardinal $\lambda$ and there exist finite sets $C^\xi_\alpha$ and integers $n_\xi$ for each $\alpha < \lambda^+$ and $\xi < \lambda$ such that for each $\xi, \alpha$

- **(i)** $C^\xi_\alpha \subseteq \alpha + 1$
- **(ii)** If $B \subset \lambda^+$ is a finite set of ordinals and $\alpha < \lambda^+$ is such that $B \subseteq \alpha + 1$, then $\{\xi : B \subseteq C^\xi_\alpha\} \in D$
- **(iii)** $\beta \in C^\xi_\alpha$ implies $C^\xi_\beta = C^\xi_\alpha \cap (\beta + 1)$
- **(iv)** $|C^\xi_\alpha| < n_\xi$

By results in [3] and [6], on the one hand, $\lambda^{<\lambda} = \lambda$ implies $\square^{fin}_{\lambda,D}$; and on the other hand, for singular strong limit $\lambda$ and any regular filter $D$ generated by $\lambda$ sets, $\square^{fin}_{\lambda,D}$ implies $\square^*_\lambda$ (For a definition of $\square^*_\lambda$ see [2, Section 5.1]).

Regularity is considered the ultimate denial of countable completeness of the filter: not only is *some* infinite intersection of filter-elements empty, but *every* infinite intersection of elements of the subset $E$, which is as big as the domain of the filter itself, is empty. We now introduce an even stronger denial. Suppose we have a filter on a set of size $\lambda$. The condition (1) below states the existence of longer and longer regular sequences $\{X_\alpha : \alpha < \beta\}$, $\beta < \lambda^+$, which moreover cohere. It turns out that the existence of such a sequence is equivalent with $\square^{fin}_{\lambda,D}$:

**Theorem 1** *Suppose $D$ is a filter on $\lambda$. Then the following conditions are equivalent:*

*(1) There are sets $\{B_{\alpha,\beta} : \alpha < \beta < \lambda^+\}$ in $D$ such that*

*(1.1) If $\alpha < \beta < \gamma$, then $B_{\alpha,\gamma} \cap B_{\beta,\gamma} = B_{\alpha,\beta} \cap B_{\beta,\gamma}$*

*(1.2) If $\alpha_n < \alpha_{n+1} < \beta$ for $n < \omega$, then $\bigcap_n B_{\alpha_n,\beta} = \emptyset$.*

*(1.3) If $\xi < \lambda$, then $\sup\{|\{\alpha < \beta : \xi \in B_{\alpha,\beta}\}| : \beta < \lambda^+\} < \aleph_0$.*

*(2) $\square^{fin}_{\lambda,D}$*

**Proof.** Let us first assume (1) and derive (2). For $\xi < \lambda, \alpha < \lambda^+$, let $C^\xi_\alpha = \{\beta < \alpha : \xi \in B_{\beta,\alpha}\} \cup \{\alpha\}$. We show that (i)-(iv) of $\square^{fin}_{\lambda,D}$ hold. Clause (i) holds by construction. To prove (ii), assume $X \subseteq \alpha + 1$ is finite. Note that

$$\bigcap_{\beta \in X} B_{\beta,\alpha} \subseteq \{\xi : X \subseteq C^\xi_\alpha\}.$$

Since $D$ is a filter, we get $\{\xi : X \subseteq C^\xi_\alpha\} \in D$. To prove (iii), assume $\alpha \in C^\xi_\gamma$. If $\delta \in C^\xi_\alpha$ and $\delta < \alpha$, then

$$\xi \in B_{\delta,\alpha} \cap B_{\alpha,\gamma} = B_{\delta,\gamma} \cap B_{\alpha,\gamma},$$

whence $\delta \in C^\xi_\gamma$. Conversely, if $\delta \in C^\xi_\gamma$ and $\delta < \alpha$,

$$\xi \in B_{\delta,\gamma} \cap B_{\alpha,\gamma} = B_{\delta,\alpha} \cap B_{\alpha,\gamma},$$

whence $\delta \in C^\xi_\alpha$. Finally, (iv) is true by condition (1.3) above, since the sets $C^\xi_\alpha$ are finite for each $\alpha$ and moreover, the number $\sup\{|C^\xi_\alpha| : \alpha < \lambda^+\}$ is finite.

