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Toward the Concept of Generalized Definability

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Attempts to formulate mathematically precise definitions of basic concepts such as causality, randomness and probability have a long history. The concept of generalized definability which is outlined in this lecture suggests that such definitions may not exist. Furthermore, it suggests that existing definitions of many basic concepts, among them those of stability, statistical independence and Pareto-optimality, may be in need of redefinition.

In essence, definability is concerned with whether and how a concept, X, can be defined in a way that lends itself to mathematical analysis and computation. In mathematics, definability of mathematical concepts is taken for granted. But as we move farther into the age of machine intelligence and automated reasoning, the issue of definability is certain to grow in importance and visibility, raising basic questions which are not easy to resolve.

To be more specific, let X be the concept of, say, a summary, and assume that I am instructing a machine to generate a summary of a given article or a book. To execute my instruction, the machine must be provided with a definition of what is meant by a summary. It is somewhat paradoxical that we have summarization programs which can summarize, albeit in a narrowly prescribed sense, without being able to formulate a general definition of summarization. The same applies to the concepts of causality, randomness and probability. Indeed, it may be argued that these and many other basic concepts cannot be defined within the conceptual framework of classical logic and set theory.

The point of departure in our approach to definability is the assumption that definability has a hierarchical structure. Furthermore, it is understood that a definition must be unambiguous, precise, operational, general and co-extensive with the concept which it defines.

The hierarchy involves five different types of definability. The lowest level is that of c-definability, with c standing for crisp. Thus, informally, a concept, X, is c-definable if it is a crisp concept, e.g., a prime number, a linear system or a Gaussian distribution. The domain of X is the space of instances to which X applies.

The next level is that of f-definability, with f standing for fuzzy. Thus, X is a fuzzy concept

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if its denotation, F, is a fuzzy set in its universe of discourse. A fuzzy concept is associated with a membership function which assigns to each point, u, in the universe of discourse of X, the degree to which u is a member of F. Alternatively, it may be defined algorithmically in terms of other fuzzy concepts. Examples of fuzzy concepts are small number, strong evidence and similarity. It should be noted that many concepts associated with fuzzy sets are crisp concepts. An example is the concept of a convex fuzzy set. Most fuzzy concepts are context-dependent.

The next level is that of f.g-definability, with g standing for granular, and f.g denoting the conjunction of fuzzy and granular. Informally, in the case of a concept which is f.g-granular, the values of attributes are granulated, with a granule being a clump of values which are drawn together by indistinguishability, similarity, proximity or functionality. f.g-granularity reflects the bounded ability of the human mind to resolve detail and store information. An example of an f.g-granular concept which is traditionally defined as a crisp concept, is that of statistical independence. This is a case of misdefinition -- a definition which is applied to instances for which the concept is not defined, e.g., fuzzy events. In particular, a common misdefinition is to treat a concept as if it were c-definable whereas in fact it is not.

The next level is that of PNL-definability, where PNL stands for Precisiated Natural Language. Basically, PNL consists of propositions drawn from a natural language which can be precisiated through translation into what is called precisiation language. An example of a proposition in PNL is: It is very unlikely that there will be a significant increase in the price of oil in the near future.

In the case of PNL, the precisiation language is the Generalized Constraint Language (GCL). A generic generalized constraint is represented by Z isr R, where Z is the constrained variable, R is the constraining relation and r is a discrete-valued indexing variable whose values define the ways in which R constrains Z. The principal types of constraints are: possibilistic (r=blank); veristic (r=v); probabilistic (r=p); random set (r=rs); usuality (r=u); fuzzy graph (r=fg); and Pawlak set (r=ps). The rationale for constructing a large variety of constraints is that conventional crisp constraints are incapable of representing the meaning of propositions expressed in a natural language -- most of which are intrinsically imprecise -- in a form that lends itself to computation.

The elements of GCL are composite generalized constraints which are formed from generic generalized constraints by combination, modification and qualification. An example of a generalized constraint in GCL is ((Z isp R) and (Z,Y) is S) is unlikely.

By construction, the Generalized Constraint Language is maximally expressive. What this implies is that PNL is the largest subset of a natural language which admits precisiation. Informally, this implication serves as a basis for the conclusion that if a concept, X, cannot be defined in terms of PNL, then, in effect, it is undefinable or, synonymously, amorphic.

In this perspective, the highest level of definability hierarchy, which is the level above PNL-definability, is that of undefinability or amorphicity. A canonical example of an amorphic concept is that of causality. More specifically, is it not possible to construct a general definition of causality such that given any two events A and B and the question, "Did A cause B?" the question could be answered based on the definition. Equivalently, given any definition of causality, it will always be possible to construct examples to which the definition would not apply or yield counter-intuitive results.

In dealing with an amorphic concept, X, what is possible -- and what we generally do -- is to restrict the domain of applicability of X to instances for which X is definable. For example, in the case of the concept of a summary, which is an amorphic concept, we could restrict the length, type and other attributes of what we want to summarize. In this sense, an amorphic concept may be partially definable or, p-definable, for short. The concept of p-definability applies to all levels of the definability hierarchy.

The theory of generalized definability is not a theory in the traditional spirit. The definitions are informal and conclusions are not theorems. Nonetheless, it serves a significant purpose by raising significant questions about a basic issue -- the issue of definability of concepts which lie at the center of scientific theories.