Infra-red renormalisation group for turbulence

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and

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A general conjecture

H.A. Rose and P.L. Sulem (1978)
G. Eyink and N. Goldenfeld (1994)

Critical Phenomena vs Turbulence

Critical Phenomena	**Turbulence**
UV cutoff M	*Integral scale* m^{-1}
Inverse correlation length	*Viscous scale*
$T - T_c$	*Viscosity* ν
Scaling regime	*Inertial range*
Anomalous conservation laws	*Dissipative anomaly*

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Do turbulent phenomena look like critical phenomena once we interchanged short and long distances or position and Fourier spaces?

Wilson's renormalisation (semi)-group

construction of the probability distribution of
scaling fields
by coarse graining degrees of freedom

Coarse graining

- Fields are stochastic variables with distribution

$$P(\phi) \propto e^{-\mathcal{A}} \qquad \mathcal{A} \equiv \mathcal{A}(\phi)$$

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- Fluctuations are integrated out

$$\mathcal{R}_\lambda \mathcal{A} = \ln \int \mathcal{D}[\delta\phi] e^{-\mathcal{A}(\tilde{\phi}+\delta\phi)}$$

Fixed point

- The original momentum support is restored

$$\tilde{\phi}(x,t) \quad \Rightarrow \quad \tilde{\phi}_\lambda(x,t) = \lambda^{d_\phi}\tilde{\phi}(\lambda^{d_x}x, \lambda^{d_t}t)$$

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if the limit

$$\lim_{\lambda\to\lambda_\star} \mathcal{R}_\lambda\left[\mathcal{A}(\phi_\lambda) + \delta\mathcal{A}(\phi_\lambda)\right] = \mathcal{A}_\star(\phi)$$

exists and is finite for some scaling dimensions d.

Scaling operators

General field functionals are renormalised according to

$$\mathcal{R}_\lambda O = \frac{\int \mathcal{D}[\delta\phi] O(\tilde{\phi}_{(1/\lambda)} + \delta\phi) e^{-\mathcal{A}(\tilde{\phi}_{(1/\lambda)} + \delta\phi)}}{\int \mathcal{D}[\delta\phi] e^{-\mathcal{A}(\tilde{\phi}_{(1/\lambda)} + \delta\phi)}}$$

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$$\lim_{\lambda\to\lambda_\star} \lambda^{-d_O} \mathcal{R}_\lambda O(\lambda^{d_x} x, \lambda^{d_t} t) = O_\star(x,t)$$

exists and it is finite with scaling dimension d_O

Infra-red vs. ultra-violet RG

ultra-violet renormalisation

Scaling at small momenta: $\tilde{\phi}$ has support for momenta in $[0; \lambda M]$ as $\lambda \downarrow 0$.

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infra-red renormalisation

Scaling at large momenta:$\tilde{\phi}$ has support for momenta in $[\lambda m, \infty]$ as $\lambda \uparrow \infty$.

Kraichnan model

Definition of the model

The passive scalar equation is

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The Kraichnan model is defined by

$$\prec f(x_1, t_1) f(x_2, t_2) \succ = \delta(t_{12}) F(x_{12})$$
$$\prec v^{\alpha}(x_1, t_1) v^{\beta}(x_2, t_2) \succ = \delta(t_{12}) D^{\alpha\,\beta}(x_{12})$$

Scaling properties

Forcing and velocity correlations are required to satisfy

$$\lim_{\lambda\uparrow\infty} \lambda^{\xi}\left[D^{\alpha\,\beta}(0) - D^{\alpha\,\beta}(x/\lambda)\right] = d_{\star}^{\alpha\,\beta}(x)$$

$$\lim_{\lambda\uparrow\infty} F(x/\lambda) = F(0)$$

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The onset of an inertial range requires scale separation

$$\frac{M}{m} >> 1 \quad \text{velocity field}$$

$$\max\left\{ \frac{1}{M}, \left(\frac{\kappa}{D}\right)^{\frac{1}{\xi}} \right\} << \min\left\{ \frac{1}{m}, \frac{1}{m_F} \right\}$$

Inertial range

The spatial structure of the velocity field is

$$D^{\alpha\,\beta}(x) = D_0 \int_{M>|q|>m} \frac{d^d q}{(2\pi)^d} \frac{e^{\imath q\cdot x}}{q^{d+\xi}} \left[\delta^{\alpha\,\beta} - \frac{q^\alpha q^\beta}{q^2}\right]$$

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The velocity correlation is well defined as M is set to infinity

$$D^{\alpha\beta}(x) \sim D m^{-\xi}\,\delta^{\alpha\beta} - D c(\xi)\,|x|^{\xi}\,T^{\alpha\beta}(\hat{x},\xi)$$

$$T^{\alpha\beta}(\hat{x},\xi) = \delta^{\alpha\beta} - \frac{\xi}{d-1+\xi}\frac{x^\alpha\,x^\beta}{x^2}$$

Hopf's equations

Due to the δ-correlation of the velocity field, equal time correlation functions

$$\mathcal{C}_{2n}(x_1, \ldots, x_{2n}; m, M) := \prec \prod_{i=1}^{2n} \theta(x_i, t) \succ$$