Let us then assume (2) and prove (1). Let $B_{\alpha,\beta} = \{\xi < \lambda : \alpha \in C^\xi_\beta\}$. By (ii) of $\square^{fin}_{\lambda,D}$, $B_{\alpha,\beta} \in D$. To prove (1.1), let $\alpha < \beta < \gamma$. If $\xi \in B_{\alpha,\gamma} \cap B_{\beta,\gamma}$, then $\alpha \in C^\xi_\gamma$ and $\beta \in C^\xi_\gamma$. By (iii), $C^\xi_\beta = C^\xi_\gamma \cap (\beta + 1)$. Thus we may conclude $\alpha \in C^\xi_\beta$ and $\beta \in C^\xi_\gamma$, i.e. $\xi \in B_{\alpha,\beta} \cap B_{\beta,\gamma}$. Conversely, if $\xi \in B_{\alpha,\beta} \cap B_{\beta,\gamma}$, then $\alpha \in C^\xi_\beta$ and $\beta \in C^\xi_\gamma$. Again by (iii), $C^\xi_\beta = C^\xi_\gamma \cap (\beta + 1)$. Thus $\alpha \in C^\xi_\gamma$ and $\beta \in C^\xi_\gamma$, i.e. finally, $\xi \in B_{\alpha,\gamma} \cap B_{\beta,\gamma}$.

To prove (1.2), assume $\alpha_n < \alpha_{n+1} < \beta$ for $n < \omega$, but $\bigcap_n B_{\alpha_n,\beta} \neq \emptyset$, say $\xi \in \bigcap_n B_{\alpha_n,\beta}$. Then each $\alpha_n$ is in $C^\xi_\beta$, which is impossible because the latter is finite. $\square$

In Theorem 7 below we will construct a regular ultrafilter which does not have the strong regularity property of Theorem 1.

# 3 A partition property

We define a particular partition property $Pr_2(\lambda, \kappa)$ which turns out to be useful when we show that the ultrafilter we construct in the next section does not have $\square^{fin}_{\lambda,D}$. A similar partition property is used in [5, Theorem 6.1].

**Definition 2** *Let $Pr_2(\lambda, \kappa)$ denote the following property of $\lambda$ and $\kappa$ with $\kappa < \lambda$ (See Figure 1):*

*Suppose $c : [\lambda]^2 \to E$, where $E$ is a filter on $\kappa$. Then there is an $i < \kappa$ such that for all $\chi < \lambda$ there is an increasing sequence $\zeta_\beta$, $\beta < \chi$, of ordinals $< \lambda$ such that for all $\beta_1 < \beta_2 < \chi$ there is $\zeta > \zeta_{\beta_2}$ such that $i \in c(\{\zeta_{\beta_1}, \zeta\}) \cap c(\{\zeta_{\beta_2}, \zeta\})$*[1] .

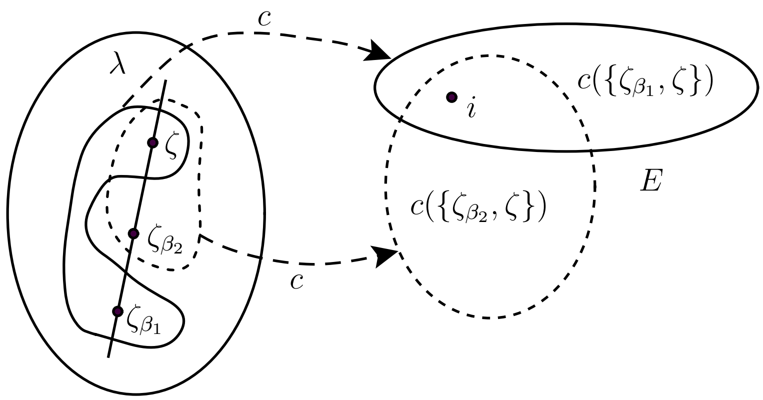


Figure 1: The partition property $Pr_2(\lambda, \kappa)$.

If $\lambda$ is weakly compact, then $Pr_2(\lambda, \kappa)$ holds for all $\kappa < \lambda$. What is interesting about $Pr_2(\lambda, \kappa)$ is that it can hold also for successor cardinals $\lambda$:

**Proposition 3** *Suppose $\kappa < \lambda$ and $\mathcal{E}$ is a $\kappa^+$-complete uniform ultrafilter on $\lambda^+$. Then $Pr_2(\lambda^+, \kappa)$.*

**Proof.** Fix $\zeta < \lambda^+$. Then $[\zeta + 1, \lambda^+) \in \mathcal{E}$. Obviously,

$$[\zeta + 1, \lambda^+) = \bigcup_{i<\kappa} b_i(\zeta) \in \mathcal{E},$$

where

$$b_i(\zeta) = \{\xi : \zeta < \xi < \lambda^+ \text{ and } i \in c(\{\zeta, \xi\})\},$$

