Second order logic, set theory and foundations of mathematics

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Abstract

The question, whether second order logic is a better foundation for mathematics than set theory, is addressed. The main difference between second order logic and set theory is that set theory builds up a transfinite cumulative hierarchy while second order logic stays within one application of the power sets. It is argued that in many ways this difference is illusory. More importantly, it is argued that the often stated difference, that second order logic has categorical characterizations of relevant mathematical structures, while set theory has non-standard models, amounts to no difference at all. Second order logic and set theory permit quite similar categoricity results on one hand, and similar non-standard models on the other hand.

1 Introduction

The distinction between first and second order logic did not arise as a serious matter before model theory was developed in the early part of the twentieth century. The current view, supported by model theory, is that first and second order logic are about as far from each other as it is possible to imagine. On the other hand, in a proof theoretic account first and second order logic behave very similarly, even though the latter is in general somewhat stronger than the former. Thus it seems necessary that in any discussion on second order logic and first order set theory one must be very clear whether the framework is model theoretic or proof theoretic. However, in a foundational discussion we should be able to make judgements that are free from frameworks. A framework is just a tool.

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When second order logic is thought of as a foundation of mathematics, it is nowadays taken in the model theoretic sense with reference to its power to characterize classical mathematical structures up to isomorphism and thereby capture our intuition in an exact way. This is often contrasted to the situation with the model theory of set theory, where the first order Zermelo-Fraenkel axioms have countable models and models with non-standard integers, contrary to our intuition about the set theoretic universe. So the huge distance between the model theory of second order logic and the model theory of first order logic manifests itself in discussions about foundations of mathematics, with second order logic appearing to emerge as the logic which more accurately captures the intended meaning of mathematical concepts.

I will argue below that despite their great apparent differences, second order logic and first order set theory turn out to be virtually indistinguishable as far as capturing mathematical concepts is concerned.

Another difference between second order logic and set theory is that the latter builds up a transfinite power-set hierarchy while the former settles with one layer of power-sets. This is a genuine difference which, unlike claims of categoricity, cannot be explained away. However, if second order logic is extended to third and higher order logics and eventually to type theory, this difference to set theory becomes respectively smaller. Still type theory maintains an explicit typing of objects while set theory is type-free. On the other hand, every set has a rank, an ordinal, which more or less works like a type. So the difference between type theory and set theory is really only in that set theory has an internal mechanism for generating higher and higher types while in type theory this is part of the set-up of the language. From the point of view of foundational questions this difference seems like a minute one.

2 Second order logic

The approach of second order logic to the foundations of mathematics is that mathematical propositions have the form

$$\mathfrak{A} \models \phi, \tag{1}$$

where $\mathfrak{A}$ is a structure, typically one of the classical structures such as integers or reals, and ϕ is a mathematical statement written in second order logic. This seems at least at first sight like an excellent approach since almost any statement in mathematics can be succinctly written as a second order property of one of the classical structures. An example is provided by Fermat's Last theorem (where exponentiation is a defined symbol):

$$\mathfrak{A} \models \forall x > 0 \forall y > 0 \forall z > 0 \forall n > 2 \neg(x^n + y^n = z^n), \tag{2}$$

where $\mathfrak{A} = (\mathbb{N}, +, \cdot, <)$.

If $\mathfrak{A}$ is one of the structures, such as $(\mathbb{N}, +, \cdot, <)$ or $(\mathbb{R}, +, \cdot, <, \mathbb{N})$, for which there is a second order sentence $\theta_{\mathfrak{A}}$ such that $\forall \mathfrak{B}(\mathfrak{B} \models \theta_{\mathfrak{A}} \leftrightarrow \mathfrak{B} \cong \mathfrak{A})$, then (1) can be expressed as a second order semantic logical truth

$$\models \theta_{\mathfrak{A}} \rightarrow \phi. \tag{3}$$

As was known already in the thirties, the second order semantic logical consequence relation $\models \psi$ is not axiomatizable (i.e. not r.e.). The Levy-hierarchy [4] of set theory is one possible way to measure how far $\models \psi$ is from being axiomatizable. As it turns out, $\models \psi$ is Π_2-complete —and thus a fortiori not Σ_2 —in set theory ([7]). In other words, the only way to become convinced of $\models \psi$ for a given ψ requires, in the unavoidable worst case, going through the entire set theoretical universe in search of evidence. To see what this means, suppose $\models \psi$ is written as $\forall x \exists y \Phi(x, y)$, where $\Phi(x, y)$ is Σ_0. Then to be convinced of the truth of $\models \psi$ one has to go through every set x and then look for y with $\Phi(x, y)$.

