Second order logic or set theory?

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Structuralism
- Some results

Anti-foundationalism
- Some results

Categoricity
- Some remarks

Second order logic

Set theory

Realism

Formalism

Foundationalism

Non-standard models

Some results

Some remarks
• **Part One:**
  – Second order logic and set theory capture mathematical concepts to the same extent of categoricity.
  – Non-standard and countable models have the same role in second order logic and set theory.

• **Part Two:**
  – Second order characterizable structures have a canonical hierarchy.
  – Second order truth cannot be expressed as truth in a particular structure.
  – Understanding second order logic seems to be essentially beyond second order logic itself.
Part One

• Second order view
• Set theory view
• Catogoricity
The second order logic view

• Mathematical propositions are of the form,

\[ M \models \phi \quad (1) \]

where \( M \) is a specific mathematical structure, like the reals, Euclidean space, etc, and \( \phi \) is a second order sentence. Or of the from

\[ \models \phi \quad (2) \]
What are the specific structures?

• Specific structures are structures $M$ that have arisen from mathematical practice:

$$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n \ldots$$
What are the specific structures?

• **Specific** structures are structures $M$ that have a second (or higher) order characterization $\vartheta_M$.

\[
M \models \theta_M
\]

\[
\forall M', M'' ((M' \models \theta_M \land M'' \models \theta_M) \rightarrow M' \cong M'')
\]
What are the **specific** structures?

- **Specific** structures are structures $M$ that have a second (or higher) order characterization $\vartheta_M$.

\[ M \models \theta_M \]
\[ \forall M', M'' ((M' \models \theta_M \land M'' \models \theta_M) \rightarrow M' \cong M'') \]

\[ M \models \phi \]  
iff  
\[ \models \vartheta_M \rightarrow \phi \]  

**Case of (1)**

**Case of (2)**
• What counts as evidence for the assertion that

\[ \models \vartheta_M \rightarrow \phi \]

holds?
Evidence for $\models \mathcal{M} \rightarrow \varphi$ is a proof of $\varphi$ from $\mathcal{M}$ (and comprehension et al. axioms).
Evidence

Evidence for $\models \nu_\mathcal{M} \rightarrow \varphi$ is a proof of $\varphi$ from $\nu_\mathcal{M}$ (and comprehension et al. axioms).

The proof tells us more than just $\models \nu_\mathcal{M} \rightarrow \varphi$. If we study a formal system in which the proof is given, then $\varphi$ holds in the entire ``cloud” of models of $\nu_\mathcal{M}$ around $\mathcal{M}$. Such models are often called non-standard.
The set theory view

• Mathematical propositions are of the form,

\[ \Phi(a_1, \ldots, a_n) \]

where \( \Phi(x_1, \ldots, x_n) \) is a formula of set theory with quantifiers ranging over all sets and \( a_1, \ldots, a_n \) are some specific definable mathematical objects.

• No (1)/(2) distinction.
What are the specific objects of set theory?

• Definable objects.
• Anything one might need in mathematics:
  \[ \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n \ldots \]
  \[ \sin(x), \zeta(x), \Gamma(x) \]
  \[ \sqrt{2}, \pi, e, \log 5, \zeta(5) \]

• Not every real is definable.
• A well-order of the reals need not be definable.
The set theory view modified

• Mathematical propositions are of the form,

\[ V_\alpha \models \Phi(a_1, \ldots, a_n) \]

where \( \alpha \) is some rather large ordinal, although anything bigger than \( \omega+5 \) (or \( \omega_1 \)) is rarely needed outside set theory itself.

• First order variables actually range over \( \alpha^{\text{th}} \) order objects over the integers.
Judgements in set theory

• What counts as evidence for the assertion that

\[ \Phi(a_1, \ldots, a_n) \]

holds?
• We can use the evidence that

\[
ZFC \models \Phi(a_1, \ldots, a_n)
\]

• Of course, this tells more than the mere assertion that \( \Phi(a_1, \ldots, a_n) \) holds in the universe of sets.
Proofs and categoricity

• Categoricity is provable from Comprehension Axioms (CA) for the classical specific structures.
  ▪ Peano(S,0,S’,0’) proves isomorphism of {S,0} and {S’,0’}.
  ▪ Peano(S,0) and Peano(S’,0’) have non-isomorphic models.

• Non-standard models of CA tell us about the nature of the evidence, not about (lack of) categoricity.

• It is the same in set theory.
  ▪ ZFC(∈,∈’) proves isomorphism of {∈} and {∈’}.
  ▪ ZFC(∈) and ZFC(∈’) have non-isomorphic models.
Part Two

• Second order characterizable structures
• Their global structure
• Their existence
Recap:

\[ M \models \theta_M \]

\[ \forall M', M'' ((M' \models \theta_M \land M'' \models \theta_M) \rightarrow M' \cong M'') \]

- \( M \) is second order characterizable \( \Rightarrow \) \( |M| \) is second order characterizable.
- If \( \kappa \) is second order characterizable, then so are \( \kappa^+ \) and \( 2^\kappa \).
- The second order theory of \( 2^\kappa \) is not Turing reducible to the second order theory of \( \kappa \).
Second order characterizable structures
• If $M$ is second order characterizable, the second order theory of $M$ is $\Delta_2$.

• The second order theory of all structures is $\Pi^2_1$-complete, hence not (Turing-reducible to) the second order theory of any particular (s. o. c.) structure.
Second order characterizable structures

\[
\begin{align*}
\models & \quad \Pi_2 \\
\cdots & \\
2^{\kappa_\alpha} & \quad \Delta_2 \\
\kappa_\alpha & \quad \Delta_2 \\
\cdots & \\
2^{\kappa_0} & \quad \Delta_2 \\
\kappa_0 & \quad \Delta_2
\end{align*}
\]
• Conclusion: In second order logic truth in all structures cannot be reduced to truth in any particular specific structure.
The existence of second order characterizable structures

• The set of second order sentences that characterize some structure is not $\Pi_2$.

• Second order characterizations depend on the propositions

  $\text{"\varphi \text{ has a model"} }$.

• This is a new form of proposition. But what counts as evidence for such propositions? A proof? Of what?

• Likely choice: $\text{ZFC} \vdash \text{"\varphi \text{ has a model";} };$ leaves second order logic behind.
Complete formulas

• A second order sentence is complete if it has a model and for any second order sentence logically implies the sentence or its negation.
• Categorical sentence are complete.
• Ajtai: Axiom of Constructibility implies that complete sentences are categorical.
• Ajtai, Solovay: Consistently, there are complete sentences that are non-categorical.
• Again, “φ is complete” is not Π₂-definable.
Part One:

– Propositions of second order logic and set theory are of a different form but both refer to real mathematical objects and use proofs as evidence.
– Second order logic and set theory capture mathematical concepts such as natural and real numbers to the same extent of categoricity.
– Second order logic and set theory both have non-standard and countable models if evidence is formalized.

Part Two:

– Second order characterizable structures have a canonical hierarchy based on cardinality.
– Second order truth cannot be expressed as truth in a particular structure.
– Obtaining second order characterizable structures seems to go beyond second order logic.

Summary
• Second order logic is the $\Sigma_2$-part of set theory. Mathematics outside set theory resides there.

• As a weaker form of set theory, second order logic is an important milestone. One can develop second order model theory.

• Set theory provides a foundation for second order logic.
Thank you!