Equicardinality on linearly ordered structures

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1. Preliminaries

We consider the following quantifiers:

1) The *equicardinality or the Härtig quantifier* I with the semantics related to I being that for every $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$,

$$\mathfrak{M} \models Ixy\,(U(x), V(y)) \iff \left|U^{\mathfrak{M}}\right| = \left|V^{\mathfrak{M}}\right|.$$

The enhanced logic is denoted by $\mathrm{FO}(I)$, where the introduction rule for the quantifier I is defined similarly as for the first-order logic FO, the clause above being naturally extended to formulas by substitution.

2) A quantifier related to the previous one is the *Rescher quantifier* R. For every $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$,

$$\mathfrak{M} \models Rxy\,(U(x), V(y)) \iff \left|U^{\mathfrak{M}}\right| \leq \left|V^{\mathfrak{M}}\right|.$$

3) For every fixed $S \subseteq \mathbb{N}$ we define a *cardinality quantifier*: For every $\mathfrak{A} \in \mathrm{Str}(\{U\})$,

$$\mathfrak{A} \models C_S x\,(U(x)) \iff \left|U^{\mathfrak{A}}\right| \in S.$$

4) A special case of the preceding is the (ordinary) *divisibility quantifier* D_n expressing that the size of a unary predicate is divisible by n.
5) A much more expressive quantifier is the *general divisibility quantifier* D expressing that the size of one unary predicate divides that of another, i.e., for every $\mathfrak{M} \in \mathrm{Str}(\{U, V\})$,

$$\mathfrak{M} \models Dxy\,(U(x), V(y)) \iff \left|U^{\mathfrak{M}}\right| \,\Big|\, \left|V^{\mathfrak{M}}\right|.$$

The logics $\mathrm{FO}(R)$, $\mathrm{FO}(C_S)$ etc. are defined similarly than $\mathrm{FO}(I)$.

In this work, the expressive power of the Härtig quantifier I on the class $\mathcal{O}$ of ordered finite structures is discussed.

1.1. Problem. Characterize $S \subseteq \mathbb{N}$ such that $\mathrm{FO}(I) \leq \mathrm{FO}(C_S)\,/\mathcal{O}$.

The results onbtained will be is in sharp contrast to the result of Kolaitis and Väänänen [KV] proving that $\mathrm{FO}(I) \not\leq FO(\mathcal{Q})\,/\mathcal{F}$ for any set $\mathcal{Q}$ of cardinality quantifiers where $\mathcal{F}$ is the class of all finite structures, showing a fundamental difference between the ordered and unordered case.

2. Motivation

2.1. Notice that

$$\mathrm{FO}(I) \equiv \mathrm{FO}(R) \equiv \mathrm{FOC} \,/\, \mathcal{O}$$

where FOC is the first order logic with counting as presented in, e.g., [Ru], i.e., first order logic enhanced by the counting construct $\exists^{\geq i} x$ where i remains a free variables (rather than being a constant).

2.2. The equicardinality is one of the most basic notions that is not definable in first order logic. Now consider the descriptive hierarchy

$$\mathrm{FO} < \mathrm{DTC} \leq \mathrm{TC} \leq \mathrm{LFP} \leq \mathrm{PFP} < \mathcal{L}^{\omega}_{\infty\omega} \,/\, \mathcal{F}$$

It is well-known that the logics in the middle correspond to complexity classes in ordered structures. Although we still have $\mathrm{FO}(I) \not\leq \mathcal{L}^{\omega}_{\infty\omega} \,/\, \mathcal{F}$, on ordered structures it holds that

$$\mathrm{FO} < \mathrm{FO}(I) \leq \mathrm{DTC} \,/\, \mathcal{O},$$

so that I is present in all but the very weakest complexity classes.

2.3. The motivation of this study comes partly from the central place of the built-in linear order in descriptive complexity theory and the fact the addition is definable in the logic $\mathrm{FO}(I)$ on $\mathcal{O}$. Furthermore, note the following general fact on the possibility of removing the built-in relations in ordered structures.

