On the "Logic without Borders" Point of View*

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> *. . . as the Bhagavad-Gita teaches, one achieves knowledge and indifference at the same time.*—Andre Weil.

> *Definability is like a wormhole from one field of logic to another.*—Jouko Väänänen, 2013.

1 Introduction

Finitism, intuitionism, constructivism, formalism, predicativism, structuralism, objectivism, platonism; foundationalism, anti-foundationalism, first orderism; constructive type theory, Cantorian set theory, proof theory; *top down* principles or *building up from below*—framework commitments, that is, ideology, permeates the logician's mathematical life. Such commitments set

*This paper is based on a series of conversations with Jouko Väänänen.

in early and are—usually—final: lines are drawn in the sand, and a working life is mapped out.

Of course, not all logicians are attracted to dogma. Some are fascinated by the space between theories, by points of data downplayed by this or that theoretical stance, or left out altogether. Their approach is pantheistic and ecumenical, and, with respect to foundations in particular, opportunistic and localized. Their attitude is critical, not toward any particular logical method, but toward the idea of *omniscience*. Neutrality is not a goal in itself; border-crossing logicians are willing to take ideology seriously where they find it effective—it is just that they rarely find it so.

We might call such a perspective the "Logic without Borders" point of view. In the below we will recount some episodes in border-crossing, pieces of mathematics chosen almost arbitrarily from the work, career and conversation of the dedicatee of this volume, Jouko Väänänen.

We will see that a key concept turns out to be definability—and indeed, what is more important than the question of what we can say? As Väänänen puts it in his "Pursuing Logic without Borders":[1]

> We were persuaded by the idea that model theory, set theory and recursion theory are just different approaches to the same goal, understanding definability.

2 First Episode: Model Theory

Should model theory have borders? Countability, or more precisely, the border between the "genuinely" uncountable as opposed to the "only apparently" uncountable, separates pure model theory, i.e. that part of the subject which is relatively free of entanglement with set theory, from the rest. Logicians

[1] in this volume

on one side ask the question: why resort to set theory for studying the apparently marginal, the outlying cases, when there is so much work to do in the cases where set-theoretic methods are (in general) not needed, i.e. in the area of (sufficiently) stable and also in the area of o-minimal models? While the border-crossing logician will take a different view of the term "marginal," at least insofar as it is used as a synonym for "entangled with set theory."

In Shelah's classification theory, for example, uncountable models of a sufficiently stable first order theory[2] can be analyzed in terms of dimension-like invariants, somewhat reminiscent of analyzing a vector space in terms of the size of its basis. Since countable first order theories with infinite models have models in all uncountable cardinalities, just as there are vector spaces and algebraically closed fields of any dimension, the (only apparently) uncountable cardinalities of such models must arise from their dimensions, in a kind of stretching from the countable case. This places constraints on how complicated such models can be. On the other hand, countable complete first order theories which fail to be sufficiently stable[3] have, in every sufficiently large cardinality, models which are non-isomorphic, but they are so close to each other that one cannot imagine analyzing the models in terms of dimension-like invariants. What "so close to each other" means can be expressed in exact terms in several ways. Originally Shelah showed that such models of size κ can be $L_{\infty\kappa}$-equivalent.[4] Using the method of transfinite Ehrenfeucht-Fraïssé games and their approximations by trees[5] the original result of Shelah has been greatly improved. In recent work by Kangas-Hyttinen-Väänänen such models are constructed in suitable cardinalities which are even $L^2_{\kappa\omega}$-equivalent.[6] This is the best possible result in the sense that for the classifi-

[2] i.e. superstable, NDOP, NOTOP

[3] i.e. they are unstable, or stable but unsuperstable, or superstable with DOP or OTOP

[4] See [14].

[5] a method developed by the Helsinki Logic Group in cooperation with Saharon Shelah

[6] See [4].

able case each model of cardinality κ (for suitable κ) can be characterized in $L^2_{\kappa\omega}$ up to isomorphism.

The phrase "problematic set-theoretical content" occurs in the literature in connection with debates about the foundational role of second order logic;[7] but it also seems to have been found useful in connection with the question, which structures should one study? Following standard mathematical practice, for model theorists avoiding pathological cases—however this may be defined in a particular context—has become the rule.