The powerful Levy Reflection Theorem [4] says that if $\Psi(x, y)$ is Σ_0, $\kappa > 0$ and $a_1, ..., a_n \in H(\kappa)$[1] such that for *some* b we have $\Psi(b, a_1, ..., a_n)$, then there is $b \in H(\kappa)$ such that $\Phi(b, a_1, ..., a_n)$. That means that if we are given $a_1, ..., a_n \in H(\kappa)$ and we want to find b such that $\Phi(b, a_1, ..., a_n)$ we need only look for such a b in $H(\kappa)$, rather than in the vast universe V of set theory.

So going back to $\models \psi$ and its Π_2-representation $\forall x \exists y \Phi(x, y)$ we first take an arbitrary x in the universe of sets and then search for y. The message of Levy's Reflection Principle is that we only need to look for y in the "neighborhood" of x (in the sense that if $x \in H(\kappa)$, $\kappa > \omega$, then we only need to look for y in $H(\kappa)$). So the good news is that we only need to hang around x, but the bad news is that x may be anywhere in the universe V of sets. For example, there is no a priori bound on the hereditary cardinality of x.

So the difference to $\models \phi$ where ϕ is first order is great for in this case we only need to look for a natural number that codes a proof of ϕ. The difference is also great to $\models \phi$ for $\phi \in L_{\omega_1\omega}$, where we only need to look for a real number that codes a proof of ϕ. Likewise with $\models \phi$ for $\phi \in L_{\omega_1\omega_1}$, where we only need[2] to look for sets of reals that code possible models of ϕ. No matter how badly behaving $L_{\omega_1\omega_1}$ otherwise is[3], at least validity in $L_{\omega_1\omega_1}$ can be checked without going further afield than sets of reals. Not so for second order logic.

It is true, to check $\models \phi$ for second order ϕ it is sufficient to check sets of hereditary cardinality less than the first supercompact cardinal (and nothing less will do) [5]. This is however a far cry from going through all natural numbers, going through all reals, or going through all sets of reals.

What does "going through" mean, for surely we cannot really "go through" all natural numbers, let alone all reals, or all sets of reals. It is all a question of how to deal with abstract objects, something where logic is supposed to help us. The great thing about natural numbers is that we can "look" at them. We can take natural numbers one by one and check what they are like. We can even imagine ourselves looking at a real number and wondering whether it codes a

[1]The set of sets that have hereditary cardinality $< \kappa$ i.e. are included in a transitive set of cardinality $< \kappa$.

[2]Thanks to the Löwenheim-Skolem Theorem of $L_{\omega_1\omega_1}$

[3]For example, its Hanf number can be bigger than the first measurable cardinal [3].

proof of something or not. But when we come to sets of real numbers there is phase transition to something infinitely more complex. We leave the world of concrete or almost concrete objects and enter the world of abstract objects.

The situation with (3) is a little simpler than with (1). In (3) one only has to search for evidence in the power-set of $\mathfrak{A}$. If $\mathfrak{A}$ is continuum-sized, then we have to search through sets of reals. What was said above about going through all sets of reals applies. It is abstract thinking.

The main result of Gödel's doctoral thesis was the Completeness Theorem for first order logic which tells us that the semantic logical consequence relation of first order logic is, in contrast to second order logic, axiomatizable in the way suggested by Hilbert and Bernays. Thus the combination of Gödel's Completeness Theorem and his Incompleteness Theorem established the sharp difference between the semantics of first and second order logics. In first order logic truth in all models can be reduced to the existence of a finite proof. In second order logic truth in all models cannot be reduced to the existence of a finite proof in any sensible way.

If (1) is the general form of a mathematical proposition, then what is the general form of a proof of (1)? Logicians have a formal concept of a proof, but we can ask more generally what is the basic form or nature of something we can assert that would make asserting (1) legitimate? An obvious answer would seem to be that if we have already grounds to assert

$$\mathfrak{A} \models \psi, \tag{4}$$

and we moreover know that every model of ψ is a model of ϕ, i.e.

$$\models \psi \to \phi, \tag{5}$$

then we can assert by the logical rule of Modus Ponens that (1) holds. But we just observed that (5) is an even more complex notion than (1). In view of our discussion above there are no rules that completely explain (5) that would be easier to present and use than (1). It is rather the opposite: we can prove in set theory that one cannot be convinced of (5) without going through the entire universe of sets (or at least up to a supercompact cardinal), while one can be convinced of (1) by "just" going through all subsets of $\mathfrak{A}$.

