

Turbulent passive advection: shapes, geometry and multiscaling in the inertial and in the large scale decay ranges.

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Part I

Turbulence

Navier–Stokes and passive advection equations

► Navier–Stokes

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\mathbf{v} - \nu \partial_{\mathbf{x}}^2 \mathbf{v} = \mathbf{f}_v - \frac{1}{\rho} \partial_{\mathbf{x}} P$$

$$\partial_{\mathbf{x}} \cdot \mathbf{v} = 0$$

Navier–Stokes and passive advection equations

▶ Navier–Stokes

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$$\partial_{\mathbf{x}} \cdot \mathbf{v} = 0$$

▶ passive advection

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\theta - \kappa \partial_{\mathbf{x}}^2 \theta = f$$

Navier–Stokes and passive advection equations

▶ Navier–Stokes

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\mathbf{v} - \nu \partial_{\mathbf{x}}^2 \mathbf{v} = \mathbf{f}_v - \frac{1}{\rho} \partial_{\mathbf{x}} P$$

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▶ passive advection

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}})\theta - \kappa \partial_{\mathbf{x}}^2 \theta = f$$

▶ Reynolds, Prandtl & Péclet numbers

$$\text{Re}_L := \frac{L V}{\nu}$$

$$\text{Pr} := \frac{\nu}{\kappa}$$

$$\text{Pe}_L := \text{Re}_L \text{Pr}$$

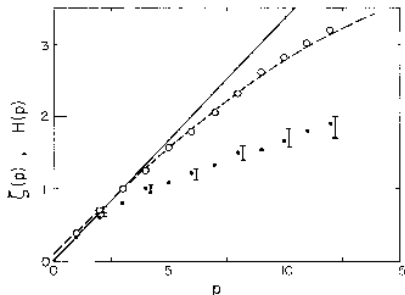
Intermittency and Multiscaling ?

$$\langle [\hat{\mathbf{r}} \cdot (\mathbf{v}(\mathbf{r}, t) - \mathbf{v}(\mathbf{0}, t))]^p \rangle \sim r^{\zeta(p)}$$

(circles)

$$\langle [(\theta(\mathbf{r}, t) - \theta(\mathbf{0}, t))]^p \rangle \sim r^{H(p)}$$

(bars)



M. H. Jensen, G. Paladin and A. Vulpiani Phys. Rev. **A** 45, 7214 - 7221 (1992).

K41 theory

- ▶ Stochastic δ -correlated forcing for the velocity field

$$\langle f^\alpha(\mathbf{x}, t) f^\beta(\mathbf{y}, s) \rangle = \delta(t - s) F^{\alpha\beta}(\mathbf{x} - \mathbf{y}), \quad d \geq 3$$

K41 theory

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- ▶ Kármán-Howarth-Monin equation $\delta v(\mathbf{x}, t) := v(\mathbf{x}, t) - v(0, t)$

$$\left(\partial_t - 2\nu \partial^2 \right) \langle v^\alpha(\mathbf{x}, t) v_\alpha(0, t) \rangle$$

$$- \frac{1}{2} \partial_\mu \langle \|\delta v\|^2(\mathbf{x}, t) \delta v^\mu(\mathbf{x}, t) \rangle = F_\alpha^\alpha(\mathbf{x})$$

K41 theory

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- ▶ Kármán-Howarth-Monin equation $\delta\mathbf{v}(\mathbf{x}, t) := \mathbf{v}(\mathbf{x}, t) - \mathbf{v}(0, t)$

$$\begin{aligned} (\partial_t - 2\nu\partial^2) \langle v^\alpha(\mathbf{x}, t) v_\alpha(0, t) \rangle \\ - \frac{1}{2} \partial_\mu \langle \|\delta\mathbf{v}\|^2(\mathbf{x}, t) \delta v^\mu(\mathbf{x}, t) \rangle = F_\alpha^\alpha(\mathbf{x}) \end{aligned}$$

