

2d inverse cascade: renormalization group theory versus numerical experiments

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and

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Introduction

- i Kolmogorov-Kraichnan's theory of 2-d turbulence

Outline

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- ii Renormalization group theory of 2-d turbulence

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full story: Phys. Rev. Lett. **99**, (2007), 144502.

Velocity and Vorticity in 2d

The Navier Stokes equation

$$(\partial_t + v \cdot \partial)v^\alpha - \nu \partial^2 v^\alpha = -\partial^\alpha P + f^\alpha, \quad \alpha = 1, 2$$

$$\partial \cdot v = 0$$

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in *two dimensions* transports the **vorticity** field

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Conservation of vorticity moments in the inviscid limit.

Kraichnan's picture of 2d turbulence

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x v^2 = -2\nu \int d^2x \omega^2 \equiv -2\nu \Omega^2$$

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- Energy conservation \implies energy transfer towards **small wavenumbers**.
- Enstrophy dissipation at **large wavenumbers**.

Kàrmàn-Howarth-Monin equation

Gaussian, time short-correlated translational invariant forcing

$$\langle f^\alpha(x, t) f_\alpha(y, s) \rangle = \delta(t - s) F(x - y, m)$$

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the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\begin{aligned} \frac{1}{2} \langle (\partial \cdot \delta v)(x) \delta v(x)^2 \rangle &= \\ &= \partial_t \langle v(x) \cdot v(0) \rangle - F(x, m) - 2\nu \langle (\partial_\alpha v)(x) (\partial^\alpha v)(0) \rangle \end{aligned}$$

Hypotheses encoding Kraichnan's theory:

- i velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \rightarrow 0} \partial_t \langle v(x) \cdot v(0) \rangle = I_{\mathcal{E}}$$

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- iii No **energy** dissipative anomalies occur:

$$\left\{ \lim_{\nu \downarrow 0} \lim_{x \downarrow 0} - \lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \right\} \nu \langle \partial_\alpha v^\beta(x, t) \partial^\alpha v_\beta(0, t) \rangle = 0$$

KHM equation and "mean field" scaling

$$\frac{1}{2} \partial_\mu S_3^\mu = \partial_t \langle v(x) \cdot v(0) \rangle - F(x, m) - 2\nu \langle (\partial_\alpha v)(x) (\partial^\alpha v)(0) \rangle$$

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$$\frac{1}{2} \partial_{\mu} S_3^{\mu} = \partial_t \langle v(x) \cdot v(0) \rangle - F(x, m) - 2\nu D_2(x)$$

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Inverse Energy Cascade

$$\langle \delta v_{\parallel}^3 \rangle = \langle \delta v_{\parallel} \delta v_{\perp}^2 \rangle \stackrel{mx \gg 1}{=} \frac{3 I_\varepsilon x}{2} \quad \text{mean field} \quad \Rightarrow \quad \delta v \sim x^{1/3}$$

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Direct Enstrophy Cascade

$$\langle \delta v_{\parallel}^3 \rangle = \langle \delta v_{\parallel} \delta v_{\perp}^2 \rangle \stackrel{\frac{x}{\ell} \ll mx \ll 1}{=} \frac{I_\Omega x^3}{8} \quad \text{mean field} \quad \delta v \sim x$$

Energy spectrum

$$\mathcal{E}(k) = \int \frac{d^d p}{(2\pi)^d} \delta(|k| - |p|) \int d^d x e^{ip \cdot x} \prec v(x) \cdot v(0) \succ$$

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G. Boffetta,
 J. Fluid Mech.
589, 253 (2007).

