On winning strategies with unary quantifiers

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Abstract

A combinatorial argument for two finite structures to agree on all sentences with bounded quantifier rank in first-order logic with any set of unary generalized quantifiers, is given. It is known that connectivity of finite structures is neither in monadic Σ^1_1 nor in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$, where $\mathbf{Q}_u$ is the set of all unary generalized quantifiers. Using this combinatorial argument and a combination of second-order Ehrenfeucht-Fraïssé games developed by Ajtai and Fagin, we prove that connectivity of finite structures is not in monadic Σ^1_1 with any set of unary quantifiers, even if sentences are allowed to contain built-in relations of moderate degree. The combinatorial argument is also used to show that no class (if it is not in some sense trivial) of finite graphs defined by forbidden minors, is in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. Especially, the class of planar graphs is not in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

1. Introduction

The expressive power of first-order logic $\mathcal{L}_{\omega\omega}$ is rather limited. This is because in $\mathcal{L}_{\omega\omega}$ it is not possible to express non-trivial recursion or counting properties. For example, PTIME computable properties like 'evenness' and 'connectivity' cannot be expressed by any first-order sentence.

Many extensions of first-order logic have been developed. The simplest way to extend $\mathcal{L}_{\omega\omega}$ and maintain its usual closure properties, is to add generalized quantifiers to the logic. The notion of generalized quantifier was introduced by Mostowski ([16]), who considered extensions of first-order logic by cardinality quantifiers like 'there exist uncountably many'. A more general way of extending first-order logic was given by Lindström ([15]). According to Lindström, any property of structures over some fixed finite vocabulary can be taken as an interpretation of a quantifier. For example, for any graph property, like planarity, there is a quantifier with the interpretation 'the graph defined by a formula is planar'. A quantifier is n-ary, if it binds at most n variables in each formula it binds. Every graph quantifier is binary and binds one formula.

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Unary generalized quantifiers have been considered in connection with the problems capturing graph properties by $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$, where $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ denotes first-order logic with the set of all unary generalized quantifiers on finite structures, and also capturing *PTIME* by a logic. In [11] it was proved that there does not exist a sentence of $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$, which would recognize all finite connected graphs. The proof used *bijective Ehrenfeucht-Fraïssé* games as a main tool. These games were introduced in [9], where they were used to prove undefinability results for infinite structures. When restricted to finite models, the rules and the power of these games in first-order and infinitary logic with any set of generalized quantifiers, were studied in [10]. In [10] Hella also extended the result in [11]. He considered least fixpoint logic *LFP* which is obtained by adding a recursion mechanism into $\mathcal{L}_{\omega\omega}$ via least fixpoints of positive formulas. Hella showed that for each natural number n, there exist a vocabulary τ and an (unary) *LFP*-definable query on τ-structures which is not expressible in the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}_n)$, where $\mathbf{Q}_n$ is the set of all n-ary quantifiers on finite structures.

On the other hand, Immerman ([13]) conjectured that fixpoint logic with counting would capture *PTIME*. In this logic counting quantifiers 'there exist at least n', are added to fixpoint logic, and structures are extended with a linearly ordered extra sort. However, Cai, Fürer and Immerman ([2]) refuted this conjecture.

The area of research that studies complexity of describing problems (e.g. computational) in some logical formalism, is called descriptive complexity theory. Perhaps the most famous result in this area was given by Fagin. He proved in [4] that *NP* consists exactly of those problems which are definable by existential second-order sentences; i.e. *NP*= Σ^1_1. This means that to solve the famous open problem in computational complexity theory, whether *NP* is closed under complement, it is enough to solve, whether $\Sigma^1_1 = \Pi^1_1$ ($\Pi^1_1 = co{-}\Sigma^1_1$) holds.

Fagin attacked this question in [5] by separating monadic Σ^1_1 and monadic Π^1_1. In these classes second-order quantification is allowed only over sets. Fagin proved that connectivity of finite graphs is not definable in monadic Σ^1_1, while it is easy to see that it is in monadic Π^1_1. Many proofs for the same result have been given afterwards. One of them was presented in [6], where a certain kind of Ehrenfeucht-Fraïssé game was used. In [8] Hanf gave a condition that guarantees a winning strategy for the duplicator, and in [6] Fagin, Stockmeyer and Vardi gave it in a form, which is more suitable for finite model theory. Furthermore, with this method and with some probabilistic arguments, Fagin, Stockmeyer and Vardi proved that connectivity of finite graphs is not in monadic Σ^1_1 even in the presence of built-in relations of moderate degree. Later Schwentick proved that this result holds even in the presence of linear order ([18]).

We recalled above that connectivity of finite graphs is definable neither in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ ([11]) nor in monadic Σ^1_1 ([6]). Neither of these results extends the other. However, the models used in boths proofs, are essentially the same. One structure consists of a big cycle and the other consists of two smaller distinct cycles. It can be asked whether these results can be combined. In this paper we see that this is indeed the case. Actually, we show a little more. We prove that Hanf's method which guarantees the duplicator a winning strategy in first-order Ehrenfeucht-Fraïssé games, actually gives the duplicator a winning strategy in bijective Ehrenfeucht-Fraïssé games. With these games and the proof presented in [6], we get the desired result: connectivity of finite graphs is not in monadic Σ^1_1 with any set of unary generalized quantifiers.

We apply Hanf's method also to other interesting graph properties and show that the classes of finite graphs with these properties are not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. In this paper we consider classes that are defined by forbidden minors.

It is known that for arbitrarily graphs $\mathbb{G}$ and $\mathbb{H}$, the problem of deciding whether $\mathbb{H}$ is a minor of $\mathbb{G}$, is *NP*-complete (see e.g. [14]). But for fixed $\mathbb{H}$, the problem is decidable in polynomial time. On the other hand, Robertson and Seymour have proved that every minor-closed class can be defined by a finite set of forbidden minors (unpublished, see [14, 17]). This yields a finite number of minor-containment tests. Thus for every minor-closed class of graphs, there exists a polynomial time algorithm which recognizes this class. For example, if $\mathcal{C}$ is the class of planar graphs, then $\mathcal{C}$ is minor-closed. By Kuratowski's theorem, the forbidden minors for $\mathcal{C}$ are $\mathbb{K}_5$ and $\mathbb{K}_{3,3}$. Polynomial time algorithms which recognize the class $\mathcal{C}$ are known (see e.g. [12]). However, it is known that there cannot exist a uniform procedure for constructing the set of forbidden minors for an arbitrary minor-closed class (see e.g. in [14]).

In [3], on the other hand, it is observed that every minor-closed set of finite graphs is definable in monadic second-order logic. Especially this means that the set of planar graphs is definable in this logic. In this paper we study the expressibility of classes defined by forbidden minors in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. It is clear that some forbidden minors define so trivial classes that they can be recognized even in $\mathcal{L}_{\omega\omega}$. But it turns out that if we exclude some forbidden minors out of consideration, the classes cannot be defined even in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. Especially, the set of planar graphs is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. The proof uses the same modification of Hanf's result, which was mentioned above.

This paper is organized as follows. In Section 2 we fix our notation concerning graphs and monadic existential second-order logic. Planar graphs and other classes of graphs defined by forbidden minors are also introduced. Section 3 consists of the rules of the well known Ehrenfeucht-Fraïssé games and the results proved in [6] that we use later. In Section 4 we recall the notion of generalized quantifier and bijective Ehrenfeucht-Fraïssé games characterizing equivalence with respect to unary quantifiers. In Section 5 we prove that Hanf's technique essentially gives the duplicator a winning strategy in bijective Ehrenfeucht-Fraïssé games with respect to unary quantifiers. Section 6 contains a proof for the result that connectivity is not in monadic Σ_1^1 with any set of unary quantifiers, not even with built-in relations of moderate degree allowed. In that section a new game characterization needed for this logic, is also given. In Section 7 we prove that no class of graphs defined by forbidden minors (except few in some sense trivial classes) is definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. This paper is concluded in Section 8 by considering some open problems.