Of course there are *some* rules that govern (5), like

$$\models \phi(P) \to \exists X \phi(X)$$

and are used all the time even if it is known that they cannot produce all cases of (5).

There are two stronger versions of (3), namely

$$ZFC \vdash \forall \mathfrak{B}(\mathfrak{B} \models \theta_{\mathfrak{A}} \to \mathfrak{B} \models \phi) \tag{6}$$

and

$$CA \vdash \theta_{\mathfrak{A}} \to \phi, \tag{7}$$

where CA is the usual axiomatization of second order logic i.e. the comprehension schema and the axiom of choice. These two conditions are Σ^0_1-properties of ϕ. Even if one wants to assert the weaker (3) it seems reasonable to give (6) or (7) as the justification. Indeed, it is the habit of mathematicians to aim at the strongest statements, whenever they seem within reach. Also, one can justify (6) or (7) by giving a proof, which is a finite object, typically even surveyable, while justifying (3) without recourse to (6) or (7) involves dealing directly with infinite objects.

One may ask, whether indeed (3) is ever asserted with certainty without the certainty arising from knowing (6) or (7). Be this as it may, there is nothing wrong in believing (3) on the basis of (6) or (7), and the recourse to (6) or (7) does not in any way render (3) meaningless.

If we compare (6) and (7), we may observe that while (7) may be harder to prove, (6) certainly gives a shorter proof. In fact, there is no recursive function h such that the following holds: If ϕ is a second order sentence and there is a proof in ZFC of $\forall M(M \models \phi)$ with n lines, then there is a proof of ϕ of $\leq h(n)$ lines from CA.[4]

I have called (6) and (7) *stronger* forms of (3) because I take it for granted that ZFC and CA are *true* axioms. It is not the main topic of this paper to investigate how much ZFC and CA can be weakened in this or that special instance of (6) and (7), as such considerations do not differentiate second order logic and set theory from each other in any essential way.

It is not easy to give an example of a ϕ such that (1) would not hold because of (6) or (7), especially since we may simply replace ZFC and CA with stronger theories if needed, e.g. if ϕ is $Con(ZFC)$ or the Paris-Harrington sentence. When one moves to topology and measure theory more examples start to emerge. Let us consider the statement ϕ_0 stating that the Lebesgue measure has an extension to a total σ-additive measure on the reals. This is a second order statement about the structure $\mathfrak{A}_0 = (\mathcal{P}(\mathbb{R}), \in, \mathbb{R}, +\cdot, <, \mathbb{N})$ (or alternatively, a third order statement about $(\mathbb{R}, +\cdot, <, \mathbb{N})$). No answer is known for the question $\mathfrak{A}_0 \models \phi_0$ nor to the question $\mathfrak{A}_0 \models \neg\phi_0$, although trivially

$$\mathfrak{A}_0 \models \phi_0 \text{ or } \mathfrak{A}_0 \models \neg\phi_0. \tag{8}$$

This is an example of the non-classical nature of the logic of correct judgements.

No generally accepted version CA^* of axiomatization of second order logic is known which would give

$$CA^* \vdash \theta_{\mathfrak{A}_0} \rightarrow \phi, \tag{9}$$

when $\phi = \phi_0$ or $\phi = \neg\phi_0$. Still the respective version of (3) is true either for $\phi = \phi_0$ or $\phi = \neg\phi_0$. So the weaker claim (3) holds one way or the other, but at the moment we cannot strengthen this observation to (9) one way or the other. All we know is that CA^* cannot be the usual CA, since ϕ_0 violates CH but is true in a model of CA constructed from a measurable cardinal. To make progress on the question of the truth of $\mathfrak{A}_0 \models \phi_0$ one would have to study *true* extensions CA^* of CA. We have the implications

[4] Joint work with Moshe Vardi.

$$CA \vdash \theta_{\mathfrak{A}_0} \to \phi \qquad \Rightarrow \qquad CA^* \vdash \theta_{\mathfrak{A}_0} \to \phi \qquad \Rightarrow \qquad \models \theta_{\mathfrak{A}_0} \to \phi. \tag{10}$$

The current situation is roughly speaking that for trivial reasons the rightmost implicant holds for either ϕ_0 or its negation, and for totally non-trivial reasons the leftmost implicant holds for neither. A lot of effort is put into trying to establish the middle implicant for either ϕ_0 or its negation, with the right choice of CA^*. The fact that the rightmost implicant holds for either ϕ_0 or its negation is, of course, of little use unless we know which case holds.