[1]Alternatively, we could define and use the partition property $Pr'_2(\lambda, \kappa)$:

Suppose $c : [\lambda]^2 \to \kappa$. Then there is an $i < \kappa$ such that for all $\chi < \lambda$ there is an increasing sequence $\zeta_\beta$, $\beta < \chi$, of ordinals $< \lambda$ such that for all $\beta_1 < \beta_2 < \chi$ there is $\zeta > \zeta_{\beta_2}$ such that $i = c(\{\zeta_{\beta_1}, \zeta\}) = c(\{\zeta_{\beta_2}, \zeta\})$.

and the union is disjoint. Since $\mathcal{E}$ is $\kappa^+$-complete, there is $f(\zeta) < \kappa$ such that $b_{f(\zeta)}(\zeta) \in \mathcal{E}$. Since $\lambda^+$ is regular, there is a stationary $Y \subseteq \lambda^+$ such that $f \restriction Y$ is constant, which we denote $i$. If $\zeta \in Y$, let $A_\zeta = b_i(\zeta)$. Suppose $\zeta_1 < \zeta_2 \in Y$. Then $b_i(\zeta_1) \cap b_i(\zeta_2) \neq \emptyset$ in $\mathcal{E}$. Let $\zeta \in b_i(\zeta_1) \cap b_i(\zeta_2)$. By definition of $b_i$, $i \in c(\{\zeta_1, \zeta\}) \cap c(\{\zeta_2, \zeta\})$. $\square$

**Remark 4** *Proposition 3 establishes $Pr_2(\lambda^+, \kappa)$ (and $Pr'_2(\lambda^+, \kappa)$) in a particularly strong form as we get even a sequence of length $\lambda^+$ with the required weak homogeneity property.*

**Corollary 5** *Suppose $\kappa < \theta \leq \lambda$ where $\theta$ is strongly compact. Then $Pr_2(\lambda^+, \kappa)$ holds.*

**Proof.** Let $F$ be the $\lambda^+$-complete filter $\{A \subseteq \lambda^+ : |\lambda^+ \setminus A| < \lambda^+\}$. By strong compactness of $\theta$, there is a $\theta$-complete uniform ultrafilter $\mathcal{E}$ on $\lambda^+$ extending $F$. Now we use Proposition 3. $\square$

In our application we need $Pr_2(\lambda, \kappa)$ in the case $\lambda$ is a successor, in fact the successor of $\Sigma_{i<\kappa}\mu_i$, $(\mu_i)_{i<\kappa}$ increasing, and in this case it suffices to consider the case $\chi = \mu_i^{++}$.

# 4 Main result

We now use the partition property $Pr_2(\lambda^+, \kappa)$ for singular $\lambda$ of cofinality $\kappa$ to construct a regular ultrafilter $D$ on $\lambda^+$ such that $\square^{fin}_{\lambda,D}$ fails.

**Definition 6** *Suppose $\lambda = \sup_{\xi<\kappa} \lambda_\xi$, $D_\xi$ is a filter on $\lambda_\xi$ for $\xi < \kappa$, and $E$ is a filter on $\kappa$. We then define*

$$\Sigma_E D_\xi = \{A \subseteq \lambda : \{\xi : A \cap \lambda_\xi \in D_\xi\} \in E\}.$$

It is easy to see that $\Sigma_E D_\xi$ is always a filter on $\lambda$, and moreover an ultrafilter, if $E$ and each $D_\xi$ are.

**Theorem 7** *Let us assume*

(a) $Pr_2(\lambda^+, \kappa)$.

(b) $\lambda = \sup\{\lambda_\xi : \xi < \kappa\}$.

*(c)* $D_\xi$ *is a regular ultrafilter on* $\lambda_\xi$ *such that* $\lambda_\xi \setminus \bigcup_{\zeta<\xi} \lambda_\zeta \in D_\xi$.

*(e)* $E$ *is a regular ultrafilter on* $\kappa$.