2. Definitions and notations

By a vocabulary σ we mean a finite set of relation symbols P_i, $1 \leq i \leq s$, each of which has an arity. A σ-structure $\mathbb{A}$ is a set A, the *universe* of $\mathbb{A}$, with a mapping associating a relation $P_i(\mathbb{A})$ over A with each $P_i \in \sigma$, where $P_i(\mathbb{A})$ has the same arity as P_i. Throughout the rest of this paper all structures considered are finite, i.e. the universe of every structure is finite.

Assume that $\mathbb{A}$ is a σ-structure and a and b are two points in $\mathbb{A}$. Then a and b are *adjacent*, if there is some P_i and tuple t, for which $t \in P_i(\mathbb{A})$, and a and b are entries in the tuple t. The *degree* $deg(a)$ of a point a is the number of points adjacent to a

but not equal to a. Whenever $X \subset A$, $\mathbb{A}\restriction X$ is the structure with universe X where the interpretation of P_i is the set of tuples t in $P_i(\mathbb{A})$ such that every entry of t is in X, for $1 \le i \le s$.

An *isomorphism* $\mathbb{A} \to \mathbb{B}$ between σ-structures $\mathbb{A}$ and $\mathbb{B}$ is a bijection $f : A \to B$, which for each $P_i \in \sigma$, satisfies $(a_1, \ldots, a_{r_i}) \in P_i(\mathbb{A})$, if and only if $(f(a_1), \ldots, f(a_{r_i})) \in P_i(\mathbb{B})$, where r_i is the arity of P_i and $a_j \in A$, for $1 \le j \le r_i$. In the case $\mathbb{A} = \mathbb{B}$, an isomorphism is called an *automorphism*. If $C \subset A$ and $D \subset B$, a bijection $p : C \to D$ is a *partial isomorphism* $\mathbb{A} \to \mathbb{B}$, if for each $P_i \in \sigma$, $(a_1, \ldots, a_{r_i}) \in P_i(\mathbb{A})$ holds if and only if $(p(a_1), \ldots, p(a_{r_i})) \in P_i(\mathbb{B})$, where r_i is the arity of P_i and $a_j \in C$, for $1 \le j \le r_i$. Every structure $\mathbb{A}$ has a trivial automorphism, namely the identity mapping $f(a) = a$ for each $a \in A$. If this is the only automorphism $\mathbb{A} \to \mathbb{A}$, the structure is called *rigid*.

The structures we consider in this paper are mainly *graphs* and *colored graphs*. Graphs are structures where the vocabulary consists of a single binary relation symbol E. A graph is denoted by $\mathbb{G} = (G, E(\mathbb{G}))$ (or shortly $\mathbb{G} = (G, E)$), where G is the universe of $\mathbb{G}$ (the set of vertices) and $E(\mathbb{G})$ is the edge relation on G (the set of edges). Two vertices $u, v \in G$ in a graph $\mathbb{G}$ are *adjacent*, if $(u, v) \in E(\mathbb{G})$. We consider only undirected graphs without multiple edges and loops, i.e. for all vertices u we have $(u, u) \notin E$ and $(u, v) \in E$ implies $(v, u) \in E$. The *degree* of a vertex u is the number of vertices v for which $(u, v) \in E$.

If $\mathbb{G}' = (G', E(\mathbb{G}'))$ is another graph, where $G' \subset G$ and $E(\mathbb{G}') \subset E(\mathbb{G})$, then $\mathbb{G}'$ is a *subgraph* of $\mathbb{G}$. If, for all $a, b \in G'$, there is an edge between a and b in $\mathbb{G}'$ if and only if there is an edge between a and b in $\mathbb{G}$, then $\mathbb{G}'$ is an *induced subgraph* of $\mathbb{G}$. Every induced subgraph of $\mathbb{G}$ is of the form $\mathbb{G}\restriction X$, where $X \subset G$.

In colored graphs there are furthermore some number k of unary relation symbols $U_1, \ldots, U_k$ in the vocabulary. In a colored graph $\mathbb{G}$, the *color* $\chi(u)$ of a point $u \in G$ is a description of which $U_i(\mathbb{G})$ the point u is a member of. Thus there are 2^k possible colors.

Consider a σ-structure $\mathbb{A}$. The neighborhood $Nbd(d, a)$ of radius d about $a \in A$ is defined recursively by

$$Nbd(1, a) = \{a\}\,;$$

$$Nbd(d + 1, a) = \{v \mid v \text{ is adjacent to some } b \in Nbd(d, a)\} \cup Nbd(d, a).$$

Thus $Nbd(d, a)$ consists of all points whose distance from a is *strictly* less than d. The *d-type* of a point a is the isomorphism type of the neighborhood of radius d about a, where a is treated as a constant. Thus two points a and b in $\mathbb{A}$ have the same d-type, if and only if $\mathbb{A}\restriction Nbd(d, a) \cong \mathbb{A}\restriction Nbd(d, b)$ under an isomorphism mapping a to b. Two structures $\mathbb{A}$ and $\mathbb{B}$ are *d-equivalent* if for every d-type τ, they have exactly the same number of points with d-type τ.

Definitions of first-order sentences and semantics are standard ones. Here equality is treated as a special relation symbol, which is not a member of the vocabulary. We call a class $\mathcal{C}$ of σ-structures first-order -definable, if there is a first-order sentence ϕ, such that for all σ-structures $\mathbb{A}$, $\mathbb{A} \models \phi$ if and only if $\mathbb{A} \in \mathcal{C}$.

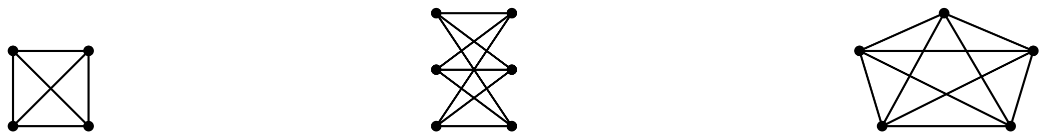
In second-order logic quantification over sets and relations is also allowed. A Σ^1_1 sentence is of the form $\exists A_1 \ldots \exists A_k \phi$, where ϕ is a first-order sentence and the A_i's are relation symbols. If each relation symbol A_i is unary, the sentence is a *monadic* Σ^1_1, or shortly *Mon* Σ^1_1, sentence. This means that the existential second-order quantifiers quantify only over sets. A class $\mathcal{C}$ of σ-structures is said to be (monadic) Σ^1_1 if it is the class of all σ-structures which satisfy some fixed (monadic) Σ_1^1 sentence. The reason that Σ_1^1 classes have been of great interest, is Fagin's result ([5]) that the complexity class *NP* consists of exactly the problems which can be defined in Σ_1^1. For example, the following sentence shows that non-connectivity of finite graphs is *Mon* Σ_1^1:

$$\exists A(\exists x \exists y(x \in A \wedge y \notin A) \wedge \forall x \forall y((x \in A \wedge y \notin A) \Rightarrow (x, y) \notin E)).$$

A class is *Mon* Π_1^1 if its complement is *Mon* Σ_1^1. As we just saw, connectivity is *Mon* Π_1^1. By Fagin's result ([5]), connectivity is not *Mon* Σ_1^1, which implies that *Mon* Σ_1^1 is not equal to *Mon* Π_1^1.

Next we recall some definitions of special kinds of graphs and classes of graphs. A graph with n vertices is *complete* (denoted by $\mathbb{K}_n$), if every vertex is adjacent to all other $n-1$ vertices. In a *bipartite* graph the vertex set can be divided into two distinct parts V_1 and V_2 such that every edge has one endpoint in V_1 and the other endpoint in V_2. If a bipartite graph with parts of sizes m and n contains all possible edges, the graph is denoted by $\mathbb{K}_{m,n}$ (see Figure 1). A graph is a *(chordless) path* of length $n-1$ with a vertex set $\{a_1, \ldots, a_n\}$, if vertices a_i and a_{i+1} are adjacent for all $1 \leq i < n$ and there is no edge between non-consecutive vertices. If $n > 2$ and also vertices a_1 and a_n are adjacent, the graph is a *(chordless) cycle*. A vertex in a graph is *isolated*, if it is not adjacent to any other vertex.