An obvious complaint about (7) in comparison to the weaker (1) is that (1) merely asserts that ϕ should hold in the standard model, while (7) seems to assert that ϕ holds in the whole pack of "non-standard" models in addition to the "standard" model $\mathfrak{A}$. While every student of logic knows how to prove this, there is a more subtle sense in which this is not really so. To see this, let us consider $\mathfrak{A} = (\mathbb{N}, +, \cdot)$. Let us consider two versions of $\phi_{\mathfrak{A}}$, one, let us call it $\phi^1_{\mathfrak{A}}$, in the vocabulary $\{+_1, \cdot_1\}$ and the other, let us call it $\phi^2_{\mathfrak{A}}$, in the vocabulary $\{+_2, \cdot_2\}$. If CA denotes the axiomatization of second order logic in a vocabulary that includes both $\{+_1, \cdot_1\}$ and $\{+_2, \cdot_2\}$, then

$$CA \vdash (\phi^1_{\mathfrak{A}} \wedge \phi^2_{\mathfrak{A}}) \to \mathrm{Isom}_{1,2}, \tag{11}$$

where $\mathrm{Isom}_{1,2}$ denotes the statement of second order logic stating that there is a bijection f such that for all x, y: $f(x +_1 y) = f(x) +_2 f(y)$ and $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$. So in this subtle sense (7) really asserts the truth of ϕ in one and only one model, namely the standard model. ([1], [6])

There are good reasons to believe that the situation described above is not characteristic of arithmetic but applies equally to other structures that can be categorically characterized in second order logic. Curiously, (2) is an example where we do not know at the moment for sure what the right CA^* would be, although it is generally believed that CA will suffice.

Naturally, CA itself has non-standard models but they should not be the concern in connection with (7) because we are not studying CA but the structure $\mathfrak{A}$. In fact the whole concept of a model of CA is out of place here as CA is used as a medium of evidence for (3). We can convince ourselves of the correctness of the evidence by simply looking at the proof given in CA very carefully. There is no infinitistic element in this.

3 Set theory

The approach of set theory to the foundations of mathematics is that mathematical propositions have the form

$$\Phi(a), \tag{12}$$

where $\Phi(x)$ is a first order formula with variables ranging over the universe of sets, and a is a set. If we compare (1) and (12), we observe that the former is restricted to one presumably rather limited structure $\mathfrak{A}$ while (12) refers to the entire universe. This is one often quoted difference between second order logic and set theory. Second order logic takes one structure at a time and asserts second order properties about that structure, while set theory tries to govern the whole universe at a time. This observation requires two qualifications.

First, while it is true that (12) refers to the entire universe, typical mathematical propositions are really statements about some V_α such that $a \in V_\alpha$. It requires some effort to find a mathematical theorem outside the realm of set theory which could not be written as a set theoretical fact about $V_{\omega+\omega}$. Borel Determinacy is one such ([2]). But what about V_{ω_1}? In fact, (1) can be easily written in set theory as a first order property of $V_{\alpha+1}$ as soon as $\mathfrak{A} \in V_\alpha$. So we can reformulate the set theoretical approach to mathematical propositions as follows: they are of the form

$$V_\alpha \models \Phi, \tag{13}$$

where Φ is a first order sentence and α is some large enough ordinal; in most areas of mathematics we can take $\alpha \leq \omega + \omega$. This demonstrates that it is not essential in (12) that the quantifiers range over the entire universe, and there is no essential difference to (1). It is a different matter if we study set theory itself. Then it is essential that (12) is not limited to any portion of the universe. Still, if we restrict (12) to formulas $\Phi(x)$ of quantifier-rank $\leq n$ for a fixed n then there is a closed unbounded class of ordinals α such that (12) is equivalent to $V_\alpha \models \Phi(a)$.