- ▶ Kolmogorov's exact results

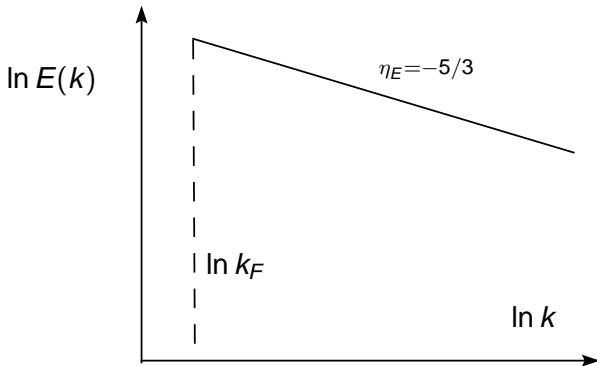
$$\lim_{\nu \downarrow 0} \lim_{x \downarrow 0} \nu \langle (\partial_\beta v^\alpha)(\mathbf{x}, t) (\partial^\beta v_\alpha)(0, t) \rangle \stackrel{\text{dissipative anomaly}}{=} \frac{F_\alpha^\alpha(0)}{2}$$

$$\lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \langle [\hat{\mathbf{x}} \cdot \delta\mathbf{v}(\mathbf{x}, t)]^3 \rangle = -\frac{6 F_\alpha^\alpha(0) x}{d(d+2)}, \quad d > 2$$

K41 and the energy "cascade" picture

Energy spectrum

$$E(q) := \int \frac{d^d k}{(2\pi)^d} \delta(q - |k|) \int d^d x e^{-i k \cdot x} C_2^\alpha(x), \quad d \geq 3$$



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Part II

Kraichnan model

Definition of the model, Obukhov (1948), Kraichnan (1968), (1994).

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$$(\partial_t + \mathbf{v} \diamond \partial_{\mathbf{x}}) \theta - \frac{\kappa}{2} \partial_{\mathbf{x}}^2 \theta = f$$

$$\partial_{\mathbf{x}} \cdot \mathbf{v} = \langle f \rangle = \langle \mathbf{v} \rangle = 0$$

$$\langle v^\alpha(\mathbf{x}_1, t) v^\beta(\mathbf{x}_2, t_2) \rangle = \delta(t_{12}) D^{\alpha\beta}(\mathbf{x}_{12}, m)$$

$$\langle f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) \rangle = \delta(t_{12}) F(m_f \mathbf{x}_{12})$$

Definition of the model, Obukhov (1948), Kraichnan (1968), (1994).

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Ito representation of the SDE

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) \theta - \frac{\kappa_{eff.}}{2} \partial_{\mathbf{x}}^2 \theta = f \quad \text{Ito form}$$

$$\kappa_{eff.} = \kappa + \frac{D_{\alpha}^{\alpha}(0)}{d} \quad \text{Taylor formula}$$

Consequences of time δ -correlation

- ▶ Galilean invariance of the statistics.

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Consequences of time δ -correlation

- ▶ Galilean invariance of the statistics.
- ▶ Closed hierarchy of equations for the **equal time correlations**

$$\left(\partial_t - \frac{\kappa}{2} \sum_{i=1}^n \partial_{\mathbf{x}_i}^2 - \mathcal{M}_n \right) \langle \theta(\mathbf{x}_1, t) \dots \theta(\mathbf{x}_n, t) \rangle =$$

$$\frac{1}{2} \sum_{lk} F(\mathbf{x}_{lk}) \langle \theta(\mathbf{x}_1, t) \dots \theta(\mathbf{x}_l, t) \dots, \theta(\mathbf{x}_k, t) \dots \theta(\mathbf{x}_n, t) \rangle$$

Consequences of time δ -correlation

- ▶ Galilean invariance of the statistics.
- ▶ Closed hierarchy of equations for the **equal time correlations**

$$\left(\partial_t - \frac{\kappa}{2} \sum_{i=1}^n \partial_{\mathbf{x}_i}^2 - \mathcal{M}_n\right) \mathbf{C}_n = \frac{1}{2} \mathbf{F} \otimes \mathbf{C}_{n-2}$$