A (finite) graph is said to be *planar*, if it embeds in the plane; i.e. we can draw a picture of the graph in the plane such that two edges never cross. In Figure 1 the first graph is planar but the next two are not.



Planar graph $K_{3,3}$ K_5

Figure 1: Planar and non-planar graphs

In an *elementary contraction*, two adjacent vertices u and v are *contracted* to form a new vertice w, that is adjacent to all vertices which were adjacent to u or to v; thus the edge between u and v is collapsed to length zero (Figure 2). Multiple edges possibly resulting in a contraction are ignored. The next definitions contain some well-known

Figure 2: Contracting edge e

concepts of graph theory (see e.g. [14]).

2.1. Definition. Let $\mathbb{G}$ and $\mathbb{H}$ be finite graphs.

- $\mathbb{H}$ is a *contraction* of $\mathbb{G}$, if $\mathbb{H}$ can be obtained from $\mathbb{G}$ by a sequence of elementary contractions or deleting isolated vertices;
- $\mathbb{H}$ is a *minor* of $\mathbb{G}$, if $\mathbb{H}$ is a contraction of a (not necessarily induced) subgraph of $\mathbb{G}$.

Assume that $\mathbb{H}$ is a finite graph and $\mathcal{H} = \{\mathbb{H}_1, \ldots, \mathbb{H}_k\}$ is a collection of finite graphs, where no $\mathbb{H}_i$ is a minor of $\mathbb{H}_j$, when $i \neq j$. We use the following notation:

$$\mathcal{C}_{\mathbb{H}} = \{\mathbb{G} \mid \mathbb{G} \text{ does not have } \mathbb{H} \text{ as a minor } \};$$

$$\mathcal{C}_{\mathcal{H}} = \{\mathbb{G} \mid \mathbb{G} \text{ has no graph in } \mathcal{H} \text{ as a minor } \}.$$

We call the graphs in $\mathcal{H}$ *forbidden minors* for $\mathcal{C}_{\mathcal{H}}$ and say that $\mathcal{C}_{\mathcal{H}}$ is defined by forbidden minors.

For a class $\mathcal{C}$ of finite graphs, we denote by $\overline{\mathcal{C}}$ the complement of $\mathcal{C}$, i.e. the class of finite graphs which are not in $\mathcal{C}$.

2.2. Definition. Let $\mathbb{H}$ be a finite graph and let $\mathcal{C}$ be a class of finite graphs.

- $\mathbb{H}$ is *minor-minimal* in $\mathcal{C}$ if $\mathbb{H} \in \mathcal{C}$ and there is no graph $\mathbb{G} \in \mathcal{C}$ such that $\mathbb{G}$ is a minor of $\mathbb{H}$ but $\mathbb{G} \neq \mathbb{H}$;
- $\mathcal{C}$ is *minor-closed*, if the minors of every graph in $\mathcal{C}$ are also in $\mathcal{C}$;
- the *obstruction set* for $\mathcal{C}$ is the set $Obstr(\mathcal{C}) = \{\mathbb{H} \mid \mathbb{H} \text{ is minor-minimal in } \overline{\mathcal{C}}\}$.

We remind the reader about some results in this area of research stated in [14], where the references can be found, too. It is known that for arbitrarily graphs $\mathbb{G}$ and $\mathbb{H}$, the problem of deciding whether $\mathbb{H}$ is a minor of $\mathbb{G}$, is *NP*-complete (see e.g. [14]). But for fixed $\mathbb{H}$, the problem is decidable in polynomial time. On the other hand, the following result by Robertson and Seymour is crucial in many applications (unpublished, see [14, 17]).

2.3. Theorem. *For every minor-closed class $\mathcal{C}$, the obstruction set for $\mathcal{C}$ is finite.*

But for every minor-closed class of graphs $\mathcal{C}$ and for every graph $\mathbb{G}$ we have that $\mathbb{G} \in \mathcal{C}$ if and only if no element in the obstruction set of $\mathcal{C}$ is a minor of $\mathbb{G}$. Thus the recognition problem for a minor-closed class of graphs is reduced to a finite number of minor-containment tests with the graphs in the obstruction set. This proves the following result.

2.4. Theorem. *For every minor-closed class of graphs, there exists a polynomial time algorithm which recognizes this class.*

The following theorem gives two well-known classes of graphs, which can be defined by forbidden minors. The first claim is obvious from the above definition. The second follows from the famous theorem of Kuratowski (for a proof, see e.g. [19, Section XI.9].

2.5. Theorem. *Let $\mathcal{C}$ be a class of finite graphs.*

- *if $\mathcal{C}$ is the class of acyclic graphs, then $\mathcal{C}$ is minor-closed and the obstruction set for $\mathcal{C}$ is $\{\mathbb{K}_3\}$;*
- *if $\mathcal{C}$ is the class of planar graphs, then $\mathcal{C}$ is minor-closed and the obstruction set for $\mathcal{C}$ is $\{\mathbb{K}_5, \mathbb{K}_{3,3}\}$.*

According to Theorem 2.4, there exist polynomial time algorithms for the classes in Theorem 2.5. The class of planar graphs is perhaps the best known example of a minor-closed class. Polynomial time algorithms which recognize this class are known (see e.g. [12]). However, it is known that there cannot exist a uniform procedure for constructing obstruction sets. This is because of the following result by Fellows and Langston (see e.g. [14]).

2.6. Theorem. *The problem of determining obstruction sets from machine descriptions of minor-closed classes of graphs, is recursively unsolvable.*

However, if we want to show that there is a polynomial time algorithm for some graph property $\mathcal{P}$, it is enough to show that the class

$$\mathcal{C} = \{\mathbb{G} \mid \mathbb{G} \text{ satisfies } \mathcal{P}\}$$

or its complement is minor-closed. If this is the case, a polynomial time algorithm must exist because of Theorem 2.4.

It should be noted, that every minor-closed class $\mathcal{C}$ can be defined in monadic second-order logic (see e.g. [3]). Namely, $\mathbb{H}$ is a minor of $\mathbb{G}$ if and only if for every vertex $u \in \mathbb{H}$ there is a connected subgraph $\mathbb{G}_u$ of $\mathbb{G}$, such that whenever $u \neq v$, G_u and G_v are disjoint and whenever $(u, v) \in E(\mathbb{H})$, there is an edge linking $\mathbb{G}_u$ and $\mathbb{G}_v$. On the other hand, we saw earlier that connectivity is in *Mon* Π^1_1. From this description it is easy to write a monadic second-order sentence expressing the class defined by the forbidden minor $\mathbb{H}$. According to Theorem 2.3, we can take a finite conjunction of these monadic second-order sentences and obtain a monadic second-order sentence that defines the class $\mathcal{C}$.

3. Ehrenfeucht-Fraïssé games

Many important results in classical model theory, such as compactness and completeness theorems, fail when consideration is restricted to finite models. However, Ehrenfeucht-Fraïssé -type games can be used also in finite model theory. This is why these games have become so important tool in this area of research. In this section we remind the reader the rules of Ehrenfeucht-Fraïssé games and the rules of the modified Ehrenfeucht-Fraïssé games introduced in [1].

