Outline

1. Background
2. Formal Semantics
3. Quantifiers
4. Boolean Operations on Quantifiers
5. The Square of Opposition
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- When some linguists started using semantic objects of higher types (Montague, Lewis), it gradually became clear that GQs occur frequently in natural languages: Barwise and Cooper (1981) and others.
- So a rather well studied logical tool was made available for the study of language, with the prospect of increased descriptive adequacy, and (sometimes) explanatory value.
- Conversely, linguistic issues led to new questions about quantifiers, questions that logic (sometimes) could answer.
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- Lecture 1: The idea of formal semantics, with GQ theory as an example
- Lecture 2: Basic properties of GQs
- Lecture 3: Monotonicity
- Lecture 4: Possessive quantification
- Lecture 5: Some logical matters

Not included are computational aspects: Robin Cooper's course next week.

Prerequisites: Basic FO, some set-theoretic notation. When in doubt: ASK!
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In basic GQ theory we ignore intensions and pragmatics: the theory is extensional.
The sentence

(1) Some students smoke

has the obvious phrase structure

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NP \rightarrow \text{Det} \rightarrow \text{some} \rightarrow \text{N} \rightarrow \text{student} \\
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As everyone knows, in FO this becomes

(2) \( \exists x (\text{student}(x) \land \text{smoke}(x)) \)

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The extension of student and smoke are sets (just as in FO). But what is the extension of every student, and of every?
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- There is another reason: Even if

  \( (3) \) Some students smoke

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  have the same truth conditions, one can prove (PW ch. 14) that no similar formalization is possible for the very similar sentence

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So we had better try another route.
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- Indeed this is just the old Aristotelian quantifier every, i.e. set inclusion:
  
  Every $A$ is $B$ $\iff$ $\text{every}(A, B) \iff A \subseteq B$
  
  No $A$ is $B$ $\iff$ $\text{no}(A, B) \iff A \cap B = \emptyset$

NB Actually Aristotle took every to have existential import ($A \neq \emptyset \land A \subseteq B$); we ignore that for now.

One thing is still missing (and often forgotten!): (6) Most students smoke may be true at Tsinghua University but false at Sun Yat-sen University. But most means the same in both places! So we need a parameter for the universe $M$: on each $M$, every $M$ is the subset relation, most $M$ is the majority relation, etc., between subsets of $M$. 

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Thus, we shall say that the global generalized quantifier *every* associates with each $M$ the local generalized quantifier $\text{every}_M$, which is a second-order relation over $M$. 

Similarly for *no*, *most*, etc. We say that these quantifiers have type $\langle 1, 1 \rangle$. The word "generalized" is usually dropped. We take words like *every*, *no*, *most* (in English) to denote global quantifiers. We still haven't said what NPs (*most students*, etc.) should denote. Looking at the tree for "Most students smoke" provides the clue: this denotation should result from combining the type $\langle 1, 1 \rangle$ quantifier *most* with the set of students. The result should in turn combine with the set of smokers to give something that can be true or false. We can obtain this by fixing the first argument of *most* to student. The result is a type $\langle 1 \rangle$ quantifier, which takes one set argument. In general, a global type $\langle 1 \rangle$ quantifier $Q$ associates with each $M$ a local type $\langle 1 \rangle$ quantifier $Q_M$: a set of subsets of $M$. 

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In general, a **global type** $\langle 1 \rangle$ quantifier $Q$ associates with each $M$ a local type $\langle 1 \rangle$ quantifier $Q_M$: a set of subsets of $M$. 

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Quantifiers

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**Definition**

A global type $\langle n_1, \ldots, n_k \rangle$ quantifier $Q$ associates with each $M$ a local type $\langle n_1, \ldots, n_k \rangle$ quantifier $Q_M$: a $k$-ary relation between relations (of the corresponding arities) over $M$. 
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- In English, they are denotations of Dets and NPs, respectively.
- Different syntactic categories may be used, but a productive system of expressions denoting type $\langle 1, 1 \rangle$ quantifiers seems common to all languages.
Some type $\langle 1 \rangle$ quantifiers

From logic:

- $\forall_M(B) \iff B = M$ (‘everything’)
- $\exists_M(B) \iff B \neq \emptyset$ (‘something’)
- $(\exists_{\geq 5})_M(B) \iff |B| \geq 5$ (‘at least 5 things’)
- $(Q_{\aleph_0})_M(B) \iff B$ is infinite
- $(Q_{\aleph_1})_M(B) \iff B$ is uncountable
- $(Q_{\text{even}})_M(B) \iff |B|$ is even
- $(Q^R)_M(B) \iff |B| > |M - B|$ (the Rescher quantifier)

Proper names (Montagovian individuals):

- $(I_a)_M(B) \iff a \in B$

NP denotations:

- $(\text{three}^A)_M(B) \iff |A \cap B| = 3$
- $(\text{most}^A)_M(B) \iff |A \cap B| > |A - B|$
Some type \( \langle 1, 1 \rangle \) quantifiers

From Aristotle:
- \( \text{all}_M(A, B) \iff A \subseteq B \)
- \( \text{some}_M(A, B) \iff A \cap B \neq \emptyset \)
- \( \text{no}_M(A, B) \iff A \cap B = \emptyset \)