Important oppositions, such as those between tame and nontame, classifiable and nonclassifiable, o-minimal or not, admitting geometric invariants or not, decidable and undecidable, or sometimes simply countable and uncountable, emerge and become harmonized with the oppositions between "nonpathological" and "set-theoretical," or, finally, "interesting" and "too general." But this is precisely where ideological borders emerge. As to undecidability, the border-crossing logician sees undecidability as a richness, a welcome elaboration of the basic picture. As for "interesting" and "too general," for the border-crossing logician "too general" is never a term of criticism, if all that is meant by "too general" is that one's reply to the question, what structures should one study? is simply "all of them."

2.1 A Remark of Sacks

Sacks expressed the conundrum thus in 1972:

> B. Dreben. . . once asked. . . "Does model theory have anything to do with logic?" It is true that model theory bears a disheartening resemblence to set theory, a fascinating branch of mathematics with little to say about fundamental logical questions, and in particular to the arithmetic of cardinals and ordinals. But the

[7] See below.

> resemblance is more of manners than of ideas, because the central notions of model theory are absolute, and absoluteness, unlike cardinality, is a logical concept.[8]

Among other things Sacks is referring to the fact that when one analyzes countable models in terms of finite partial isomorphisms, one uses countable ordinals as invariants and the set-theoretic aspect is submerged, because the concept of an ordinal is absolute.[9] In detail, consider the Ehrenfeucht-Fraïssé game of length ω between non-isomorphic countable models $\mathfrak{A}$ and $\mathfrak{B}$. Clearly player I[10] has a winning strategy because he can enumerate the models in ω moves and then he must win because no isomorphism exists. Consider now the restriction to games of finite length, not to any fixed finite length, but modifying the game by adding the clause that player I has to count down an ordinal α while he plays. This ordinal is like a clock which ticks down from the ordinal α and stops when it hits zero. He can count down only finitely many steps so the game is finite, but it is potentially infinite in the sense that there is no bound on the length of the game.[11] In general, for "small" α player I will not have a winning strategy. So how big must α be in order that I wins this harder game? A simple argument shows that if player II[12] has a winning strategy for all countable α, then II has a winning strategy in the original Ehrenfeucht-Fraïssé game of length ω, and the models are isomorphic. As the models are not isomorphic, there must be a countable α such that player II does not have a winning strategy, and then by the Gale-Stewart theorem, which implies that these games are determined, player I does have a winning strategy. The smallest such α measures the distance

[8] [11]

[9] See Scott, [12].

[10] the anti-isomorphism player, sometimes called the "spoiler"

[11] The idea of thinking of ordinals as measures of potential infinity and of trees as measures of potential countability, is presented first in [6]; see also [17].

[12] the isomorphism player, sometimes called the "duplicator"

of the models $\mathfrak{A}$ and $\mathfrak{B}$ from being isomorphic. The bigger α is the closer they are to being isomorphic. This gives a hierarchy in terms of countable ordinals. On each level α of the hierarchy there are the pairs of countable models where α is the "watershed," the boundary where the advantage in the Ehrenfeucht-Fraïssé game slides from player II to player I. With all clocks less than α player II, the "isomorphism player," is able to survive without losing, but once the clock is started from α (or bigger), player I is able to find the difference in the models and manifest the non-isomorphism by winning the game. All elements of this game are quite absolute.

The point is that this fails in uncountable models. When one uses countable partial isomorphisms to investigate uncountable models, one needs trees (an analogue of ordinals) which have non-trivial set theoretic properties. So set theory becomes entangled here with model theory.

To see this, consider the Ehrenfeucht-Fraïssé game of length ω_1 between non-isomorphic models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_1$. Clearly player I has a winning strategy, as before, because he can list the models in ω_1 moves and then he must win because no isomorphism exists. In analogy to the countable case we again modify the game by adding the clause that player I has to go up a tree, which has no uncountable branches, while he plays. This tree is like a clock which ticks up the tree and stops when the branch ends. He can go up only countably many steps so the game is countable, but it is potentially uncountable in the sense that there is no bound on the countable length of the game. As before, in general, for small trees player I will not have a winning strategy. How big must the tree be in order that I wins this harder game? It can be shown that for any non-isomorphic $\mathfrak{A}$ and $\mathfrak{B}$ there are trees such that player I wins, and of course there are trees S such that II wins (because one can start with small trees). Clearly we are thinking of trees here as analogues of ordinals. How far this analogy reaches is an interesting set-theoretical question. The structure of the class of trees with no uncountable branches is much more complicated than the structure of the class of all ordinals but as before, there is a hierarchy in terms of such trees and one can use properties of such trees to chart the area where the advantage in the Ehrenfeucht-Fraïssé game (with a tree as a clock) moves from II to I. Interestingly, there is a gray area where neither player has a winning strategy because the Gale-Stewart theorem (or Borel Determinacy) does not give determinacy for these games.