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- ▶ Hypoelliptic "Hopf" operators

$$\mathcal{M}_n = \sum_{i \neq j=1}^n d^{\alpha\beta}(\mathbf{x}_{ij}) \frac{\partial}{\partial \mathbf{x}_i^\alpha} \frac{\partial}{\partial \mathbf{x}_j^\beta}$$

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- ▶ Structure tensor of the velocity field

$$d^{\alpha\beta}(\mathbf{x}) := D^{\alpha\beta}(0) - D^{\alpha\beta}(\mathbf{x})$$

Infinite inertial range of the advection

$$\text{Re} \gg 1 \quad \& \quad \text{Pr} \ll 1$$

$$\lim_{m \downarrow 0} d^{\alpha\beta}(\mathbf{x}) = D_1 x^\xi \mathcal{I}^{\alpha\beta}(\hat{\mathbf{x}}, \xi) + O(mx)^2$$

$\mathcal{I}^{\alpha\beta}$ incompressibility projector

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Inertial range of the passive scalar

$$\ell = \left(\frac{\kappa}{D_1} \right)^{\frac{1}{\xi}} \ll x \ll \text{Min} \left(\frac{1}{m}, \frac{1}{m_f} \right)$$

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Decay: thermal equilibrium range ?

$$\frac{1}{m_f} \ll x \ll \frac{1}{m}$$

Exact results

- ▶ Existence and uniqueness of solutions Hakulinen (2000)
relaxing to a **steady state**, assuming:

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Exact results

- ▶ Existence and uniqueness of solutions Hakulinen (2000) relaxing to a **steady state**, assuming:
 1. *vanishing* κ
 2. self-similar advection.
 3. **physically relevant** boundary conditions.
- ▶ Existence of homogeneous translational invariant **zero modes**

$$\mathcal{M}_n Z(\mathbf{x}_1, \dots, \mathbf{x}_n) = 0$$

$$Z(\lambda \mathbf{x}_1, \dots, \lambda \mathbf{x}_n) = \lambda^{\zeta_n} Z(\mathbf{x}_1, \dots, \mathbf{x}_n)$$

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- ▶ Zero modes yield inertial range asymptotics of the solutions as residues of the Mellin transform

$$\tilde{\mathcal{C}}_{n;z}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_0^\infty \frac{dw}{w} \frac{\mathcal{C}(w \mathbf{x}_1, \dots, w \mathbf{x}_n; m_f, m)}{w^z}$$

Two point correlation

► Hopf equation

$$-d^{\alpha\beta}(\mathbf{x})\partial_{\alpha}\partial_{\beta}C_2(\mathbf{x}, m_f) = F(m_f\mathbf{x})$$

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Two point correlation

- ▶ Hopf equation

$$-d^{\alpha\beta}(\mathbf{x})\partial_{\alpha}\partial_{\beta}C_2(\mathbf{x}, m_f) = F(m_f\mathbf{x})$$

- ▶ Mellin transform:

$$\begin{aligned} \int_0^{\infty} \frac{dw}{w} \frac{1}{w^z} \left[-d^{\alpha\beta}(w\mathbf{x}) \frac{\partial}{\partial w\mathbf{x}^{\alpha}} \frac{\partial}{\partial w\mathbf{x}^{\beta}} C_2(w\mathbf{x}, m_f) \right] \\ = \int_0^{\infty} \frac{dw}{w} \frac{1}{w^z} F(wm_f\mathbf{x}) \end{aligned}$$

Two point correlation

- ▶ Hopf equation

$$-d^{\alpha\beta}(\mathbf{x})\partial_{\alpha}\partial_{\beta}C_2(\mathbf{x}, m_f) = F(m_f\mathbf{x})$$

- ▶ Mellin transform:

$$d^{\alpha\beta}(\mathbf{x})\partial_{\alpha}\partial_{\beta}(m_f\mathbf{x})^{z+2-\xi}\tilde{C}(z+2-\xi) = \frac{(m_f\mathbf{x})^z\tilde{F}(z)}{-m_f^{2-\xi}}$$