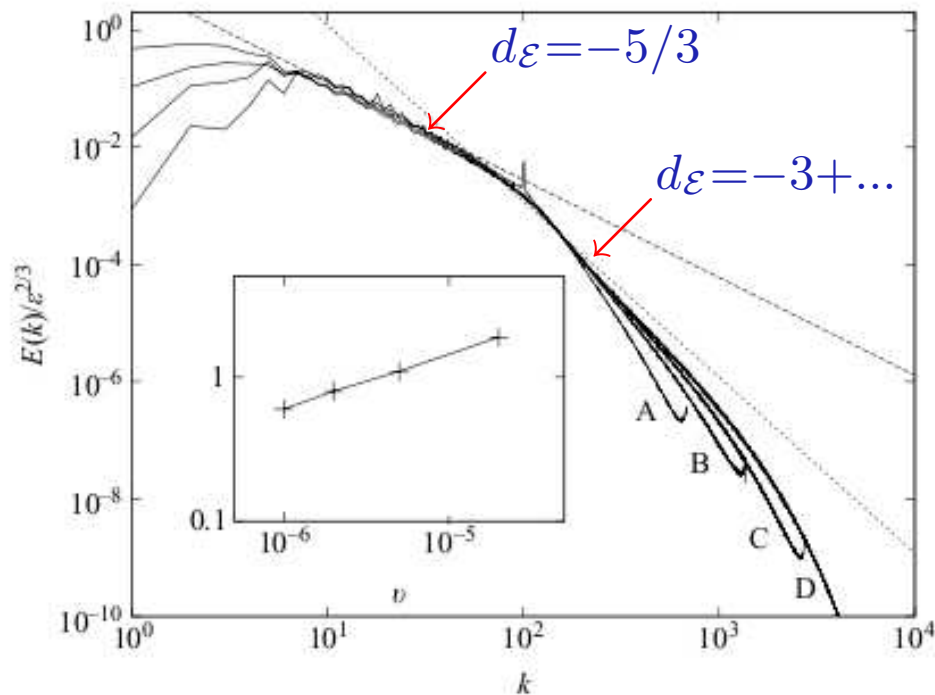


FIGURE 2. Energy spectra for the two simulations for the different resolutions (labels as in figure 1). Dashed and dotted lines represent the two predictions $Ck^{-5/3}$ with $C=6$ and k^{-3} respectively. Inset: correction δ to the Kraichnan exponent -3 as a function of viscosity, computed by fitting the spectra with a power law $k^{-(3+\delta)}$ in the range $100 \leq k \leq 400$.

UV Renormalization Group analysis

P. Olla (1991),(1994)
Honkonen et al. (1998)

Power law forcing

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with

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with *power law spectrum* $d = 2$

$$\check{F}(p) = \frac{g_1 \nu^3 h_1(p, M, m)}{p^{d-4+2\varepsilon}}$$

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- τ seems to play no role for RG !

RG prediction

Energy spectrum

$$\mathcal{E}(q) = \varepsilon^{1/3} g_1^{2/3} \nu^2 q^{1 - \frac{4\varepsilon}{3}} R \left[\varepsilon, \frac{m}{q}, \left(\frac{q_b}{q} \right)^{2 - \frac{2\varepsilon}{3}} \right] \quad (\star)$$

$$q_b \propto \left[\frac{\varepsilon}{\nu^3 \tau^3} \right]^{\frac{1}{6 - 2\varepsilon}}$$

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If R has a limit for $q \downarrow 0$ the scaling prediction of (\star) coincides with the **scale by scale** balance prediction

$$d_v - d_t = -\frac{d_t}{2} - d_x (2 - \varepsilon)$$

$$2 d_v - d_x = -\frac{d_t}{2} - d_x (2 - \varepsilon)$$

RG prediction: 3d case

The energy spectrum prediction

$$\mathcal{E}(q) \sim q^{1 - \frac{4\varepsilon}{3}} \quad \varepsilon \leq 2$$

appears consistent with numerics in *3d*:

- A. Sain, Manu and R. Pandit, Phys. Rev. Lett. **81**, 4377 (1998).
- L. Biferale, A. Lanotte and F. Toschi, Phys. Rev. Lett. **92**, 094503 (2004).

Numerics

Model

We numerically integrate the *vorticity* equation

$$\omega = \epsilon_{\alpha\beta} \partial^\alpha v^\beta$$

$$\partial_t \omega + v \cdot \partial \omega = (-1)^{p+1} \nu \Delta^p \omega + \frac{(-1)^{q+1}}{\tau} \Delta^{-q} \omega + f_\omega$$