Here is an example of a non-trivial and novel set-theoretical analysis which was and is needed to work out the properties of such trees, and this has immediate implications for the model theory of uncountable structures. For example, as the work of Hyttinen, Shelah, Tuuri and others has shown, the extent to which the non-isomorphism of uncountable elementarily equivalent models can measured by trees is closely related to the stability theoretic properties of the first order theory of the models.[13]

3 Second Episode: The Symbiosis between Model Theory and Set Theory

Symbiosis is the relationship between model theory and set theory in which one on the one hand exploits set-theoretical results to prove theorems in model theory, and on the other hand, one uses model-theoretic considerations to force interesting concepts and problems in set theory out into the open. Symbiosis was developed by Väänänen in order to, as he puts it, "expose the nature of the logic"; to "uncover the set-theoretical commitments of the logic, its content, its strength, even its reference." Symbiosis signifies *co-dependence*—in the benign sense of the term—and is a form of *entanglement*. Recent debates about the foundational virtues of second order logic vs. set

[13]See [5].

theory, for example, decry the entanglement of set theory with second order logic, insofar as it is admitted to exist at all.[14] In fact, as Väänänen shows, not only is there nothing pernicious here, second order logic (denoted SOL) is actually symbiotic with set theory, even in the technical sense of the term defined below—a predicament, possibly, for those who feel compelled, on foundational grounds, to make a choice between the two formalisms.[15]

The technical definition of symbiosis is as follows. First some notation. By a "predicate" we mean a formula of set theory, typically "x is a cardinal" or "x is the power-set of y". If a predicate P is added to the language of set theory as a (definable) new symbol, then a $\Sigma_1(P)$-predicate means a Σ_1-formula in the vocabulary $\{\in, P\}$. A $\Delta_1(P)$-predicate is a $\Sigma_1(P)$-predicate $Q(x_1, \ldots, x_n)$ for which there is another $\Sigma_1(P)$-predicate $Q'(x_1, \ldots, x_n)$ such that $\forall x_1 \ldots x_n(Q(x_1, \ldots, x_n) \leftrightarrow Q'(x_1, \ldots, x_n))$ is true (in V). If φ is a sentence of a logic L^* and L' is a subset of the (many-sorted) vocabulary of φ, then the *projection* of φ to L' is the class of reducts of models of φ to L'. A model class is said to be Δ-*definable* in L^* if it is a projection of a sentence of L^* and also its complement is. Now the definition:

Definition 1. *A logic L^* is symbiotic with a predicate P of set theory if the predicate "$\varphi \in L^*$" and the predicate "$M \models_{L^*} \varphi$" are $\Sigma_1(P)$ and $\Delta_1(P)$ respectively, and in addition, a model class K_P describing P (see below) is Δ-definable in L^*.*

What symbiosis tells us about a logic L^* is that its truth predicate is "recursive" in the predicate P, in the generalized sense of being $\Delta_1(P)$. The class K_P is defined as follows:

Definition 2. *Suppose P is n-ary. The model class K_P consists of models $(M, E, a_1, ..., a_n)$ isomorphic to some $(M', \in, a'_1, ..., a'_n)$ such that M' is a*

[14] See for example [13].

[15] See below for the proof that SOL is symbiotic with set theory. See also [15].

transitive set and $P(a_1', ..., a_n')$ *holds.*

Barwise's concept of an *absolute logic* is related to symbiosis but is not the same.[16] An absolute logic as defined by Barwise requires the satisfaction predicate to be Δ_1, but without extra predicates. In the generalization of the concept introduced by Väänänen one adds the predicate P as a kind of "oracle."