In r-round first-order Ehrenfeucht-Fraïssé game between two finite σ-structures $\mathbb{A}$ and $\mathbb{B}$, there are two players, the spoiler and the duplicator. For each round $i \leq r$, the spoiler first chooses an element in one of the structures. Then the duplicator has to choose an element from the other structure. Let a_i be the element chosen from $\mathbb{A}$ and b_i the one chosen from $\mathbb{B}$. This continues for r rounds. The duplicator wins this $EF^r(\mathbb{A}, \mathbb{B})$ -game, if $\mathbb{A}\restriction\{a_1, \ldots, a_r\}$ is isomorphic to $\mathbb{B}\restriction\{b_1, \ldots, b_r\}$ under the isomorphism $a_i \mapsto b_i$, for $1 \leq i \leq r$. Otherwise the spoiler wins. The importance of this game comes from the following well-known result. Assume that $\mathcal{C}$ is a class of finite structures and $\overline{\mathcal{C}}$ is the complement of $\mathcal{C}$, i.e. the class of σ-structures which are not in $\mathcal{C}$. Then we have the following result.

3.1. Theorem. *A class $\mathcal{C}$ is first-order definable if and only if there is r such that whenever $\mathbb{A} \in \mathcal{C}$ and $\mathbb{B} \in \overline{\mathcal{C}}$, then the spoiler has a winning strategy in $EF^r(\mathbb{A}, \mathbb{B})$.*

Many modifications of this game have been developed. In [1] and [6] the following Ajtai-Fagin -game was used. This game applies to monadic existential second-order logic. The game is played over a class $\mathcal{C}$ of finite σ-structures. In this $AF^{c,r}(\mathcal{C})$-game there are c colors and r rounds. The rules of the game are as follows.

- The duplicator selects a member of $\mathcal{C}$ to be $\mathbb{A}$.
- The spoiler colors $\mathbb{A}$ with the c colors.
  Let $\mathbb{A}'$ be the resulting colored structure.
- The duplicator selects a member of $\overline{\mathcal{C}}$ to be $\mathbb{B}$.
- The duplicator colors $\mathbb{B}$ with the c colors.
  Let $\mathbb{B}'$ be the resulting colored structure.
- The game is concluded by $EF^r(\mathbb{A}', \mathbb{B}')$.

If $\mathcal{C}$ or $\overline{\mathcal{C}}$ is empty, the spoiler wins, since the duplicator can not make the first or the third move. Using this game Ajtai and Fagin prove the next result.

3.2. Theorem. *A class $\mathcal{C}$ of σ-structures is Mon Σ_1^1 if and only if there are c and r such that the spoiler has a winning strategy in $AF^{c,r}(\mathcal{C})$.*

Hanf proved in [8], that a winning strategy for the duplicator in the game $EF^r(\mathbb{A}, \mathbb{B})$ (for finite and infinite structures), is guaranteed by counting d-types. In [6] this condition was presented in a form, which is more suitable for finite model theory.

3.3. Lemma. *Let r be a positive integer. There is a positive integer d such that whenever $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent structures, then the duplicator has a winning strategy in $EF^r(\mathbb{A}, \mathbb{B})$.*

In Section 5 we will see that under these assumptions the duplicator has a winning strategy also in a game corresponding to a stronger logic. With Lemma 3.3 as an important tool the following result was proved in [6].

3.4. Theorem. *Connectivity of finite graphs is not in Mon Σ_1^1.*

In [6] Fagin, Stockmeyer and Vardi used quite difficult probabilistic arguments for extending this theorem also to the case, where structures are allowed to contain built-in relations of moderate degree. We use the same notations as presented in [6]. Let $V_n = \{v_0, \ldots, v_{n-1}\}$ be a universe of size n. Built-in relations are specified by a vocabulary $\{P_1, \ldots, P_s\}$ and for each $n \geq 1$ and $i \leq s$, an interpretation $\hat{P}_{n,i}$ of P_i as a relation on V_n. Let $\mathbb{P}_n$ denote the structure with domain V_n and relations $\hat{P}_{n,1}, \ldots, \hat{P}_{n,s}$. Let $f(n)$ be the maximum degree of a point v in the structure $\mathbb{P}_n$. Built-in relations have *moderate degree* if $f(n) = (\log n)^{o(1)}$, i.e. there is a function $\sigma(n)$ with $\lim_{n\to\infty} \sigma(n) = 0$ such that $f(n) \leq (\log n)^{\sigma(n)}$ for all n (the base of the logarithm is 2). With these notations, the following extension of the previous theorem was proved in [6].

3.5. Theorem. *Connectivity is not in Mon Σ_1^1, even in the presence of built-in relations of moderate degree.*

4. Generalized quantifiers

In this section we introduce the notion of unary generalized (or Lindström) quantifiers and results for elementary equivalence in first-order logic with these quantifiers.

Assume that $s_k = (1, \ldots, 1)$ is a sequence of length k. A structure is of type s_k if it is of the form $\mathbb{A} = (A, R_1(\mathbb{A}), \ldots, R_k(\mathbb{A}))$, where $R_i(\mathbb{A}) \subset A$ for $i = 1, \ldots, k$. We associate any class $\mathcal{K}$ of structures of type s_k, which is closed under isomorphisms, with a unary generalized quantifier $Q_{\mathcal{K}}$. The set of formulas of the logic $\mathcal{L}_{\omega\omega}(Q_{\mathcal{K}})$ is defined as for first-order logic with the following additional rule:

> If $\psi_1, \ldots, \psi_k$ are formulas and x_i is a variable for $i = 1, \ldots, k$, then $Q_{\mathcal{K}} x_1, \ldots, x_k(\psi_1, \ldots, \psi_k)$ is a formula.

The semantics of the quantifier $Q_{\mathcal{K}}$ is determined by the class $\mathcal{K}$ as follows:

> $\mathbb{A} \models Q_{\mathcal{K}} x_1, \ldots, x_k(\psi_1(x_1, \bar{b}_1), \ldots, \psi_k(x_k, \bar{b}_k))$ if and only if
> $(A, \psi_1(\mathbb{A}, \bar{b}_1), \ldots, \psi_k(\mathbb{A}, \bar{b}_k) \in \mathcal{K}$, where
> $\psi_i(\mathbb{A}, \bar{b}_i) = \{a \in A \mid \mathbb{A} \models \psi_i(a, \bar{b}_i)\}$ for each $i = 1, \ldots, k$.

The logic $\mathcal{L}_{\omega\omega}(\mathbf{Q})$ for $\mathbf{Q}$ a set of generalized quantifiers is defined similarly. Throughout the rest of this paper, we denote by $\mathbf{Q}_u$ the set of all unary quantifiers on finite structures.

The quantifier rank $qr(\phi)$ of an $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$-formula ϕ is defined as follows:

- $qr(\phi) = 0$ if ϕ is atomic
- $qr(\neg\phi) = qr(\phi)$
- $qr(\phi \wedge \psi) = qr(\phi \vee \psi) = \max\{qr(\phi), qr(\psi)\}$
- $qr(\exists x \phi) = qr(\forall x \phi) = qr(\phi) + 1$
- $qr(Q_{\mathcal{K}} x_1, \ldots, x_k(\phi_1, \ldots, \phi_k)) = \max\{qr(\phi_1), \ldots, qr(\phi_k)\} + 1$ for $Q_{\mathcal{K}} \in \mathbf{Q}_u$ a quantifier of type s_k.

If $\mathbb{A}$ and $\mathbb{B}$ are models of the same vocabulary satisfying exactly the same $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ sentences of quantifier rank at most r, then we write $\mathbb{A} \equiv^r_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)} \mathbb{B}$.

4.1. Example. *(i)* The existential quantifier $\exists$ corresponds to the class of all finite structures (A, P), where $\emptyset \neq P \subset A$. Similarly, the universal quantifier $\forall$ can be identified with the unary generalized quantifier, which is defined by the class of all structures (A, P), where $P = A$. The type of these quantifiers is (1).

(ii) We define $\mathcal{K}_{EVEN}$ to be the class of all finite structures (A, P), where the subset P of A contains an even number of elements. Similarly, the class $\mathcal{K}_{\geq \frac{1}{2}}$ is defined to be the class of all finite structures (A, P), where $P \subset A$ and $|P| \geq \frac{1}{2}|A|$. The corresponding quantifiers are of type (1).