- Resolution up to 2048^2

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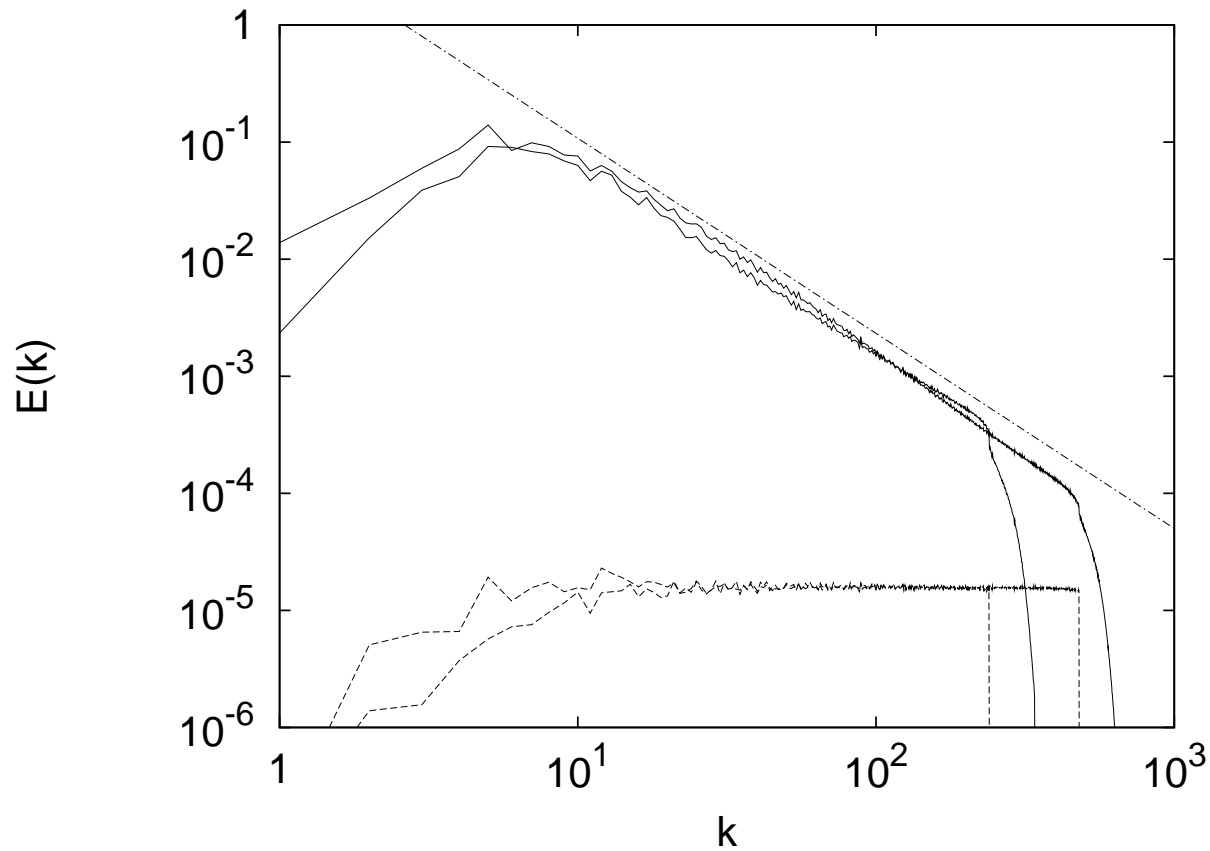
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- $(p, q) = (4, 0)$ for $\varepsilon \leq 2$
- $(p, q) = (4, 1)$ for $\varepsilon \geq 3$ to inquire the inverse cascade.