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- ▶ Poles and zero modes

$$C_2(\mathbf{x}, m_f) \propto$$

$$\frac{1}{m_f^{2-\xi}} \sum_{j\mathbf{l}} \int_{z^*-i\infty}^{z^*+i\infty} \frac{dz}{(2\pi i)} \frac{\tilde{F}_{j,\mathbf{l}}(z-2+\xi)(m_f\mathbf{x})^z Y_{j\mathbf{l}}(\hat{\mathbf{x}})}{(z-\zeta_2^{(s.s.)}(j))(z-\zeta_2^{(l.s.)}(j))}$$

Asymptotic analysis of the two point correlation

$$C_2(\mathbf{x}, m_f) \propto$$

$$m_f^{-2+\xi} \sum_{j\mathbf{l}} \int_{z^*-l\infty}^{z^*+l\infty} \frac{dz}{(2\pi l)} \frac{\tilde{F}_{j,\mathbf{l}}(z-2+\xi) (m_f \mathbf{x})^z Y_{j\mathbf{l}}(\hat{\mathbf{x}})}{(z - \zeta_2^{(s.s.)}(j))(z - \zeta_2^{(l.s.)}(j))}$$

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Asymptotic analysis of the two point correlation

$C_2(\mathbf{x}, m_f) \propto$ Canonical dimension

$$m_f^{-2+\xi} \sum_{j|l} \int_{z^*-l\infty}^{z^*+l\infty} \frac{dz}{(2\pi l)} \frac{\tilde{F}_{j,l}(z-2+\xi) (m_f \mathbf{x})^z Y_{j|l}(\hat{\mathbf{x}})}{(z - \zeta_2^{(s.s.)}(j))(z - \zeta_2^{(l.s.)}(j))}$$

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- Small scales (inertial range)

$$\zeta_2^{(s.s.)}(j) = \frac{2-d-\xi}{2} \left\{ 1 - \sqrt{1 + \frac{4j(d-2+j)(d+\xi-1)}{(d-1)(d+\xi-2)^2}} \right\}$$

$$\xrightarrow{\xi \downarrow 0} j + \frac{j(j-1)\xi}{(d-1)(d-2+2j)} + O(\xi^2)$$

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- ▶ Small scales (inertial range)

$$C_2(\mathbf{x}, m_f) \simeq m_f^{-2+\xi} - c x^{2-\xi} F(0) + \dots$$

Asymptotic analysis of the two point correlation

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$$\zeta_2^{(l.s.)}(j) = -\frac{d+\xi-2}{2} \left\{ 1 + \sqrt{(d+\xi-2)^2 + \frac{4j(d-2+j)(d+\xi-1)}{d-1}} \right\}$$

$$\xrightarrow{\xi \downarrow 0} (2-d-j) \left\{ 1 + \frac{(d+j-1)\xi}{(d-1)(d+2j-2)} \right\} + O(\xi^2)$$

Asymptotic analysis of the two point correlation

$$C_2(\mathbf{x}, m_f) \propto$$

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$$C_2(\mathbf{x}, m_f) \simeq x^{2-\xi} (m_f x)^{-d} \tilde{c}_0 \check{F}(0)$$

$$+ x^{2-\xi} (m_f x)^{\zeta_2^{(l.s.)}(2) - \zeta_2^{(0)l.s.}} \tilde{c}_2 \check{F}_2(0) + \dots$$

Asymptotic analysis of the two point correlation

$$C_2(\mathbf{x}, m_f) \propto$$

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- ▶ Small scales (inertial range)

$$C_2(\mathbf{x}, m_f) \simeq m_f^{-2+\xi} - c x^{2-\xi} F(0) + \dots$$

- ▶ Large scale decay ($m_f^{-1} \ll x \ll m^{-1}$)

$$C_2(\mathbf{x}, m_f) \simeq x^{2-\xi} (m_f x)^{-d} \tilde{c}_0 \check{F}(0) \leftarrow \text{Corssin integral}$$

$$+ x^{2-\xi} (m_f x)^{\zeta_2^{(l.s.)}(2) - \zeta_2^{(0)l.s.}} \tilde{c}_2 \check{F}_2(0) + \dots$$

Passive advection by Navier–Stokes

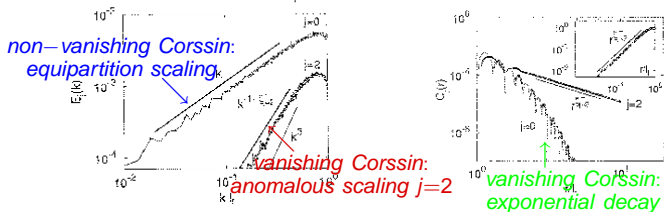
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A. Celani & A. Seminara, (2006).

