

Gutzwiller's trace formula and functional determinants

P. Muratore-Ginanneschi, Math. Dept. University of Helsinki

Density of quantum states in the semiclassical limit

$$\rho(E) = \lim_{\Im z \downarrow 0} \int_0^\infty \frac{dT}{\pi i} e^{\frac{izT}{\hbar}} \text{Tr} \hat{K}(T)$$

$\Re z = E$ Energy

$$\text{Tr} \hat{K}(T) := \int dx K(x, t+T | x, t)$$

Gutzwiller trace formula

$$\rho(E) \sim \int \frac{dp dq}{(2\pi\hbar)^d} \delta(E - H(p, q))$$
$$+ \mathfrak{I} \frac{i}{\pi\hbar} \sum_{p,r} \frac{T_p e^{\frac{i r S_p(E)}{\hbar} + i\frac{\pi}{2}\mu_{p,r}}}{\det_{\perp}(1 - M_{p,r})}$$

p,r Sum over periodic orbits and their repetitions

$\mu_{p,r}$ **Maslov index**: topological stability of the orbit

$M_{p,0}$ **Monodromy matrix**: only transversal fluctuations

Path integral formalism

Integration over the **loop space**

(by analytic continuation from the Wiener measure)

$$\text{Tr } \hat{K}(T) = \int_{q(t+T)=q(t)} D[q(t)] e^{\frac{iS}{\hbar}}$$

In the neighborhood of classical trajectories

$$\delta^2 S = \delta q \left(\partial_{\dot{q}} L \right)_{q_{cl}} \Big|_t^{T+t} + \frac{\hbar}{2} \int_t^{T+t} ds \delta q D_{q_{cl}}^2 L \delta q + o(\hbar)$$

The boundary form is zero if the classical trajectory belongs to the loop space

Mathematical definition of the functional determinant

For finite dimensional matrices:

$$\log \text{Det } O = \left[\frac{d}{ds} \text{Tr } O^{-s} \right]_{s=0}$$

For infinite dimensional operators $\text{Tr } O^{-s}$ converges for $\Re s$ large enough. Theorems by Seeley insure analytical continuation to a meromorphic function in the complex plane holomorphic in $s=0$

$$O^{-s} = \frac{1}{2\pi i} \int_{\gamma} d\lambda \frac{\lambda^{-s}}{O - \lambda}$$

Families of trace class operators

Let O_u be a one-parameter u family of elliptic operators on a d -dimensional manifold M with $(d-1)$ -dimensional boundary where O_u satisfies some boundary conditions B . Then

$$\frac{d}{du} \log \text{Det } O_u = \text{Tr} \frac{dO_u}{du} O_u^{-1}$$

Hereby we suppose u to range over operators in the same homotopy class. Furthermore we assume the trace-class condition

$$\left| \text{Tr} \frac{dO_u}{du} O_u^{-1} \right| < \infty$$

Forman's theorem

(Invent. Math. 88, 447-493, 1987.)

If **A** and **B** are elliptic boundary conditions on **M** with linear complement **CA** and **CB**

$$\frac{d}{du} \log \frac{\text{Det } O_{u,A}}{\text{Det } O_{u,B}} = \frac{d}{du} \log \det \Phi_{O_u}$$

$$\Phi_{O_u} : CB \rightarrow CA \quad \Phi_{O_u} = T_{CA} P_{CB}$$

P is the Poisson map of **O** whilst **v** is vector field on ∂M and transverse to it at each point

$$[T(f)](x) := \left(f(x), \frac{\partial}{\partial v} f(x), \dots, \frac{\partial^{n-1}}{\partial v^{n-1}} f(x) \right)$$

Second order differential operators

Quadratic fluctuations in configuration space are governed by the **Sturm-Liouville** operator

$$D_{q_{cl}}^2 L = O := \frac{d}{dt} \left(L_{\dot{q}\dot{q}} \frac{d}{dt} + L_{\dot{q}q} \right) + L_{q\dot{q}} \frac{d}{dt} + L_{qq}$$

The Poisson map $OP=0, [T_{CB}(P)] = h_{CB}$ can be lifted to TM

$$\left[\begin{array}{cc} \frac{d}{dt} & 1 \\ L_{qq} \left(\frac{d}{dt} L_{\dot{q}q} \right) & \frac{d}{dt} L_{\dot{q}\dot{q}} + L_{\dot{q}q} \end{array} \right] T(P) = 0$$

Bott pair description of boundary conditions

In **d-dimensions**, the Bott pair associated to some elliptic boundary conditions **A** is specified by two **2 d x 2 d** matrices (A_0, A_1) with rank of $[A_0, A_1]$ equal **2 d**

$$f \in A \iff \begin{bmatrix} [T f(t_0)] \\ [T f(t_1)] \end{bmatrix} = \begin{bmatrix} A_0 x \\ A_1 x \end{bmatrix}$$

The boundary conditions are **self-adjoint** if

$$A_0^\dagger J A_0 = A_1^\dagger J A_1$$

for **J** is the **symplectic pseudo-metric**.