Numerical Energy Spectra

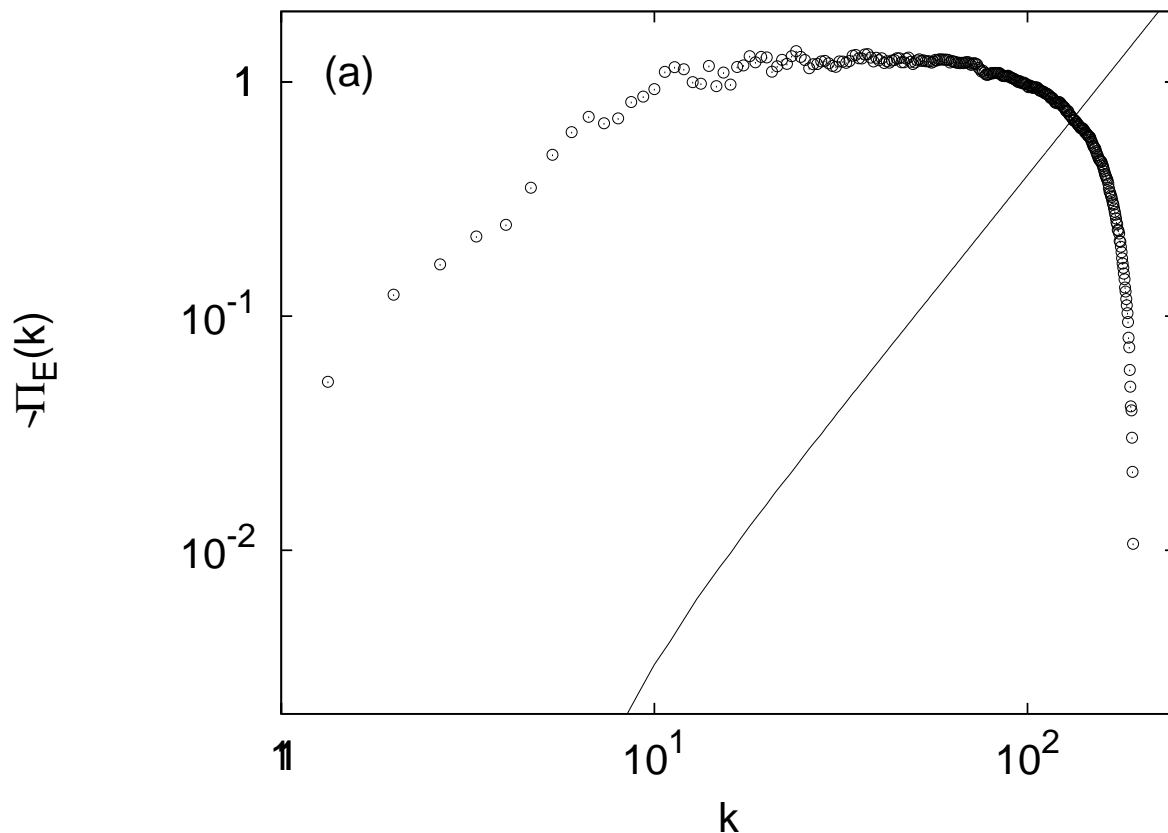
“Small scale forcing” $\varepsilon \leq 2$



Energy flux at $\varepsilon = 1$

Energy flux for $\varepsilon = 1$ and integrated *energy input*

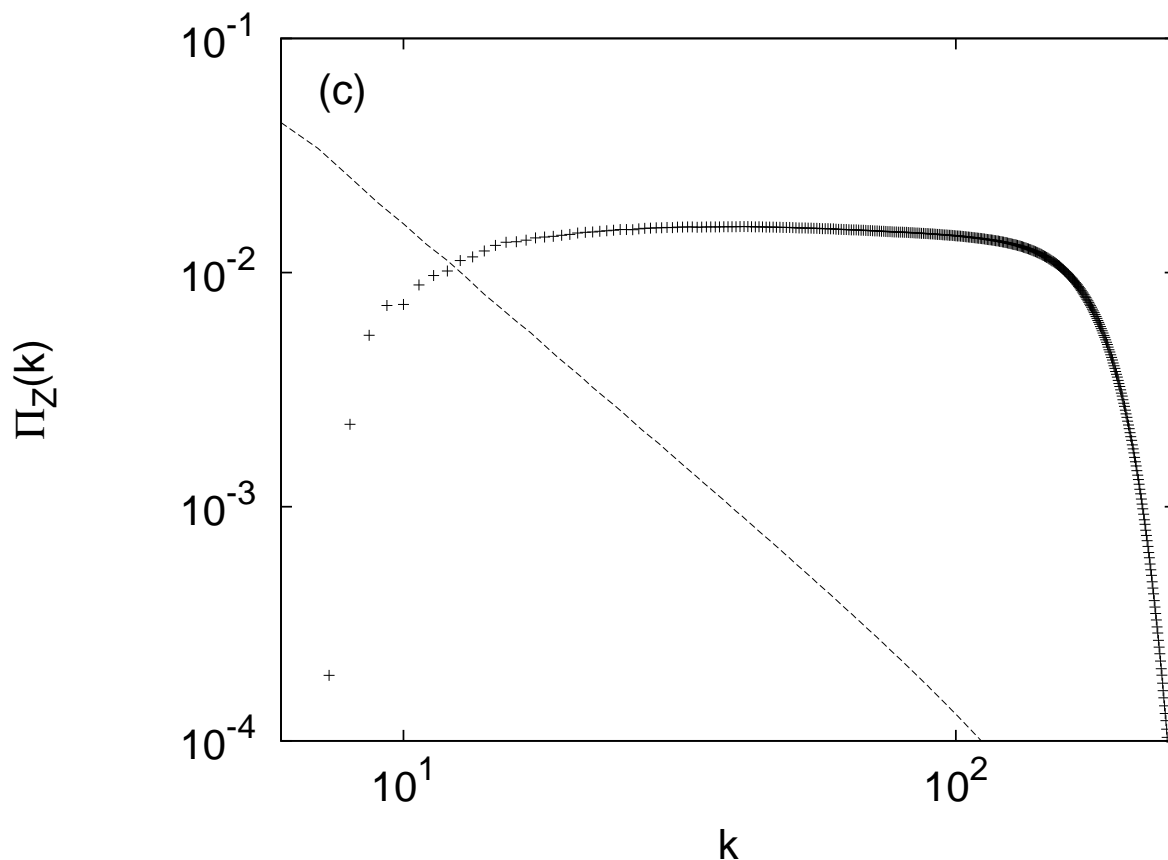
$$\int_{k \leq p} \frac{d^d k}{(2\pi)^d} \frac{1}{k^{d-4+2\varepsilon}} \Big|_{d=2, \varepsilon=1} \sim p^2$$