For example, there is a symbiosis between the Härtig-quantifier and the predicate $x = Cd(y)$ ("the cardinality of y is x"). First of all, as Lindström showed in [9], the class of well-ordered models is a relativized reduct (i.e. a reduct in the sense of many-sorted logic) of a model class definable by means of the Härtig-quantifier, for a linear order $(A, <)$ is a well-order if and only if there are sets X_a, $a \in A$, such that for all $a, b \in A$: $a < b \leftrightarrow |X_a| < |X_b|$. The satisfaction predicate "the assignment s satisfies the formula φ in the model M" for the extension of first order logic by the Härtig-quantifier can be defined in set theory by a $\Sigma_1(Cd)$-formula. On the other hand, the class of models (up to isomorphism) of the form $(M, \in, a, b)$, where M is transitive and $a = |b|$, is Δ-definable in the extension of first order logic by the Härtig-quantifier. We get hold of transitive models $(M, \in)$ because we can characterise well-foundedness. We can make sure that $a = |b|$ because we can use the Härtig-quantifier.

Let us take an example of an application of set theory in model theory via symbiosis. One of the cornerstones of set theory is the Levy Reflection Principle, which states that the hereditarily countable sets (denoted HC) form a Σ_1 elementary submodel of V. One can now observe that if a logic L^* is symbiotic with set theory per se (i.e. without any added predicate), then it has the Löwenheim-Skolem Theorem down to $\aleph_0$. To see this, note that if $\varphi \in L^* \cap HC$ has a model, then the statement "φ has a model" is a true Σ_1-sentence of set theory, hence it is true in HC. Hence φ has

[16] See [2].

a countable model. In general, if symbiosis exists with a predicate, then we have a Löwenheim-Skolem Theorem down to κ as soon as $H(\kappa)$[17] has enough reflection. For example, we noted that there is a symbiosis between the Härtig-quantifier and the predicate $x = Cd(y)$ ("the cardinality of y is x"). Thus as soon as $H(\kappa)$, which is always a Σ_1 submodel of V, is also a Σ_1(Cd)-elementary submodel of V, we have a Downward Löwenheim-Skolem theorem for the Härtig quantifier: If $\varphi \in L(H)$ has a model, then φ has a model of cardinality $< \kappa$.

Let us consider an example in the other direction, from model theory to set theory. The compactness of the infinitary language $L_{\kappa\kappa}$ is a well-known case. This logic is symbiotic with the predicate "x is the set of sequences of length $< \kappa$ of elements of y". There are two different versions of compactness for this logic. The first says that any theory T of size κ all subsets of smaller cardinality of which have a model, has itself a model. The other drops the cardinality assumption that T has size κ. The first concept leads to the concept of a weakly compact cardinal, the second leads to the concept of strongly compact cardinal. Both concepts have become important in set theory, independently of the relation to infinitary languages.

A kind of ultimate symbiosis is the symbiosis of second order logic with the power-set operation "x is the power set of y," as was mentioned, and as is proved below.

Once one begins to look for symbiosis, one finds it everywhere! For example, as Väänänen points out, symbiosis is not limited to infinite models, but happens in the area of finite models too. To see this, recall that a class of finite models is recursive iff it is Δ-definable in the above sense in first order logic (denoted FO) in the context where all models considered are finite.

So here a computational definability concept, namely "recursive," coincides with a model-theoretic definability concept. Moreover, due to a result

[17] $H(\kappa)$ denotes the set of sets of hereditary cardinality less than κ. Thus HC is $H(\aleph_1)$.

of Fagin, a class of finite models is existential second order definable iff it is NP.[18] Finally, a class of finite ordered models is definable in Fixpoint logic iff it is PTIME.[19]

A beautiful case of symbiosis for a logic which is strictly between first order logic and second order logic is the extension of first order logic by the well-ordering quantifier

$$WxyA(x,y) \leftrightarrow \{(a,b) : A(a,b)\}\text{is a well-ordering}$$

denoted $FO(W)$. This logic is symbiotic with set theory per se, i.e. with the predicate $x = x$. This is because a model class is Δ_1 if and only if it is Δ-definable in $FO(W)$. The well-ordering quantifier is simple and yet it hits exactly the Δ_1 level in set theory.

Symbiosis often goes deeper than just the level of model classes. In many examples model-theoretic invariants, such as Löwenheim number, Hanf number, decision problems and so on have natural set theoretic characterizations. For example, the Löwenheim number of second order logic is the supremum of all Π_2-definable ordinals, and similar results hold for the Hanf number and the decision problem of second order logic. Conversely, in many cases set-theoretic invariants such as cardinals where one has reflection, have natural model theoretic applications. For example, by a result of Magidor, the first supercompact cardinal is the first cardinal where second order logic has a strong form of a Löwenheim-Skolem Theorem.[20] The symbiosis here is informal, but what is also interesting is that in each case one can look at the symbiosis more carefully and sharpen the results about Löwenheim-Skolem Theorems, Hanf numbers and so on to be able to prove symbiosis in the technical sense also.