(iii) The *Rescher quantifier* R is defined to be the class of all structures (A, P, S), where $P, S \subset A$ and $|P| \leq |S|$. In the definition of the *Härtig quantifier*, the last condition is replaced by $|P| = |S|$. These quantifiers are of type $(1, 1)$.

A class $\mathcal{C}$ of finite σ-structures is said to be $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$-definable, if there is an $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$-sentence ϕ, such that for every σ-structures $\mathbb{A}$, we have $\mathbb{A} \models \phi$ if and only if $\mathbb{A} \in \mathcal{C}$.

In [11] it was proved that unary generalized quantifiers are not strong enough to distinguish connected and non-connected finite graphs. The proof uses the next lemma ([9]) as a tool.

4.2. Lemma. *Let $\mathbb{A}$ and $\mathbb{B}$ be models of the same vocabulary. Assume that there exist nonempty sets I_m, $m \leq r$, of partial isomorphisms $p : \mathbb{A} \to \mathbb{B}$ satisfying the bijective extension condition:*

(BE) *if $p \in I_{m+1}$, $m+1 \leq r$, then there is a bijection $f_p : A \to B$ such that $p \cup \{(a, f_p(a))\} \in I_m$ for every $a \in A$.*

Then $\mathbb{A} \equiv^r_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)} \mathbb{B}$.

The game theoretical form of this result goes as follows. For each round $i \leq r$ the duplicator chooses first a bijection $f_i : A \to B$ and the spoiler answers by choosing a point $a_i \in A$. The duplicator wins this $BEF^r(\mathbb{A}, \mathbb{B})$ -game over structures $\mathbb{A}$ and $\mathbb{B}$ if

$$p = \{(a_i, f_i(a_i)) \mid i \leq r\}$$

is a partial isomorphism $\mathbb{A} \to \mathbb{B}$; otherwise, or if $\mathbb{A}$ and $\mathbb{B}$ are of different cardinalities, the spoiler wins. Now we have the following result ([9]).

4.3. Lemma. *Let $\mathbb{A}$ and $\mathbb{B}$ be models of the same vocabulary. Assume that the duplicator has a winning strategy in $BEF^r(\mathbb{A}, \mathbb{B})$. Then $\mathbb{A} \equiv^r_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)} \mathbb{B}$.*

With this lemma we can easily prove the following result, that is used repeatedly in the following sections.

4.4. Theorem. *Let $\mathcal{C}$ be a class of σ-structures. If for each natural number r, there are σ-structures $\mathbb{A}$ and $\mathbb{B}$, such that $\mathbb{A} \in \mathcal{C}$ and $\mathbb{B} \notin \mathcal{C}$, and the duplicator has a winning strategy in $BEF^r(\mathbb{A}, \mathbb{B})$, then $\mathcal{C}$ is not definable in the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.*

Proof. Assume on the contrary, that there is an $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$-sentence ϕ, that defines the class $\mathcal{C}$ and let $qr(\phi) = r$. Let $\mathbb{A}$ and $\mathbb{B}$ be σ-structures for this r, such that $\mathbb{A} \in \mathcal{C}$ and $\mathbb{B} \notin \mathcal{C}$ and the duplicator has a winning strategy in $BEF^r(\mathbb{A}, \mathbb{B})$. According to Lemma 4.3 we have $\mathbb{A} \models \phi$ if and only if $\mathbb{B} \models \phi$. Because of the definition of ϕ, we have $\mathbb{B} \in \mathcal{C}$, which is a contradiction. □

Using Lemma 4.2, the following proposition was proved in [11].

4.5. Proposition. *For each natural number r there are finite graphs $\mathbb{A}$ and $\mathbb{B}$ such that $\mathbb{A}$ is connected, $\mathbb{B}$ is non-connected, and $\mathbb{A} \equiv^r_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)} \mathbb{B}$.*

The models used in the proof of this proposition are similar to those used in the proof of Theorem 3.4, a big cycle and two disjoint smaller cycles. However, we do not go into details here. This proposition implies the following result ([11]).

4.6. Theorem. *Connectivity of finite graphs is not definable in* $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

It should be noted that this theorem does not extend Fagin's result (Theorem 3.4). Namely, the class of non-connected graphs is Mon Σ_1^1 but by the above theorem this class cannot be expressed in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. On the other hand, $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ can express non-recursive properties that is not possible even in full second-order logic. Hence these results are incomparable. In Section 6 we prove a result which extends both these theorems.

5. Modifications of Hanf's technique

We recalled in Section 3 a modification of Hanf's technique to finite models presented in [6]. The condition in Lemma 3.3 guarantees a winning strategy for the duplicator in first-order Ehrenfeucht-Fraïssé games. But the condition gives in fact more. In this section we shall see that this condition gives the duplicator a winning strategy also in bijective Ehrenfeucht-Fraïssé -games described in the previous section.

While playing a *BEF*-game, the duplicator should in every round be able to choose a bijection f such that no matter how the spoiler chooses a point u in the first structure, the resulting function defined by already chosen points and additionally the pair $(u, f(u))$, is always a partial isomorphism. The following lemma shows that this is possible under the assumptions defined in Section 3. The j-matching condition in the proof is essentially the same as was used in [6].

5.1. Proposition. *Let r be a positive integer. There is a positive integer d, that depends only on r, such that whenever $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent structures, the duplicator has a winning strategy in the game $BEF^r(\mathbb{A}, \mathbb{B})$.*

Proof. Choose $d = 3^r$. Let $\mathbb{A}$ and $\mathbb{B}$ be d-equivalent structures. We show that the duplicator can ensure after j rounds, where $1 \le j \le r$, that the following j-matching condition holds:

$$\theta_j : \mathbb{A}\restriction(\cup_{i\le j} Nbd(3^{r-j}, a_i)) \cong \mathbb{B}\restriction(\cup_{i\le j} Nbd(3^{r-j}, f_i(a_i))),$$

where θ_j is an isomorphism that maps a_i to $f_i(a_i)$ for $1 \le i \le j$, when points $a_1, \ldots, a_j$ are selected by the spoiler and bijections $f_1, \ldots, f_j$ are selected by the duplicator.

Let the duplicator first choose a bijection $f_1 : A \to B$ such that for all $a \in A$ the d-type of a is the same as the d-type of $f_1(a)$. The structures are d-equivalent and thus such a bijection exists. The spoiler then chooses a point $a_1 \in A$. Denote $p_1 = \{(a_1, f_1(a_1))\}$. Recall that the structures are d-equivalent and $d = 3^r$, so they are also 3^{r-j}-equivalent for all $1 \le j \le r$ (for a proof, see [6]). Hence the 1-matching condition holds and obviously p_1 is a partial isomorphism.

In general, in round j, where $1 \le j < r$, assume that we have a partial isomorphism $p_j = \{(a_i, f_i(a_i)) \mid i \le j\}$. Furthermore, assume that the bijections f_i, where $i \le j$, have been chosen such that the j-matching condition holds. We show that the duplicator can maintain this condition in round $j + 1$.