From model theory:
- \( I_M(A, B) \iff |A| = |B| \) (the H"artig quantifier)
- \( \text{more}_M(A, B) \iff |A| > |B| \)

From natural language:
- \( \text{between two and five}_M(A, B) \iff 2 \leq |A \cap B| \leq 5 \)
- \( \text{finitely many}_M(A, B) \iff A \cap B \text{ is finite} \)
- \( \text{all but at most three}_M(A, B) \iff |A - B| \leq 3 \)
- \( \text{most}_M(A, B) \iff |A \cap B| > |A - B| \)
- \( \text{more than } p/q \text{ of the}_M(A, B) \iff |A \cap B|/|A| > p/q \)
- \( \text{the ten}_M(A, B) \iff |A| = 10 \& A \subseteq B \)
- \( \text{no } - \text{ except John}_M(A, B) \iff A \cap B = \{j\} \)
- \( \text{Mary’s}_M(A, B) \iff \emptyset \neq \{b \in A : R(m, b)\} \subseteq B \quad (R(x, y) \text{ iff } x \text{ ‘owns’ } y) \)
- \( \text{at least two of every girl’s}_M(A, B) \iff \text{girl} \cap \{a : \exists b \in A \ R(a, b)\} \subseteq \{a : |\{b \in A : R(a, b)\} \cap B| \geq 2\} \)
Boolean operations: conjunction and disjunction

There are obvious ways to form conjunctions and disjunctions of quantifiers. E.g. in the type \( \langle 1, 1 \rangle \) case:

- \( (Q \land Q')_M(A, B) \iff Q_M(A, B) \text{ and } Q'_M(A, B) \)
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(Discussion in PW chs. 3.2.3, 4.3, and 4.5.5.) Compare:

(9) *Not most students smoke
(10) At most half of the students smoke
Languages have different ways of expressing negation, but logically there are two distinct ways of negating a quantifier.
Boolean operations: negation

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- Here is the type $\langle 1, 1 \rangle$ case:

**Definition**

(a) outer negation: $(\neg Q)_M(A, B) \iff \neg Q_M(A, B)$

(b) inner negation: $(Q\neg)_M(A, B) \iff Q_M(A, M - B)$ (VP negation)

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- NB \(\neg\neg Q = Q\neg\neg = (Q^d)^d = Q\), so \(\neg(Q^d) = Q\neg\), \((Q\neg)^d = \neg Q\), etc.
Boolean Operations on Quantifiers

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- Compare

  (11) Not every student passed the exam
  (12) Every student didn’t pass the exam (ambiguous)
  (13) It is not the case that some student passed the exam
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\[ 0 = \neg 1 \]

where 1 is the **trivially true** quantifier \( 1_M(A, B) \) for all \( A, B \subseteq M \).
The (modern) square of opposition

Q and its negations can be arranged in a familiar way:

- **all** and **no** are **inner negation**
- **dual** and **not all** are **outer negation**
- **some** and **dual** are **dual negation**
Aristotle’s square is slightly different:

- **A** (affirmative) - universal
- **E** (negative) - no
- **I** (subaltern) - some
- **O** (not all) - not all

**Contradictory**

**Contrary**

**Subcontrary**

**Subaltern**
Comparing the squares

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- The difference is clear as soon as we consider squares of other quantifiers.
Another square

all but at most five $A$ are $B$ \hspace{1cm} at most five $A$ are $B$
\[ |A - B| \leq 5 \hspace{1cm} |A \cap B| \leq 5 \]

at least six $A$ are $B$ \hspace{1cm} “all but at least six $A$ are $B$”
\[ |A \cap B| \geq 6 \hspace{1cm} |A - B| \geq 6 \]

NB For example, *all but at most five* $(A, B)$ and *at most five* $(A, B)$ are not contraries if $|A| < 10$.

More on the square

Define:

\[ \text{square}(Q) = \{ Q, \neg Q, Q\neg, Q^d \} \]

Fact

(a) \( \text{square}(0) = \text{square}(1) = \{0, 1\} \).
(b) If \( Q \) is non-trivial, so are the other quantifiers in its square.
(c) If \( Q' \in \text{square}(Q) \), then \( \text{square}(Q) = \text{square}(Q') \). So any two squares are either identical or disjoint.
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- These are equivalent! (Exercise; we come back to a principled explanation)
Type $\langle 1, 1 \rangle$ vs. type $\langle 1 \rangle$ quantifiers

- We can go from $\langle 1, 1 \rangle$ to type $\langle 1 \rangle$ by fixing or freezing the first argument. PW ch. 4.5.5.2 argue that this should be defined as follows:
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  - This is a common operation in logic: introduce an extra (set) argument that serves as the universe. But there are many examples from language:
    - $all = \forall^{rel}$
    - $some = \exists^{rel}$
    - $at least five = \exists_{\geq 5}^{rel}$
    - $most = (Q^R)^{rel}$ ($Q^R$ is the Rescher quantifier)
    - $infinitely many = \exists_{\mathbb{N}_0}^{rel}$
    - $an even number of = Q_{even}^{rel}$
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$$\chi_R(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } R(x_1, \ldots, x_n) \\ 0 & \text{otherwise} \end{cases}$$
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E.g. wh-questions can be construed as functions whose values are sets rather than truth values: “Which students passed the exam?”