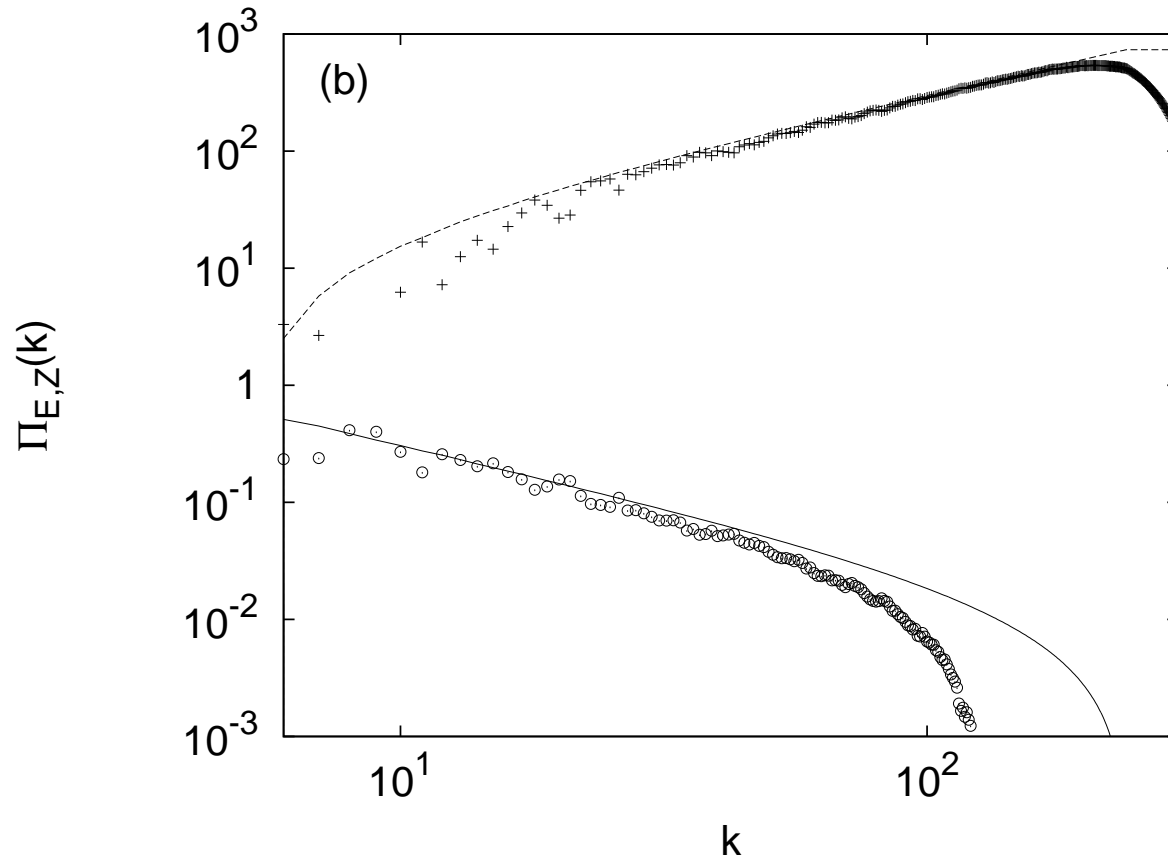
Enstrophy flux at $\varepsilon = 4$

Enstrophy flux for $\varepsilon = 4$ and integrated *enstrophy input*

$$\int_{k \geq p} \frac{d^d k}{(2\pi)^d} \frac{k^2}{k^{d-4+2\varepsilon}} \Big|_{d=2, \varepsilon=4} \sim p^{-2}$$



$$\varepsilon = 2.5$$



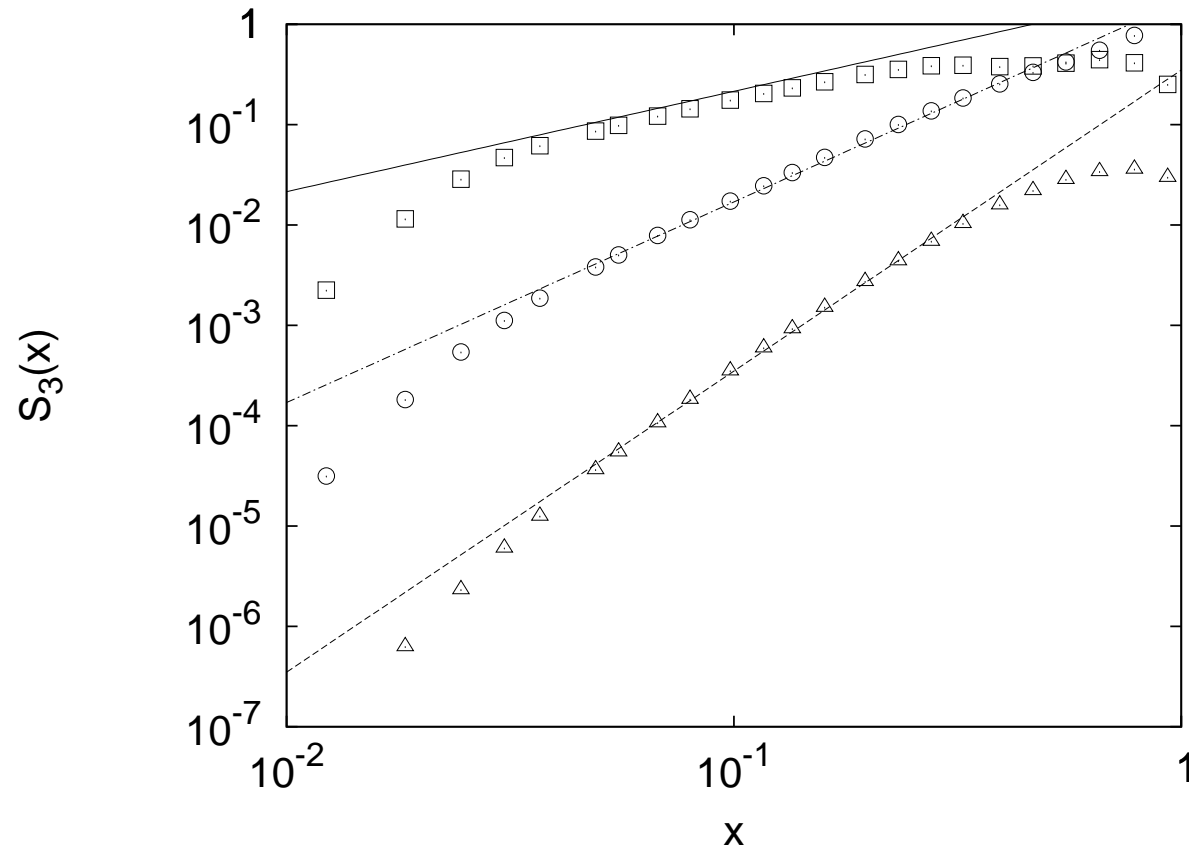
● circles: \mathcal{E} -flux

line: integrated \mathcal{E} -input

● crosses: Ω -flux

dashes: integrated Ω -input

Third order velocity structure functions



- $\epsilon = 1$ (squares): $\delta v \sim x^{1/3}$ (inverse cascade).
- $\epsilon = 2.5$ (circles): $\delta v \sim x^{-1 + \frac{2}{3}}$ (dimensional).
- $\epsilon = 4$ (triangles): $\delta v \sim x$ (direct cascade).