The duplicator chooses now a bijection $f_{j+1} : A \to B$ such that

$$f_{j+1}\restriction \cup_{i\le j} Nbd(2 \cdot 3^{r-j-1}, a_i) \subset \theta_j,$$

where θ_j is the isomorphism given by the j-matching condition. Because $\theta_j \restriction Nbd(3^{r-j-1}, a)$ is an isomorphism for every $a \in \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i)$, for each 3^{r-j-1}-type τ there are equally many elements with type τ in the sets

$$\cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i) \quad \text{and} \quad \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, f_i(a_i)).$$

Since $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent structures, this holds also for the complements of these sets. Thus for each 3^{r-j-1}-type τ, if $a_1^\tau, \ldots, a_{k_\tau}^\tau$ is a list of all elements in the set $A \setminus \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i)$ with type τ, there are elements $b_1^\tau, \ldots, b_{k_\tau}^\tau$ in $B \setminus \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, f_i(a_i))$ with the same type τ. Now we define $f_{j+1} : A \to B$ by

$$f_{j+1}(a) = \begin{cases} \theta_j(a), & \text{if } a \in \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i) \\ b_i^\tau, & \text{if } a \notin \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i) \text{ and} \\ & \quad a = a_i^\tau \text{ for some } 3^{r-j-1}\text{-type } \tau \text{ and } i \leq k_\tau. \end{cases}$$

Then f_{j+1} is a bijection. The spoiler then chooses a point $a_{j+1} \in A$. By construction, $p_{j+1} = \{(a_i, f_i(a_i)) \mid i \leq j+1\}$ is a partial isomorphism. We see also that the $(j+1)$-matching condition holds. Namely, if $a_{j+1} \in \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i)$, then $Nbd(3^{r-j-1}, a_{j+1}) \subset \cup_{i \leq j} Nbd(3^{r-j}, a_i)$. Hence, the $(j+1)$-matching condition follows from induction hypothesis, that the j-matching condition holds. If $a_{j+1} \notin \cup_{i \leq j} Nbd(2 \cdot 3^{r-j-1}, a_i)$, then no point in $Nbd(3^{r-j-1}, a_{j+1})$ is adjacent to any point in $\cup_{i \leq j} Nbd(3^{r-j-1}, a_i)$. This together with the j-matching condition implies, that the $(j+1)$-matching condition holds. This means that also p_r is a partial isomorphism and thus the duplicator has a winning strategy in $BEF^r(\mathbb{A}, \mathbb{B})$. □

5.2. Corollary. *A class $\mathcal{C}$ of finite σ-structures is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$, if for each d there exist σ-structures $\mathbb{A}$ and $\mathbb{B}$, such that $\mathbb{A} \in \mathcal{C}$ and $\mathbb{B} \notin \mathcal{C}$, and $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent.*

Proof. We show that conditions of Theorem 4.4 are satisfied. Let r be a positive integer and let d be given by Proposition 5.1 for this r. Suppose $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent structures, such that $\mathbb{A} \in \mathcal{C}$ and $\mathbb{B} \notin \mathcal{C}$. Then it follows from Proposition 5.1, that the duplicator has a winning strategy in $BEF^r(\mathbb{A}, \mathbb{B})$. According to Theorem 4.4, $\mathcal{C}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. □

The condition given by Proposition 5.1 can be used as a tool to prove non-expressibility results in the logics $\mathcal{L}_{\omega\omega}$ and *Mon* Σ_1^1 with unary generalized quantifiers. The class of connected finite graphs and classes of graphs defined by forbidden minors, are considered in the following two sections. We illustrate the use of this condition in the following examples.

5.3. Example. It is well known that there is a first-order sentence, which says that a graph is a union of disjoint cycles. Namely, let $\psi_k(x)$ be the following first-order formula, which says that a vertex x has degree at most k:

$$\forall y_1 \ldots \forall y_{k+1}(xEy_1 \wedge \ldots \wedge xEy_{k+1}) \implies (y_1 = y_2 \vee y_1 = y_3 \vee \ldots \vee y_k = y_{k+1})).$$

A graph is a union of disjoint cycles, if and only if it satisfies the first-order sentence $\forall x(\psi_2(x) \wedge \neg\psi_1(x))$.

We show, however, that the property 'a graph contains a cycle' is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. Let $\mathcal{C}$ be the class of all finite graphs which contain a cycle. Assume d is a positive integer. Let $\mathbb{A}$ consist of two disjoint parts, a cycle of length $2d+1$ and a path of length $2d$; then $\mathbb{A} \in \mathcal{C}$. Let the points in the path be $\{a_1, \ldots, a_d, a_{3d+2}, \ldots, a_{4d+2}\}$ and the points in the cycle $\{a_{d+1}, \ldots, a_{3d+1}\}$. Assume next that $\mathbb{B}$ is a path of length $4d+1$ and denote $B = \{b_1, \ldots, b_{4d+2}\}$. Obviously, $\mathbb{B} \notin \mathcal{C}$. In both structures there are $4d+2$ points and for each i $(1 \leq i \leq 4d+2)$ the points a_i and b_i have the same d-type. Thus $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent, and by Corollary 5.2, $\mathcal{C}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

Observe, that the same example shows also that connectivity and 2-colorability of finite graphs are not in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$, either. The first argument is easy to see because $\mathbb{A}$ is not connected but $\mathbb{B}$ is. The second one follows from our choice of the length of the cycle in $\mathbb{A}$. For odd cycles three colors are needed to color vertices such that no adjacent points have the same color. However, obviously any path can be colored with two colors.

5.4. Example. Let $\mathcal{C}_{\mathcal{R}}$ be the class of all finite rigid graphs. Recall from Section 2 that every graph in $\mathcal{C}_{\mathcal{R}}$ has only the trivial automorphism. We show that $\mathcal{C}_{\mathcal{R}}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. Again, assume that d is a positive integer. We now construct d-equivalent graphs $\mathbb{A} \in \mathcal{C}_{\mathcal{R}}$ and $\mathbb{B} \notin \mathcal{C}_{\mathcal{R}}$.

The vertex sets of $\mathbb{A}$ and $\mathbb{B}$ are $A = \{a_i \mid 1 \leq i \leq 4d+9\}$ and $B = \{b_i \mid 1 \leq i \leq 4d+9\}$. The edge set of $\mathbb{A}$ is

$$\begin{aligned} E(\mathbb{A}) = \{&(a_1, a_2), (a_2, a_4), (a_3, a_4)\} \cup \{(a_i, a_{i+1}) \mid 4 \leq i < 2d+4\} \\ &\cup \{(a_{2d+5}, a_{2d+6}), (a_{2d+6}, a_{2d+8}), (a_{2d+7}, a_{2d+8})\} \\ &\cup \{(a_i, a_{i+1}) \mid 2d+8 \leq i < 4d+9\}. \end{aligned}$$

Thus in both components of $\mathbb{A}$ there is one point with degree three and this point is an endpoint of three paths of different lengths. Because the longest path in the components are of different length $2d$ and $2d+1$, it can be easily seen that $\mathbb{A}$ is rigid (see Figure 3). The structure $\mathbb{B}$ is obtained from the structure $\mathbb{A}$ by taking an isomorphic copy of

Figure 3: Rigid and non-rigid structure

$\mathbb{A}$ and replacing the edges (b_{d+4}, b_{d+5}) and (b_{3d+8}, b_{3d+9}) by the edges (b_{d+4}, b_{3d+8}) and (b_{3d+9}, b_{d+5}) (see Figure 3). But now the other component of $\mathbb{B}$ is a path, and thus $\mathbb{B}$ is not rigid. However, it is easy to see that each vertex a_i and b_i have the same d-type, for $1 \leq i \leq 4d+9$. Thus we can conclude as above, that $\mathbb{A}$ and $\mathbb{B}$ are d-equivalent graphs and because $\mathbb{A}$ is rigid and $\mathbb{B}$ is not, $\mathcal{C}_{\mathcal{R}}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

6. Connectivity and *Mon* $\Sigma_1^1(\mathbf{Q}_u)$

In Sections 3 and 4 we mentioned that connectivity of finite graphs cannot be expressed in the logic *Mon* Σ_1^1 or in the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. The proofs of the main theorems in Section 3 use

Lemma 3.3 as an important tool. But if we replace the use of Lemma 3.3 by Proposition 5.1 in the proofs, we can extend the results to the logic *Mon* Σ_1^1 with unary generalized quantifiers. This is done in the present section. Hence, the result extends on one hand that in [5] and on the other hand the one in [11].

First we fix some notations. A *Mon* $\Sigma_1^1(\mathbf{Q}_u)$ sentence is of the form $\exists A_1 \ldots \exists A_k \phi$, where ϕ is a $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ sentence and each A_i is unary. A class $\mathcal{C}$ of finite graphs is said to be *Mon* $\Sigma_1^1(\mathbf{Q}_u)$, if it is the class of all finite graphs which satisfy some fixed sentence of *Mon* $\Sigma_1^1(\mathbf{Q}_u)$.