[18]See Fagin, [3].
[19]See N. Immerman [7] and M. Vardi [19] and [8].
[20]See Magidor, [10].

4 Third Episode: The Entanglement of Second Order Logic with Set Theory

That second order logic is entangled with set theory, is, under this perspective, of no concern. Väänänen's precisification of the notion of entanglement is the above-mentioned concept *symbiosis*. The concept was isolated in his 1978 [**?**], in which the following is proved: Second order logic is symbiotic with the predicate "$y = P(x)$" (the powerset operation). This means that a model class (closed under isomorphism) is definable in set theory by a Δ_1 formula in the vocabulary which includes the binary predicate "$y = P(x)$" in addition to $\in$, if and only if the model class is Δ-definable in second order logic.

Why is second order logic symbiotic with the power-set operation? We reproduce here Väänänen's reply to this question,[21] together with some of his thoughts on the first order case:

> First of all, since second order logic is closed under negation, it is enough to show that the satisfaction predicate is Σ_1 with respect to the power-set operation. This is entirely standard. The power-set operation is needed for the semantics of the second order quantifiers. Conversely, we have to show that the class of transitive sets equipped with the power-set operation is definable in second order logic with extra predicates. The first observation is that well-foundedness, and hence transitivity, can be defined (up to isomorphism) in second order logic. After this is it easy to use a second order quantifier to say that that power-set operation on the transitive set is really the full power-set operation.
>
> In the first order case the symbiosis of FO with set theory takes

[21] personal communication

> place on the level of Kripke-Platek set theory with urelements, denoted KPU^-, without the axiom of infinity.[22] So a model class is definable in first order logic (when we think the universe of each model consisting of urelements) iff it is Δ_1 in KPU^-. The moral of the story is that first order logic is so weak that its symbiosis with set theory takes place on the level of the weak set theory KPU^-.

Another way of presenting the entanglement of second order logic with set theory is laid out by Väänänen in his 2001 and 2012 papers [16], and [18]. Here set theory and second order logic are presented as *analogs*, or reflections of each other—*avatars*, to use Michael Harris's terminology[23]—kindred logical productions crystallized by the intention of the logician as he inclines toward this or that precisification of the mathematician's natural language discourse. As Väänänen puts it in that paper:

> We study two metatheories of mathematics: first order set theory and second order logic. It is often said (e.g. in [13]), that second

[22]See J. Akkanen, [1].

[23]and Grothedieck's, Deligne's and others. Apparently the word has become a term of art in the field. Harris quotes Grothedieck on avatars:

> Inspired by certain ideas of Serre, and also by the wish to find a certain common "principle" or "motif" for the various purely algebraic "avatars" that were known, or expected, for the classical Betti cohomology of a complex algebraic variety, I had introduced towards the beginning of the 60s the notion of "motif".

and Weil (translated from the French by Harris):

> . . . those obscure analogies . . . disturbing reflections of one theory in another."

Though as Harris notes, the reference to avatars is only implicit in the Weil quote. See Harris's forthcoming "Not Merely Good, True, and Beautiful," Princeton University Press.

> order logic is better than first order set theory because it can in its full semantics axiomatize categorically N and R, while first order axiomatization of set theory admits non-standard, e.g. countable models. We show below that this difference is illusory. If second order logic is construed as our primitive logic, one cannot say whether it has full semantics or Henkin semantics, nor can we say whether it axiomatizes categorically N and R.[24] So there is no difference between the two logics: first order set theory is merely the result of extending second order logic to transfinitely high types.

The border-crossing logician does not pronounce, he proves. Which is to say, this is just the starting point of the analysis. For example, Väänänen asks, how do we recognize second order characterizable structures? Which structures are second order characterizable in the first place?[25] What does it really mean to say that a theory fails to be categorical?[26]

We briefly consider the last two questions. Recall that a theory is κ-categorical if all models of the theory of size κ are isomorphic. A key observation in [18] is this: the question whether arithmetic in its second order construal is categorical is, as it stands, *ill-defined*. The well-defined question is whether two Henkin models (of second order arithmetic) which have a common expansion to a model of the Comprehension Axiom are isomorphic. If such an extension exists, then the two models are easily proved to be isomorphic in that common extension, that is, without any set-theoretical

[24]Of course as Väänänen notes in [18], one can prove *internal* categoricity in these cases. See below.