2.4. Proposition. *For every query q with domain* $\mathrm{Str}(\{\leq\})$, *there is a monadic quantifier Q_q such that for any regular $\mathcal{L} \geq \mathrm{FO}(I) \,/\, \mathcal{O}$, q is expressible in $\mathcal{L}$ iff $\mathcal{L} \geq \mathrm{FO}(Q_q) \,/\, \mathcal{O}$.* $\square$

For example,

$$\begin{aligned} \mathtt{unif} - \mathtt{TC}_0 &\equiv \mathrm{FO}(I, Q_{\mathrm{BIT}}) \\ &\equiv \mathrm{FO}(I, Q_+, Q_\times) \\ &\equiv \mathrm{FO}(I, Q_\times) \\ &\equiv \mathrm{FO}(I, D) \\ &\equiv \mathrm{FO}(D) \,/\, \mathcal{O}, \end{aligned}$$

where the first equivalence is a re-formulation of a result by Barrington, Immerman and Straubing [BIS], the second is an observation of Immerman [I], and the rest are easy observations.

3. Results

3.1. Definition. Let $S \subseteq \mathbb{N}$. The sets U and V are *S-equally large* in A, if for all $X \subseteq A \setminus (U \cup V)$, we have that

$$|U \cup X| \in S \iff |V \cup X| \in S.$$

There is a natural way to express this notion in ordered structures. Obviously sets of the same cardinality are S-equally large. If sets of different sizes are S-equally large, this gives rise to some periodicity of S.

3.2. Definition. a) Let $f\colon \mathbb{N} \to \mathbb{N}$ and $\Delta \subseteq \mathbb{N}$ be an interval. Then f is *periodic on* Δ *with period* $\omega \in \mathbb{N}$ if $f(x) = f(x+\omega)$ whenever $x, x+\omega \in \Delta$. A set $S \subseteq \mathbb{N}$ is *periodic on* Δ *with period* ω if its characteristic function χ_S is, i.e., $x, x+\omega \in \Delta$ implies $x \in S \iff x+\omega \in S$.
b) Let $S \subseteq \mathbb{N}$. The functions $f_S, \omega_S\colon \mathbb{N} \to \mathbb{N}$ are defined in the following way. Let $n \in \mathbb{N}$. Then $f_S(n)$ is the least $\ell \in \mathbb{N}$ such that for some $\omega \in \mathbb{N}$, $0 < \omega \leq \ell$, the set S is periodic on the interval $\{\, i \in \mathbb{N} \mid \ell - \omega \leq i \leq n - (\ell - \omega) \,\}$ with period ω. Furthermore, $\omega_S(n)$ is the least $\omega \in \mathbb{N}$ such that S is periodic on the interval $\{\, i \in \mathbb{N} \mid f_S(n) - \omega \leq i \leq n - (f_S(n) - \omega) \,\}$ with period ω.

3.3. Theorem. *Let* $S \subseteq \mathbb{N}$*. Suppose that there are* $k, \ell \in \mathbb{N}$ *such that*

$$k \cdot f_S(n) \cdot \omega_S(n)^{\ell} \geq n.$$

Then $\mathrm{FO}(I) \leq \mathrm{FO}(C_S)\,/\mathcal{O}$. $\square$

This condition demonstrates that usually a cardinality quantifier is capable of expressing I on $\mathcal{O}$, and if not, it shows some resemblance to the divisibility quantifiers D_n. For example, $\mathrm{FO}(I) \leq \mathrm{FO}(C_{\mathbb{P}})\,/\mathcal{O}$ with $\mathbb{P}$ the set of prime numbers, and $\mathrm{FO}(I) \leq \mathrm{FO}(C_S)\,/\mathcal{O}$ for a randomly (with uniform distribution) chosen $S \subseteq \mathbb{N}$.

It seems that the condition above is not only sufficient, but also necessary (for future developments, follow my webpage ⟨`http://www.math.helsinki.fi/~kluosto`⟩).

References

[BIS] D. Barrington, N. Immerman and H. Straubing: *On uniformity within* NC_1. Journal of Computer and System Sciences, 1990, 274–306.

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[Ru] Matthias Ruhl: *Counting and addition cannot express deterministic transitive closure.* In Proceedings of 14th IEEE Symposium on Logic in Computer Science, 1999, 326–334.