Next we need a similar game over a class of structures as was introduced in Section 3 but which is tailored for the logic *Mon* $\Sigma_1^1(\mathbf{Q}_u)$. In the resulting game there are a set C of c distinct colors and a class $\mathcal{C}$ of finite graphs given. But instead of an Ehrenfeucht-Fraïssé game, we use a *BEF*-game. This game is denoted by *c-BEF*$^r(\mathcal{C})$ and the rules of the game are as follows.

- The duplicator selects a member of $\mathcal{C}$ to be $\mathbb{A}$.
- The spoiler colors $\mathbb{A}$ with the c colors.
  Let $\mathbb{A}'$ be the resulting colored structure.
- The duplicator selects a member of $\overline{\mathcal{C}}$ to be $\mathbb{B}$.
- The duplicator colors $\mathbb{B}$ with the c colors.
  Let $\mathbb{B}'$ be the resulting colored structure.
- The spoiler and the duplicator play the game *BEF*$^r(\mathbb{A}', \mathbb{B}')$.

The following theorem shows that this is exactly the kind of game we are going to need.

6.1. Theorem. *Let $\mathcal{C}$ be a class of finite σ-structures. If for all c and r, the duplicator has a winning strategy in c-BEF$^r(\mathcal{C})$ -game, then $\mathcal{C}$ is not in Mon $\Sigma_1^1(\mathbf{Q}_u)$.*

Proof. Assume on the contrary, that $\theta = \exists A_1 \ldots \exists A_k \phi$, where $\phi \in \mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$, is such a sentence that for all finite σ-structures $\mathbb{A}$ we have $\mathbb{A} \in \mathcal{C}$ if and only if $\mathbb{A} \models \theta$. Choose $c = k$ and $r = qr(\phi)$. Let the duplicator select a σ-structure $\mathbb{A}$ such that $\mathbb{A} \models \theta$. The spoiler chooses now the relations A_i (giving the coloring) such that $\mathbb{A}' \models \phi$, where $\mathbb{A}' = (\mathbb{A}, A_1, \ldots, A_c)$. This means that for each A_i in θ the spoiler colors the subset of $\mathbb{A}$ which corresponds that existential quantification in θ. Then the duplicator selects a structure $\mathbb{B}$ in $\overline{\mathcal{C}}$. We assume that $|A| = |B|$ (otherwise the spoiler wins trivially). Let the duplicator color the vertices of $\mathbb{B}$ with the c colors according to his winning strategy and denote the resulting structure $(\mathbb{B}, B_1, \ldots, B_c)$ by $\mathbb{B}'$. Then the spoiler and the duplicator play the game *BEF*$^r(\mathbb{A}', \mathbb{B}')$. Because the duplicator has a winning strategy, it follows from Lemma 4.3 that $\mathbb{A}' \equiv^r_{\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)} \mathbb{B}'$ and hence $\mathbb{B}' \models \phi$. But this means that $\mathbb{B} \models \theta$, which is a contradiction. □

Now we have the following theorem.

6.2. Theorem. *Connectivity is not in Mon $\Sigma_1^1(\mathbf{Q}_u)$.*

Proof. We replace the use of Lemma 3.3 by Proposition 5.1 and Theorem 3.2 by Theorem 6.1 in the proof of Theorem 3.4 (in Section 3). Otherwise the proof is exactly the same (for details, see [6]). □

In [6] Fagin, Stockmeyer and Vardi considered connectivity in *Mon* Σ_1^1 and proved some probabilistic results which hold even if built-in relations of moderate degree are allowed. Using these results, we can imitate the proof of Theorem 3.5 (for details, see [6]) and prove a similar theorem with unary generalized quantifiers.

6.3. Theorem. *Connectivity is not in Mon* $\Sigma_1^1(\mathbf{Q}_u)$*, even in the presence of built-in relations of moderate degree.*

It should be noted that *BEF*-games cannot be applied in the presence of a linear order. Namely, when a bijection chosen by the duplicator maps a point $a \in A$ to a point $f(a) \in B$, the duplicator has to make sure that the number of points less than a in $\mathbb{A}$ must be equal to the number of points in $\mathbb{B}$ which are less than $f(a)$, with respect to that linear order. Otherwise the spoiler would win in the next two moves. Thus the chosen bijections must preserve the linear order. Hence there is essentially only one bijection which the duplicator can choose. This means that the bijections must be isomorphisms between $\mathbb{A}$ and $\mathbb{B}$. In other words, in the presence of linear order the duplicator has a winning strategy only if the structures $\mathbb{A}$ and $\mathbb{B}$ are isomorphic.

7. Forbidden minors

We saw in Section 5 that the condition given in [6] to guarantee a winning strategy for the duplicator in first-order Ehrenfeucht-Fraïssé game, gives a winning strategy for the duplicator in bijective Ehrenfeucht-Fraïssé games, too. This condition can be applied to many graph properties to show that they are not expressible in first-order logic with unary generalized quantifiers. Our aim is to prove as general results as possible. Hence, we consider classes of graphs defined by forbidden minors. Especially, the following results imply that planarity of finite graphs cannot be expressed in first-order logic with unary generalized quantifiers.

Recall the concept of forbidden minor and the definition of classes $\mathcal{C}_{\mathbb{H}}$ and $\mathcal{C}_{\mathcal{H}}$ from Section 2. We consider now logical definability of $\mathcal{C}_{\mathcal{H}}$. First we separate a few cases given in the following definition.

7.1. Definition. A finite graph $\mathbb{H}$ is *trivial*, if it is

- acyclic, and
- every point has degree at most three, and
- each connected component contains at most one point with degree exactly three.

A class $\mathcal{C}_{\mathbb{H}}$ is *trivial*, if $\mathbb{H}$ is, and a class $\mathcal{C}_{\mathcal{H}}$ is *trivial* if all graphs in $\mathcal{H}$ are trivial.

In other words, a finite graph $\mathbb{H}$ is trivial, if each connected component of $\mathbb{H}$ is of one of the following forms:

- an isolated point,
- a (chordless) path,
- three (chordless) paths with one endpoint in common.

In Figure 4 there are examples of trivial graphs. In the following lemmas some easy

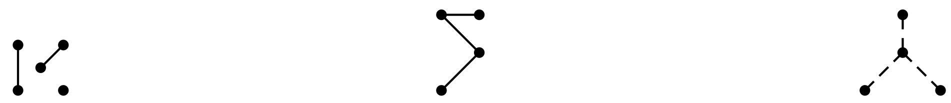


Figure 4: Trivial graphs

observations about trivial graphs are given.

7.2. Lemma. *Let u and v be vertices in a (not necessarily trivial) graph $\mathbb{H}$, such that deg$(u) \leq k$ and deg$(v) \leq \ell$. If w is the new vertice formed by contracting vertices u and v, then deg$(w) \leq k + \ell - 2$. The degrees of all other vertices do not increase.*

Proof. The vertex u was connected to at most $k - 1$ vertices not equal to v and the vertex v was connected to at most $\ell - 1$ vertex not equal to u. Thus the degree of w is at most $k + \ell - 2$. The second claim is obvious. □

7.3. Lemma. *If $\mathbb{H}$ is a trivial graph, then every minor of $\mathbb{H}$ is again trivial.*

Proof. Because only adjacent vertices are contracted, every minor of a trivial graph is acyclic as well. Every connected component of $\mathbb{H}$ contains at most one vertex with degree three, and thus according to Lemma 7.2, every minor of a connected component of $\mathbb{H}$ contains at most one point with degree three. Hence every connected component of a minor of $\mathbb{H}$ contains at most one vertex with degree three and the degree of the other vertices is at most two. Thus every minor of $\mathbb{H}$ is again trivial. □

Consider a graph $\mathbb{G}$ and assume that a trivial graph $\mathbb{H}$ is a minor of $\mathbb{G}$. According to Lemma 7.2, the degree of vertices that are adjacent to only vertices with degree two, does not increase in elementary contractions. Thus for every point in $\mathbb{H}$ with degree three, there must be such a point in $\mathbb{G}$ already. And every path in $\mathbb{H}$ must of course occur also in $\mathbb{G}$. Hence, $\mathbb{H}$ is in fact a subgraph of $\mathbb{G}$. On the other hand, if $\mathbb{H}$ is a subgraph of $\mathbb{G}$, then it is also a minor of $\mathbb{G}$. Namely, $\mathbb{H}$ is a minor of the subgraph of $\mathbb{G}$, where no edges are contracted. We have proved the following lemma.