[25]A structure M is said to be second order characterizable if there is a second order sentence θ such that M is, up to isomorphism, the only model of θ, that is, θ categorically characterizes M.

[26]p. 97, [18].

metatheory in the background.[27] If no such expansion exists, that is if the language in question is so impoverished that it cannot describe both structures, e.g. by lacking certain predicates, then the question whether the two structures are isomorphic makes no sense. Put another way, if we cannot even say what the two structures are, then how can we ask whether the one is isomorphic to the other?

Which structures are second order characterizable? Answer: almost all of the structures the mathematician encounters in his or her working life are second order characterizable. It is consistent that some are not, e.g. it is consistent that the reals with a Hamel basis, when the reals are considered as a vector space over the rationals, is not second order characterizable.[28] Second order characterizable structures actually form a hierarchy. If a structure of cardinality κ is second order characterizable, then so is κ (as a structure of the empty vocabulary), and all second order characterizable structures of the same cardinality are Turing equivalent in the sense given in [18]. In fact the following are proved in that paper:

Proposition 3. *The first inaccessible (Mahlo, weakly compact, Ramsey) cardinal is second order characterizable. If κ is the first measurable cardinal, then 2^κ is second order characterizable. All second order characterizable cardinals are below the first strong cardinal.*[29]

Proposition 4. *If κ is second order characterizable, then so are κ^+, 2^κ, $\aleph_\kappa$ and $\beth_\kappa$. More generally, if κ and λ are second order characterizable, then so is κ^λ.*

[27] This is called "internal categoricity" in [18].

[28] The result uses Cohen forcing. See [4] for details.

[29] The result concerning inaccessible cardinals is due to Zermelo, and the result on measurable cardinals is due to D. Scott. The result concerning strong cardinals is due to Magidor. The large cardinal sequence begins above ω. See [10].

Thus, as Väänänen observes, the first non-second order characterizable cardinal is a singular strong limit cardinal.

The analysis goes deeper. One can ask, of the predicate "φ is a second order characterization of a structure," what is its complexity? It is shown in [18] that the predicate is the conjunction of a Σ_2-complete and a Π_2-complete property of φ. As such, it is not itself Σ_2 or Π_2, a fact which has very important foundational consequences:

> So recognizing whether a candidate second order sentence is a second order characterization of some structure is so complex a problem that it cannot (by the Proposition above) be reduced to truth [of the form] $A \models \varphi^*$ in any particular second order characterizable structure A. It encodes a solution to propositions of the type $\not\models \varphi^*$. So in complexity it is above all the particular truths $A \models \varphi^*$ and on a par with, but not equivalent to $\models \varphi^*$. The whole framework of the second order view takes the concept of a second order characterizable structure as its starting point. In the case of familiar classical structures we can easily write the second order characterizations. But if we write down an arbitrary attempt at a second order characterization, the problem of deciding whether we were successful is in principle harder than the problem of finding what is true in the structure, if the sentence indeed characterizes some structure.

This actually supports, because it emphasizes the individuality of particular structures, forms of structuralism which specialize to the second order view as laid out in [18]—a view which is built on the idea that second order validity is not reducible to truth in any one second order characterizable structure. At the same time a weakness of the view becomes visible: does the number theorist who is looking for integer solutions to Diophantine equations really work with a completely different structure than the analyst who works with complex numbers? Are not the integers rather a substructure of the complex numbers? It seems counterintuitive that, e.g. Dedekind's embedding of the natural numbers into the integers, and of these into the rationals, and of these into the reals, and finally of these into the complex numbers, is a wrong picture.

5 Conclusion

We began with Weil's recommendation of a passage from the Bhagavad-Gita. We want to say that here too one achieves knowledge and indifference at the same time—or perhaps it would be more correct to say that for the border-crossing logician, his indifference is actually the *source* of his knowledge. To him, the fine-structural, set-theoretical focus both on logic, and on logics; the development of logical frameworks which are not so much *groundings* as systems of *avatars*; the attempt to expose matters of reference and of content, while all the while remaining unmoved by ideological pressures... *this* is the logical life worth living.

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