Let us now turn to the question when can we assert justifiably (12)? The complexity of (12) is of course beyond description, as (12) is undefinable in set theory. Even the stronger

$$\forall\alpha(V_\alpha \models \Phi) \tag{14}$$

is a Π_2-complete property of (the Gödel number of) Φ. Just as with (1) there is a different stronger formulation of (12):

$$ZFC \vdash \Phi(a), \tag{15}$$

where we assume that a is a definable term. As in our above discussion on truth and provability in second order logic we can view (15) as potentially surveyable evidence for (12) without drawing the conclusion that (15) is the meaning of (12). The most famous example of a difference between (12) and (15) is the Continuum Hypothesis CH. One of the intriguing problems of set theory is to find a *true* extension ZFC^* of ZFC which decides CH. For any extension ZFC^* of ZFC we have the analogue of (10):

$$ZFC \vdash \Phi \qquad \Rightarrow \qquad ZFC^* \vdash \Phi \qquad \Rightarrow \qquad \Phi. \tag{16}$$

Here Φ can be any mathematical statement. For both CH and $\neg CH$ we know that the leftmost implicant is false. The rightmost implicant holds for CH or $\neg CH$ but we do not know for which. Kreisel seems to be the first to pay attention to this curious situation.

Discussing implications like (10) and (16) does not mean that we are anywhere near finding CA^* or ZFC^*, or even that it is unproblematic to suggest that CA^* or ZFC^* can be found. The point is that the situation is entirely *similar* in second order logic and in set theory. In both cases we equally far or equally close to a solution.

All the usual mathematical structures can be characterized up to isomorphism in set theory by appeal to their second order characterization but letting the second order variables range over sets that are subsets of the structure to be characterized.

The only difference to the approach of second order logic is that in set theory these structures are indeed explicitly defined while in second order logic they are merely described. In this respect second order logic is closer to the standard mathematical practice of not paying attention to what the "objects" e.g. complex numbers really are, as long as they obey the right rules. However, it is important also in second order logic to prove the *existence* of the structure to be characterized. After the existence has been proved, the object can be forgotten. In set theory the existence is proved by defining the object and showing that the definition is legitimate. When we move on to more counter-intuitive structures this difference disappears. Take for example the structure

$$(\mathcal{P}(\omega), <) \tag{17}$$

where $<$ is a well-order of order-type $2^{\aleph_0}$. There is a second order sentence θ such that (17) is the only model of θ, up to isomorphism. Neither second order logic nor set theory can define such a well-ordering.

If set theory is formalized with two $\in$-relations, say $\in_1$ and $\in_2$, and the ZFC axioms are adopted in the common vocabulary $\{\in_1, \in_2\}$, then the equation

$$F(x) = \{'F(y) : y \in' x\}' \tag{18}$$

defines a class function F which is an isomorphism between the $\in_1$-sets and the $\in_2$-sets. In this sense set theory, like second order logic, has *internal categoricity.*

4 Foundations of mathematics

Which is the right way to do mathematics: second order logic or set theory? Let us leave aside the question whether the higher ordinals that exist in set theory are really needed. The point is that set theory is just a "taller" version of second order logic, and if one does not need (or like) the tallness, then one can replace set theory by second (or higher) order logic. However, this does not yield more categoricity, for both second order logic and set theory are equally "internally categorical". If we look at second order logic and set theory from the outside we enter metamathematics. Then we can build formalizations of the semantics of either second order logic or set theory and prove their categoricity in "full" models as well as their non-categoricity in "Henkin" models.

But is there an "outside" position for the language of mathematics? If there is a framework where one can position oneself outside mathematics, what is that framework? If we are going to prove metamathematical results in that framework, the framework has to involve a lot of mathematics itself. It would seem natural to identify that supposedly outside framework with mathematics itself and construe the metamathematics in some other way. An alternative approach to metamathematics is that in our language of mathematics we formalize the various languages that we are interested in, including their semantics. Then we use our mathematics to prove illuminating (completeness, incompleteness, categoricity, non-categoricity, etc) results about those formal languages and about their semantics. These results tell something about our "real" language of mathematics to the extent that our formalizations reflect this "real" language. However, properties of this reflection can only be observed, never proved.

The "real" language is not a mathematical concept, even less its semantics. Only the formal languages and their semantics are mathematically defined and can be subjected to mathematical proofs.

There is no outside point of view in foundations of mathematics. Formalization does not take us outside but rather inside. Formalization does not give a more general view but a more restricted view. Therefore foundational conclusions made from mathematical results concerning formal systems have always an element of doubt. Still formalization endowed with conceptual analysis is a correct way to deepen our understanding of foundations of mathematics, and has been highly successful in explaining why some mathematical questions have turned out so difficult to solve and why some cannot be solved with current methods.

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