*Then* $D = \Sigma_E D_\xi$ *is a regular ultrafilter on* $\lambda$ *with* $\neg\Box^{fin}_{\lambda,D}$. *Moreover,* (1.1) *and* (1.2) *of Theorem 1 cannot be satisfied.*

**Proof.** We first show that $D$ is regular. Let $\Lambda_\xi = \lambda_\xi \setminus \bigcup_{\zeta<\xi} \lambda_\zeta$. Let $\{D^\xi_\alpha : \alpha < \lambda_\xi\}$ be the regular family in $D_\xi$. W.l.o.g., $\forall \alpha < \lambda_\xi (D^\xi_\alpha \subseteq \Lambda_\xi)$. Let $\{J_\xi : \xi < \kappa\}$ be the regular family in $E$. W.l.o.g. $J_\xi \subseteq [\xi, \kappa)$. For $\alpha < \lambda_\xi$, $\xi < \kappa$, let

$$H^\xi_\alpha = \bigcup_{\beta \in J_\xi} D^\beta_\alpha.$$

Let $H^\xi$ be the set of all $H^\xi_\alpha$, $\alpha < \lambda_\xi$, and $H = \bigcup_{\xi<\kappa} H^\xi$. Clearly, $|H| = \lambda$. To prove $H \subseteq D$, let $H^\xi_\alpha \in H$. Now

$$\{\zeta < \kappa : H^\xi_\alpha \cap \lambda_\zeta \in D_\zeta\} \supseteq J_\xi.$$

Thus $H^\xi_\alpha \in D$. Finally, we show that $H$ is a regular family. For this end, suppose $M$ is an infinite subset of $H$ such that $\bigcap M \neq \emptyset$. Let $\delta \in \bigcap M$. Let $\beta$ be the unique $\beta$ for which $\delta \in \Lambda_\beta$. Let us first assume $M \subseteq H^\xi$ for some $\xi < \kappa$. Then $\delta$ is in $D^\beta_\alpha$ for infinitely many $\alpha$, contrary to the regularity of the set $\{D^\beta_\alpha : \alpha < \lambda_\beta\}$. Thus we may assume that there is an infinite set $\{\xi_n : n < \omega\}$ such that $M$ meets each $H^{\xi_n}$. So for each $n$ there is $\beta_n \in J_{\xi_n}$ and $\alpha_n < \lambda_{\xi_n}$ such that $\delta \in D^{\beta_n}_{\alpha_n}$. By the choice of $\beta$, $\beta_n = \beta$ for all $n$. But this contradicts the regularity of the set $\{J_\xi : \xi < \kappa\}$.

We assume now $D$ satisfies condition (2) of Theorem 1, and derive a contradiction. Let $B_{\alpha,\beta}$, $\alpha < \beta < \lambda^+$, be as in Theorem 1. Since $B_{\alpha,\beta} \in D$,

$$a(\alpha, \beta) =_{df} \{\xi < \kappa : B_{\alpha,\beta} \cap \mu_\xi \in D_\xi\} \in E.$$

**Claim 1:** If $\zeta_1 < \zeta_2 < \zeta_3$, then $a(\zeta_1, \zeta_3) \cap a(\zeta_2, \zeta_3) \subseteq a(\zeta_1, \zeta_2)$.

To prove the Claim, assume $\zeta_1 < \zeta_2 < \zeta_3$ and $\xi \in a(\zeta_1, \zeta_3) \cap a(\zeta_2, \zeta_3)$. Thus $B_{\zeta_1,\zeta_3} \cap \mu_\xi \in D_\xi$ and $B_{\zeta_2,\zeta_3} \cap \mu_\xi \in D_\xi$. Hence $B_{\zeta_1,\zeta_3} \cap B_{\zeta_2,\zeta_3} \cap \mu_\xi \in D_\xi$. Now we use the fact that $B_{\zeta_1,\zeta_3} \cap B_{\zeta_2,\zeta_3} = B_{\zeta_1,\zeta_2} \cap B_{\zeta_2,\zeta_3}$, i.e. (1.1) of Theorem 1, to conclude that $B_{\zeta_1,\zeta_2} \cap \mu_\xi \in D_\xi$, and thereby $\xi \in a(\zeta_1,\zeta_2)$. The Claim is proved.

Let $c(\{\zeta,\xi\}) = a(\zeta,\xi) \in E$. Recalling that $\mu_i^{++} < \lambda$, by $Pr_2(\lambda^+,\kappa)$ there are an $i < \kappa$ and an increasing sequence $\zeta_\beta$, $\beta < \chi$, $\chi = \mu_i^{++}$, of ordinals $< \lambda^+$ such that for all $\beta_1 < \beta_2 < \chi$ we have $i \in c(\{\zeta_{\beta_1},\zeta\}) \cap c(\{\zeta_{\beta_2},\zeta\})$, for some $\zeta \in [\zeta_{\beta_2},\lambda^+)$. Let $Y = \{\zeta_\beta : \beta < \chi\}$.

**Claim 2:** If $\zeta_1 < \zeta_2$ in $Y$, then $B_{\zeta_1,\zeta_2} \cap \mu_i \in D_i$, i.e. $i \in a(\zeta_1,\zeta_2)$.