KHM for power law forcing: $0 \leq \varepsilon \leq 2$

$$-\frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle =$$
$$\frac{\Sigma_2}{(2\pi)^2} \frac{F_0 A(-4 + 2\varepsilon, 2)}{(4 - 2\varepsilon) x^{4-2\varepsilon}} - \partial_t \langle v_\beta(x) v^\beta(0) \rangle$$

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By hypotheses (i), (ii)

$$\partial_t \langle v_\beta(x) v^\beta(0) \rangle = \frac{\Sigma_2}{(2\pi)^2} \frac{M^{4-2\varepsilon} \bar{F}_{4-2\varepsilon}}{4 - 2\varepsilon}$$

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The ratio between forcing and energy injection rate is

$$\frac{\partial_t \langle v_\beta(x) v^\beta(0) \rangle}{F^\alpha_\alpha(x)} \propto (Mx)^{4-2\varepsilon} \gg 1$$

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$$\langle \delta v_{\parallel}^3 \rangle \simeq M^{4-2\varepsilon} F_0 c'_1 x + \frac{c'_2 F_0}{x^{3-2\varepsilon}} + \dots$$

KHM for power law forcing: $2 < \varepsilon < 3$

$$\partial_t \langle v_\beta(x) v^\beta(0) \rangle - \frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle = \alpha$$

$$\frac{m^{4-2\varepsilon} \bar{F}_{4-2\varepsilon}}{(2\varepsilon-4)} + \frac{m^{4-2\varepsilon} (mx)^2 \bar{F}_{6-2\varepsilon}}{4(6-2\varepsilon)} - \frac{F_0 A(2\varepsilon-4, 2) x^{2\varepsilon-4}}{(2\varepsilon-4)} + \dots$$

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The injection rate is

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The KHM reduces to

$$-\frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle = \frac{\Sigma_2}{(2\pi)^2} \left\{ \frac{m^{4-2\varepsilon} (mx)^2 \bar{F}_{6-2\varepsilon}}{4(6-2\varepsilon)} - \frac{F_0 A(2\varepsilon-4, 2) x^{2\varepsilon-4}}{(2\varepsilon-4)} + \dots \right\}$$

KHM for power law forcing: $2 < \varepsilon < 3$

$$\partial_t \langle v_\beta(x) v^\beta(0) \rangle - \frac{1}{2} \partial_\mu \langle \delta v^\mu(x) \delta \mathbf{v}^2(x) \rangle \propto \alpha$$

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The structure function scales as

$$\langle \delta v_{\parallel}^3 \rangle \stackrel{(mx) \ll 1}{\simeq} F_0 c_2' x^{2\varepsilon-3} \{ 1 + c_1' (mx)^{6-2\varepsilon} + \dots \}$$

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Conjecture

Unusual bifurcation of RG fixed points ?

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- Critical Phenomena for depinning of interfaces.
- Interfaces subject only to thermal fluctuations.
- Upper critical dimension $d_u = 3$.
- $d < d_u$: two lines of fixed point coalescing at $d = 3$ into a **line** of "drifting" fixed points:

$$\mathcal{R} \{ f^*(l) \} = \mathcal{R}(f^*(l - \delta^* l)) \quad (d = 3)$$

Lipowski-Fisher bifurcation

$$\mathcal{R}\{f^*(l)\} = f^*(l) \quad d < 3$$

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- $d = 3$: $\zeta = 0$ **marginal** perturbation.
- $d = 3$: **stationary** fixed points can be ruled out.

Conclusions

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RG:

- No obvious O.P.E. corrects the prediction.
- Fixed point does not bifurcate from the Gaussian in the marginal limit.