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At zero molecular viscosity κ

$$[\mathcal{M}_{2n}] = [x^{\xi} \partial^2]$$

Zero modes

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i.e. to the first residues of the Mellin transform

$$\tilde{C}_{2n}(\{x_i\}_{i=1}^{2n}; z) = \int_0^\infty dw\, \frac{C_{2n}(\{w x_i\}_{i=1}^{2\,n})}{w^z}$$

Anomalous scaling

Structure functions in the inertial range

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exhibit anomalous scaling (*Gawedzki and Kupiainen (1995), Chertkov et al. (1995)*)

$$\mathcal{S}_{2n}(x; m, M) = |x|^{n(2-\xi)} (m\,|x|)^{-\rho_{2n}} s_{2n}(mx, Mx)$$

with

$$\lim_{m \downarrow 0} \lim_{M \uparrow \infty} s_{2n}(mx, Mx) = s^{\star}_{2n} = \text{finite}$$

Field theory

The Martin Siggia Rose action of the Kraichnan model

$$\mathcal{A} = -\imath \langle \bar{\theta} \,, \left(\partial_t + v \cdot \partial - \frac{\nu}{2} \partial^2 \right) \theta \rangle$$

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- $\frac{\delta\theta(x,t)}{\delta f(y,s)} \overset{\text{law}}{\sim} \theta(x,t)\bar{\theta}(y,s)$ (response field)

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- $\theta(x, t)$ (original field)
- $\frac{\delta \theta(x,t)}{\delta f(y,s)} \overset{\text{law}}{\sim} \theta(x, t) \bar{\theta}(y, s)$ (response field)
- Ito discretisation implies
  1. $\prec \theta(x, t) \bar{\theta}(y, t) \succ = 0$
  2. $\nu = \kappa + D(m^{-\xi} - M^{-\xi})$

What I.R. renormalisation does?

- Assume Gaussian scaling

$$d_x = -1 \qquad d_t = -2$$
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- The IRG-flow of the action at zero forcing tends to the *non-local fixed point*

$$\lim_{\lambda\uparrow\infty} \mathcal{R}_\lambda \mathcal{A}\big|_{u=0} =$$
$$-\imath \, \langle \bar\theta \, , \left(\partial_t - \frac{\varkappa\, \lambda^\xi}{2} \partial^2 \right) \theta \rangle + \frac{g\, \varkappa}{2} \frac{\lambda^\xi - 1}{\xi} \langle \bar\theta \partial_\alpha \theta \, , \bar\theta \partial^\alpha \theta \rangle$$

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- Relevant terms in the action describe the advection of a scalar field by a Gaussian velocity field constant in space and δ-correlated in time

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- General response functions factor in products of Gaussians times exponentials of the $\mathcal{M}_{2n}$

E.G. the 2-points response function has the form $(t_2 > t_1)$

$$\prec \prod_{i=1}^{2n} \theta(x_i, t_i)\bar{\theta}(y_i, s_i) \succ = H(s_2 - t_1) \prod_{i=1}^{2} e^{\mathcal{M}_1(t_i - s_i)}$$
$$+H(t_1 - s_2)H(s_2 - s_1)e^{\mathcal{M}_1(t_2 - t_1)}e^{\mathcal{M}_2(t_1 - s_2)}e^{\mathcal{M}_1(s_2 - s_1)}$$
$$+ \dots$$

Scaling observables

- The forcing contribution to the structure functions coincides with its fluctuations

$$S_{2n}(x,t) = [\theta(x,t) - \theta(0,t)]^{2n} \langle f\,, \bar{\theta} \rangle^{2n}$$

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$$S_{2n}(x,t) = [\theta(x,t) - \theta(0,t)]^{2n} \langle f\, , \bar{\theta} \rangle^{2n}$$

ii They are IRG scaling operators

$$\lim_{\lambda \uparrow \infty} \lambda^{-\zeta_{2n}} \mathcal{R}_{\lambda} S_{2n} \left(\lambda^{d_x} x, \lambda^{d_t} t \right) = S^{\star}_{2n} \left(x, t \right)$$

$$\zeta_{2n} = n(2-\xi) - \frac{2\, n\, (n-1)}{d+2} + O(\xi)$$

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- large scale forcing does not fluctuate under the U.V. renormalisation group flow of the structure functions

$$\mathcal{S}_{2n}(x,t) \overset{\text{R.G. flow}}{\sim} [\theta(x,t) - \theta(0,t)]^{2\,n}$$

Scaling and U.V. singularities

- At finite U.V. cut-off M:

$$\lim_{|x|\downarrow 0} \frac{\prec [\theta(x,t) - \theta(0,t)]^{2\,n} \succ}{|x|^{2n}}$$

$$= \lim_{|x|\downarrow 0} (m\,|x|)^{-\rho_{2\,n}}\; s_{2n}(m\,x, M\,x)$$

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ii The existence of inertial range implies

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What U.V. renormalisation does ?

iii it follows

$$\lim_{|x| \downarrow 0} \frac{\mathcal{S}_{2n}(x; m, M)}{|x|^{2n}} \propto M^{n\xi} \left(\frac{M}{m} \right)^{\rho_{2n}}$$

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iv U.V. R.G. studies the blow up rate with M of the local field functionals

$$G_{2n} = [x^{\alpha}\partial_{\alpha}\theta(0,t)]^{2n}$$

Antonov et al (1998)

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- Isotropic and anisotropic scaling exponents recovered within second order accuracy.