Perturbative results (small ξ)

- ▶ Expansion of the Hopf operators

$$\mathcal{M}_n = \mathcal{M}_n^{(0)} + \xi \mathcal{M}_n^{(1)} + \mathcal{O}(\xi^2)$$

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Perturbative results (small ξ)

- ▶ Expansion of the Hopf operators

$$\mathcal{M}_n = \mathcal{M}_n^{(0)} + \xi \mathcal{M}_n^{(1)} + O(\xi^2)$$

- ▶ Free theory in the translational invariant sector

$$\mathcal{M}_n^{(0)} = \frac{D_\alpha^\alpha(0, m)}{2d} \sum_{i=1}^n \partial_i^2$$

$$\mathcal{M}_n^{(0)} \mathcal{G}_{\partial_n, n}^{(0)} = 1 \quad \partial_n := d(n-1)$$

admits a well-known **explicit** zero mode expansion

$$\mathcal{G}_{\partial_n, n}^{(0)}(\mathbf{r} - \mathbf{w}) = \sum_{j \mathbf{l}}^{\infty} H_{j, \mathbf{l}}(\mathbf{r}) (\mathcal{K} \circ H_{j, \mathbf{l}})(\mathbf{w}) + \mathbf{r} \Leftrightarrow \mathbf{w}$$

$H_{j, \mathbf{l}}(\mathbf{r}) =$ Harmonic polynomial

$(\mathcal{K} \circ H_{j, \mathbf{l}})(\mathbf{r}) =$ Kelvin transform of $H_{j, \mathbf{l}}(\mathbf{r}) = \frac{1}{r^{\partial_n - 2}} H_{j, \mathbf{l}}\left(\frac{\mathbf{r}}{r^2}\right)$

$(\mathbf{r}, \mathbf{w}) =$ **Jacobi coordinates**

Perturbative results (small ξ)

- ▶ Free theory zero mode properties

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Perturbative results (small ξ)

- ▶ Free theory zero mode properties
 - ▶ Zero modes are classified by (j, \mathbf{l}) , $SO(d_n)$ Gelfand-Zetlin patterns.
 - ▶ *reducible zero modes*: $\exists x_i^\alpha$ such that $\frac{\partial H_{j\mathbf{l}}}{\partial x_i^\alpha} = 0$
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$$\mathcal{M}_n^{(1)} = \sum_{i \neq j}^n d_{(1)}^{\alpha\beta}(\mathbf{x}_{ij}) \partial_{i\alpha} \partial_{j\beta}$$

$$d_{(1)}^{\alpha\beta}(\mathbf{x}) = D_\alpha^\alpha(0, 1) \left\{ \ln(m\mathbf{x}) \delta^{\alpha\beta} - \frac{1}{d-1} \frac{\mathbf{x}^\alpha \mathbf{x}^\beta}{x^2} \right\} + O(m\mathbf{x})^2$$

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The symmetry goes down to

$$SO(\mathfrak{d}_n) \rightarrow \Sigma_n \times SO(d)$$

Zero modes are found by *diagonalisation* Gawędzki & Kupiainen

(1995)

Inertial range anomalous scaling

$$S_{2n}(x) := \langle [\theta(\mathbf{x}, t) - \theta(0, t)]^{2n} \rangle \simeq \delta Z_{2n,ir} + O(mx)$$

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$$\zeta_n = n(1 - \xi/2) - \rho_n, \quad \rho_n > 0$$