Linear complement to the boundary conditions

$$A \sim (A_0, A_1)$$

Boundary conditions A

$$CA \sim (CA_0, CA_1)$$

Linear complement of A

$$(CA_0 A_0 + CA_1 A_1) x = 0$$

Examples:

$$\textit{Cauchy} \sim (0, 1)$$

$$\textit{Periodic} \sim (1, 1)$$

$$\textit{Dirichlet} \sim \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

Forman's theorem for (second order) differential operators

$$\frac{\text{Det } O_{u_0, A} \text{ Det } O_{u_1, B}}{\text{Det } O_{u_0, B} \text{ Det } O_{u_1, A}} = \det \frac{[T_{CA}(P_{CB})_{u_0}]}{[T_{CA}(P_{CB})_{u_0}]}$$

$$[T_A(P_B)] := A_0[T(P_B)](t_0) + A_1[T(P_B)](t_1)$$

For Sturm-Liouville operators

$$\text{Det } O_{u_0, \text{Cauchy}} = \text{Det } O_{u_1, \text{Cauchy}}$$

Equivalence between T*M and TM descriptions

Fluctuations around the stationary phase approximations in phase and configuration space generate the same functional

$$x = R \tilde{x} \quad R = \begin{bmatrix} 1 & 0 \\ L_{\dot{q}q} & L_{\dot{q}\dot{q}} \end{bmatrix} \quad \begin{array}{l} x = (\delta q, \delta p) \\ \tilde{x} = (\delta q, \delta \dot{q}) \end{array}$$

$$\begin{aligned} & \det [R^{-1} (CA_0 + CA_1 F_{u_1}(t_1, t_0)) R] \\ & = \det [CA_0 + CA_1 F_{u_1}(t_1, t_0)] \end{aligned}$$

Examples of fluctuation determinant

Let \mathbf{F} the fundamental solution of the linear dynamics
in phase space

$$\frac{\text{Det } O_{u_1, A}}{\text{Det } O_{u_0, A}} = \det \frac{CA_0 + CA_1 F_{u_1}(t_1, t_0)}{CA_0 + CA_1 F_{u_0}(t_1, t_0)}$$

$$(CA_0, CA_1) \sim \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \quad \text{Dirichlet b.c}$$

$$\mathbf{1}, \mathbf{1right} \\ (CA_0, CA_1) \sim \mathbf{i} \quad \text{Periodic b.c.}$$

Determinant homotopy class and Morse index

If $D_{q_{cl}}^2 L_B$ with **b.c. B** is self adjoint the **spectrum** is real. If the classical kinetic energy is strictly **positive definite** the spectrum is **bounded from below**.

$$\text{sign Det}(D_{q_{cl}}^2 L_B) = e^{i\pi\mu_B}$$

$\mu_B =$ Number of negative eigenvalues

\equiv Morse index for the **b.c. B**

Equal fluctuation determinants in T^*M and TM

$$\mu_B = \eta_B$$

Intersection forms and Morse index

Duistermaat (Adv. in Math. 21, 173-195, 1976) : Morse index from intersections of Lagrangian planes.

Salamon & Zehnder (Com. in Pure and Appl. Math. XLV, 173-195, (1992) : (infinite dimensional) Morse theory for periodic solutions of Hamiltonian systems (Conley and Zehnder index).

Robbin & Salamon (Bull. LMS 27, 1995): spectral-flow of self-adjoint operators

Y. Long (Adv. in Math. 154, 76-131, 2000) : explicit iteration formulae for the Conley and Zehnder index in terms of the stability blocks of the linear flow.

Faddeev-Popov method

In the presence of zero modes of $D_{q_{cl}}^2 L_B$ the stationary phase approximation can be applied only to fluctuations orthogonal to the nullspace. The constraint is imposed by writing

$$1 = \int \prod_{\alpha} d\tau_{\alpha} \delta(F_{\alpha}) \left| \det \frac{\partial F_{\beta}}{\partial \tau_{\gamma}} \right|$$

$\tau_{\alpha} =$ Moduli of the invariant (sub)-group **G**

$$F_{\alpha} = \left\langle q \quad q_{cl, [\tau]}, \frac{\partial q_{cl, [\tau]}}{\partial \tau_{\alpha}} \right\rangle \quad \text{Orthogonality condition}$$

$$\text{Tr } \hat{K}(T) \approx e^{\frac{iS_{cl}}{\hbar}} \int \mu(dG) \int d[c] | \det Z | \prod_n \delta(c_n) e^{\frac{i\delta^2 S}{\hbar}}$$

Coleman's regularisation method

The Morse index of $D_{q_{cl}}^2 L_B$ is invariant under a sign definite small perturbation of the potential under which zero eigenvalues acquire a positive values $\lambda_n(\epsilon)$. If **B** stands for periodic b.c.

$$\lim_{\epsilon \downarrow 0} \left| \frac{\text{Det } D_{q_{cl}}^2 L_{per., \epsilon}}{\det Z^2 \prod_{n \in ker} \lambda_n(\epsilon)} \right| = \left| \det V \det_{\perp} (1 - M) \right|$$

det V depends on the symmetry group and on the the parabolic (longitudinal) block of the monodromy matrix **M**. If the energy is the only conserved quantity

$$|\det V| = \left| \frac{dT}{dE} \right|$$

Conclusions

- Forman's theorem offers a general dimensional reduction method to compute functional determinants of elliptic, not necessarily trace class operators.
- The computation of functional determinants through homotopy transformations naturally relates the Morse index (configuration space) to the Conley and Zehnder index (phase space) of periodic classical extremal.