To prove the Claim, assume $\zeta_1 < \zeta_2$ in $Y$. Then for some $\zeta > \zeta_1,\zeta_2$ we have $i \in a(\zeta_1,\zeta) \cap a(\zeta_2,\zeta)$. By Claim 1, $a(\zeta_1,\zeta) \cap a(\zeta_2,\zeta) \subseteq a(\zeta_1,\zeta_2)$, whence $i \in a(\zeta_1,\zeta_2)$ i.e. $B_{\zeta_1,\zeta_2} \cap \mu_i \in D_i$. The Claim is proved.

Let $\xi \in Y$ such that $|Y \cap \xi| > \mu_i$, and for $\alpha < \mu_i$ let

$$Z_\alpha = \{\zeta \in Y \cap \xi : \alpha \in B_{\zeta,\xi} \cap \mu_i\}.$$

**Claim 3:** $Y \cap \xi = \bigcup\{Z_\alpha : \alpha < \mu_i\}$.

To prove this, assume $\zeta \in Y \cap \xi$. By Claim 2, $i \in a(\zeta,\xi)$, i.e. $B_{\zeta,\xi} \cap \mu_i \in D_i$, which implies that we may pick $\alpha \in B_{\zeta,\xi} \cap \mu_i$. Now $\zeta \in Z_\alpha$. Claim 3 is proved.

As $|Y \cap \xi| > \mu_i$, there is $\alpha$ such that $Z_\alpha$ is infinite. Let $\alpha_0 < \alpha_1 < ...$ be an infinite increasing sequence in $Z_\alpha$. Then $\alpha \in \bigcap_n B_{\alpha_n,\xi}$. This contradicts (1.2) of Theorem 1, as $\bigcap_n B_{\alpha_n,\xi} = \emptyset$. $\square$

**Corollary 8** *Suppose $\theta$ is strongly compact. Then every cardinal $\lambda > \theta$ of cofinality $< \theta$ has a regular ultrafilter $D$ such that $\square^{fin}_{\lambda,D}$ fails.*

# 5 Model theory

The background of $\square^{fin}_{\lambda,D}$ is the following question, asked by Chang and Keisler as Conjecture 18 in [1]:

> Let $M$ and $N$ be structures of cardinality $\leq \lambda$ in a language of size $\leq \lambda$ and let $D$ be a regular ultrafilter over $\lambda$. If $M \equiv N$, then $M^\lambda/D \cong N^\lambda/D$.

The question is a natural one as most of the model theory regarding ultrapowers is centered on the regular ultrafilters. It is reasonable to assume GCH in this question, although it is not part of the question.

Another open problem that motivated the formulation of $\square^{fin}_{\lambda,D}$ is Conjecture 19 of [1]:

> If $D$ is a regular ultrafilter over $\lambda$, then for all infinite $M$, $M^\lambda/D$ is $\lambda^{++}$-universal.

The original motivation for the study of $\square^{fin}_{\lambda,D}$ was its equivalence with the above conjectures:

**Theorem 9 ([4])** *Assume $D$ is a regular ultrafilter on $\lambda$. Then the following conditions are equivalent:*

*(i)* $\square^{fin}_{\lambda,D}$.

*(ii) If $M$ and $N$ are elementarily equivalent models of a language of cardinality $\leq \lambda$, then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length $\lambda^+$ on $M^\lambda/D$ and $N^l/D$.*

*(v) If $M$ is a structure in a language of cardinality $\leq \lambda$, then $M^\lambda/D$ is $\lambda^{++}$-universal.*

By means of Theorem 7 we can get the relative consistency of the failure of the above conjectures:

**Corollary 10** *Suppose $\lambda$ is a singular strong limit cardinal of cofinality $\kappa$, and $Pr_2(\lambda^+,\kappa)$ holds. Then $\lambda$ has a regular ultrafilter $D$ such that for some structure $M$ in a language of cardinality $\leq \lambda$ the reduced product $M^\lambda/D$ is not $\lambda^{++}$-universal.*

**Corollary 11** *Suppose $\lambda$ is a singular strong limit cardinal of cofinality $\kappa$, and $Pr_2(\lambda^+,\kappa)$ holds. Then $\lambda$ has a regular ultrafilter $D$ such that for some elementarily equivalent structures $M$ and $N$ of cardinality $\lambda$ in a language of cardinality $\leq \lambda$, the reduced products $M^\lambda/D$ and $N^\lambda/D$ are non-isomorphic.*

In the above corollaries the vocabularies of the structures $M$ and $N$ can taken to be finite.

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