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$$\rho_n = \xi \frac{n(n-2)}{2(d+2)} + O(\xi^2)$$

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- Chertkov, Falkovich, Kolokolov and Lebedev (1995), (1996)

$$\rho_n = \xi \frac{n(n-2)}{2d} + O\left(\frac{1}{d}\right)^2$$

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- ▶ OPE: U.V. R.G. of the **local field functionals**

$$G_{2n} = [x^\alpha \partial_\alpha \theta(0, t)]^{2n}$$

determines the ρ_{2n} 's **perturbatively for $\xi \downarrow 0$**

Antonov et al (1998), (2001), Kupiainen and Muratore-Ginanneschi (2006) .

Inertial range scaling and U.V. singularities

- ▶ Canonical scaling analysis yields

$$S_{2n}(x, mx, Mx) = x^{(2-\xi)n} (m|x|)^{-\rho_{2n}} s_{2n}(mx, Mx)$$

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- ▶ Inversion of limits Ansatz (OPE)

$$\lim_{|x| \downarrow 0} \frac{S_{2n}(x; m, M)}{|x|^{2n}} \propto M^{n\xi} \left(\frac{M}{m}\right)^{\rho_{2n}}$$

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Self-symmetry breaking by the I.R. scale

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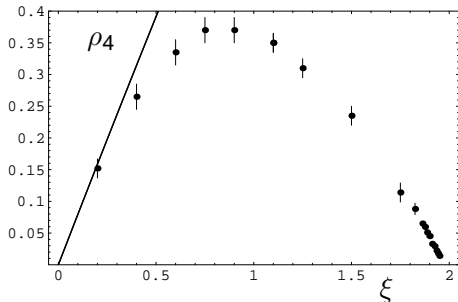
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$$S_{2n}(x) = \sum_{\sigma \in S_{2n} / S_n \times S_2^{2n}} \sigma \int \prod_{j=1}^n dt_j \langle \prod_{i=1}^{n-1} F(x_{i+1}(t_j)) \rangle$$



Frisch Mazzino and Vergassola (1999), Mazzino and Muratore-Ginanneschi (2001)

Large scale zero modes

$$Z_{2n,l} = \frac{\mathcal{Y}_{nl}}{R^{d_n-2+\sigma_{2n,l}}} \quad \text{large scale zero mode}$$

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- Four point by explicit diagonalisation Muratore-Ginanneschi (2008)

$$\sigma_{4,0}([4, 0]) = -\xi \left\{ \frac{2(d+4)}{d+2} - 1 \right\} + O(\xi^2) \quad (\text{"dual" to irreducible})$$

$$\sigma_{4,0}([4, 2]) = -\xi \left\{ \frac{2(d-2)}{d-1} - 1 \right\} + O(\xi^2) \quad (\text{"dual" to reducible})$$

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$$Z_{2n,1} = \frac{Y_{n1}}{R^{d_{n-2} + \sigma_{2n,1}}} \quad \text{large scale zero mode}$$

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- RG type analysis

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U.V. singularities and large scale decay of zero *modes*

► For

$$\frac{1}{m_f} \ll \frac{1}{m} \ll x$$

canonical scaling analysis yields

$$Z_{2n}(x, mx, Mx) = \frac{g_{2n}(m_f x, mx, Mx)}{x^{d_n - 2 + 2n}}$$

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$$\lim_{x \uparrow \infty} g_{2n}(m_f x, mx, Mx) = \tilde{S}'_{2n} \left(\frac{m}{m_f}, \frac{M}{m_f} \right)$$

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- ▶ Operator product expansion

$$\lim_{m_f \downarrow 0} \tilde{S}_{2n} \left(\frac{m}{m_f}, \frac{M}{m_f} \right) \propto M^{n\xi} \left(\frac{M}{m} \right)^{\rho_{2n}}$$

yields a prediction for the scaling in m_f .

The probabilistic meaning of zero modes

Bernard, Gawędzki and Kupiainen (1996), Falkovich, Gawędzki and Vergassola (2001)

- The **scale-invariant** semigroup

$$P_n(\mathbf{X}, t | \mathbf{X}_0, t_0) = e^{(t-t_0)\mathcal{M}_n}(\mathbf{X}, \mathbf{X}_0)$$

$$P_n(\lambda \mathbf{X}, \lambda^{2-\xi} t | \lambda \mathbf{X}_0, \lambda^{2-\xi} t_0) = \lambda^{-nd} P_n(\mathbf{X}, t | \mathbf{X}_0, t_0)$$

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- is the transition probability density of the **Markov process** $\kappa \downarrow \mathbf{0}$ Le Jan and Ramon (1998)

$$d\mathbf{x}_t = \mathbf{v}(\mathbf{x}_t, t) dt$$

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- governs the evolution of Lagrangian averages

$$\langle f \rangle_{(t|\mathbf{x}_0, t_0)} = \int d\mathbf{X}_0 f(\mathbf{X}) P_n(\mathbf{X}, t | \mathbf{X}_0, t_0)$$

Coherent structures preserved by the flow

- $f(\lambda \mathbf{X}) = \lambda^\sigma f(\mathbf{X}) \quad \Rightarrow \quad \langle f \rangle(t|\mathbf{X}_0, t_0) \sim t^{\frac{\sigma}{2-\xi}}$

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$$P_n(\mathbf{X}, t | \lambda \mathbf{X}_0, 0) \stackrel{\lambda \uparrow \infty}{\simeq} \sum_{ij} \lambda^{\zeta_{n,p}} \psi_{n,p}(\mathbf{X}, t) \phi_{n,p}(\mathbf{X}_0)$$

$$0 = (\partial_t - \mathcal{M}_n) P_n = P_n (\partial_t - \mathcal{M}_n) \quad \text{implies}$$

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- $\mathcal{M}_n \phi_{n,p+1}(\mathbf{X}) = \phi_{n,p}(\mathbf{X})$
- $\partial_t \psi_{n,p+1}(\mathbf{X}, t) = -\psi_{n,p}(\mathbf{X}, t)$ (towers of slow modes)
- $\zeta_{n,p} = \zeta_{n,0} + p(2 - \xi)$

Particles and Shapes

- ▶ Lagrangian average of scaling functions

$$\phi(\lambda \mathbf{r}) = \lambda^\sigma \phi(\mathbf{r})$$

$$\langle \phi(\boldsymbol{\rho}(t)) \rangle = \int d^{d_n} \rho \phi(\boldsymbol{\rho}) p_n(\boldsymbol{\rho}, t | \mathbf{r}, 0)$$

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- ▶ Dimensional analysis prediction

$$\langle (\mathbf{r}_1 - \mathbf{r}_2)^2 \rangle \stackrel{t \uparrow \infty}{\sim} t^3 \quad \text{Richardson law}$$

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would imply

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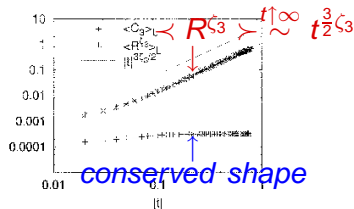
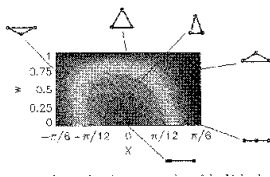
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Harmonic decomposition of C_3

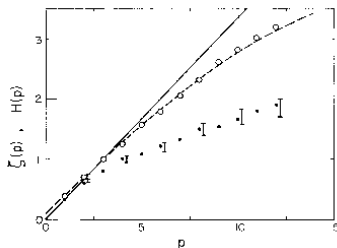
$$C_3 = \langle R_t^\zeta f(\chi_t, w_t) \cos \psi_t \rangle + \dots$$



A. Celani & M. Vergassola (2001)

Conclusions

- ▶ Genuine anomalous scaling in passive advection from first principles.



Conclusions

- ▶ Genuine anomalous scaling in passive advection from first principles.
- ▶ Existence of statistical conservation laws observed in realistic models Mydlarski, Pumir, Shraiman, Siggia, and Warhaft (1998), Celani and Vergassola (2001), Celani and Seminara (2006)