7.4. Lemma. *A graph $\mathbb{G}$ has a trivial graph $\mathbb{H}$ as a minor if and only if $\mathbb{H}$ is a (not necessarily induced) subgraph of $\mathbb{G}$.*

The following lemma explains the chosen naming in the above definition.

7.5. Lemma. *If a class $\mathcal{C}_{\mathcal{H}}$ is trivial, then it is first-order definable.*

Proof. It follows from previous lemma that a graph $\mathbb{G}$ is in $\mathcal{C}_{\mathcal{H}}$ if and only if $\mathbb{G}$ has no graph in $\mathcal{H}$ as a subgraph.

Let $\mathbb{H} \in \mathcal{H}$, where $H = \{a_1, \ldots, a_n\}$, and let $\phi_{\mathbb{H}}$ be a first-order sentence, which says that a graph has $\mathbb{H}$ as a subgraph. That is,

$$\phi_{\mathbb{H}} = \exists x_1 \ldots \exists x_n (\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{n} x_i \neq x_j \wedge \bigwedge_{\substack{i,j=1 \\ (a_i,a_j) \in E(\mathbb{H})}}^{n} E(x_i, x_j))$$

Now we have

$$\mathbb{G} \in \mathcal{C}_{\mathcal{H}} \quad \text{if and only if} \quad \mathbb{G} \models \bigwedge_{\mathbb{H} \in \mathcal{H}} \neg \phi_H.$$

Hence $\mathcal{C}_{\mathcal{H}}$ is first-order definable. □

The trivial cases considered above are first-order definable. It turns out that in a way the other cases can be reduced to these trivial ones. Our aim is to show that no class of graphs defined by forbidden minors, except the ones with a trivial graph in the obstruction set, can be defined in first-order logic even with any set of unary generalized quantifiers.

Consider a connected non-trivial graph $\mathbb{H}$ with 5 vertices, where one vertex w has degree four and the degree of the other vertices is one. The vertex w can be subdivided to two vertices with degree three (see Figure 5). The rightmost graph $\mathbb{H}'$ in Figure 5

Figure 5: Graphs $\mathbb{H}$ and $\mathbb{H}'$

is also non-trivial, since it is connected and has two vertices with degree three. Note however, that we can not get rid of vertices with degree three by subdividing them to several vertices.

Next add two new vertices u^e and v^e corresponding vertices u and v in $\mathbb{H}'$, and an edge between u^e and v^e. The new vertices are not adjacent to any other vertices in $\mathbb{H}'$. Let $d \geq 1$ and consider the graph $\mathbb{H}_d$ obtained from this graph by expanding each edge to a path of length $2d$ (see Figure 6). This graph has $\mathbb{H}'$, and hence also $\mathbb{H}$, as a minor, since each path can be contracted to an original edge and auxiliar connected component can be contracted to one isolated vertex and that vertex can be deleted. Denote the path

Figure 6: Graphs $\mathbb{H}_d$ and $\mathbb{H}'_d$

corresponding edge e between vertices u and v by $(g_1, \ldots, g_{2d+1})$ and the path between vertices u^e and v^e by $(g_1^e, \ldots, g_{2d+1}^e)$. Consider the graph $\mathbb{H}'_d$ obtained when edges (g_d, g_{d+1}) and (g_d^e, g_{d+1}^e) are removed and they are replaced by edges (g_d, g_{d+1}^e) and (g_d^e, g_{d+1}) (see the rightmost graph in Figure 6). This graph is acyclic and in both connected components only one vertex has degree three and the degree of the other vertices is at most two. Hence, $\mathbb{H}'_d$ is a trivial graph. So by Lemma 7.3, $\mathbb{H}$ is not a minor of $\mathbb{H}'_d$.

However, there are as many vertices in both graphs $\mathbb{H}_d$ and $\mathbb{H}'_d$. Moreover, for each vertex x in one of the graphs, $Nbd(d, x)$ contains at most one vertex with degree not equal to two. This is because the distance between the vertices with degree not equal to two, is at least $2d$. There are as many points x in both graphs, where $Nbd(d, x)$ contains a vertex with degree three and as many points y, where $Nbd(d, y)$ contains a vertex with degree one. For other vertices z, $Nbd(d, z)$ is a path of length $2d - 2$. Hence, we can conclude that $\mathbb{H}_d$ and $\mathbb{H}'_d$ are d-equivalent.

It follows from Corollary 5.2, that $\mathcal{C}_{\mathbb{H}}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. This is because, as we saw, for every d the graph $\mathbb{H}_d$ has $\mathbb{H}$ as a minor, but $\mathbb{H}'_d$ does not have. Hence $\mathbb{H}_d \notin \mathcal{C}_{\mathbb{H}}$ and $\mathbb{H}'_d \in \mathcal{C}_{\mathbb{H}}$ and, $\mathbb{H}_d$ and $\mathbb{H}'_d$ are d-equivalent.

The same method described above applies to any non-trivial graph.

7.6. Lemma. *For every non-trivial graph $\mathbb{H}$, and every positive integer d, there are d-equivalent graphs $\mathbb{G}$ and $\mathbb{G}'$ such that $\mathbb{H}$ is a minor of $\mathbb{G}$ and $\mathbb{G}'$ has no non-trivial minors.*

Proof. First, for every vertex in $\mathbb{H}$ with degree greater than three, subdivide this vertice to several vertices with degree at most three in such a way that the resulting graph $\mathbb{H}'$ has $\mathbb{H}$ as a minor. Next add for each edge e between u and v in $\mathbb{H}'$ two new vertices u^e and v^e and an edge between u^e and v^e. The new vertices are not adjacent to any other vertices. Then expand all the edges in this graph to a path of length $2d$. Distinct edges correspond disjoint paths, i.e. there are no common vertices or edges, except possible common endpoints. It is obvious that the resulting graph $\mathbb{G}$ has $\mathbb{H}$ as a minor.

Our next task is to form such a graph $\mathbb{G}'$, that $\mathbb{G}'$ has no non-trivial minors. Let $(g_1, \ldots, g_{2d+1})$ be the path of length $2d$ corresponding an edge e between vertices u and v in $\mathbb{H}'$, and let $(g^e_1, \ldots, g^e_{2d+1})$ be the path between auxiliar vertices u^e and v^e. For each this kind of pair of paths, we remove edges (g_d, g_{d+1}) and (g^e_d, g^e_{d+1}) and replace them by edges (g_d, g^e_{d+1}) and (g^e_d, g_{d+1}). The other edges are exactly as in $\mathbb{G}$.

For every vertex u with $deg(u) = k$ in $\mathbb{H}'$, there is a connected acyclic component in $\mathbb{G}'$, where the vertex corresponding u has degree k and the degree of the other vertices is at most two. Every vertex in $\mathbb{H}'$ has degree at most three, and thus $\mathbb{G}'$ is a trivial graph. So by Lemma 7.3, $\mathbb{G}'$ has no non-trivial minors.

Both graphs $\mathbb{G}$ and $\mathbb{G}'$ have the same vertex set and each vertex has the same d-type in both graphs. Thus $\mathbb{G}$ and $\mathbb{G}'$ are d-equivalent. □

Now we are able to prove the following theorem for arbitrary class $\mathcal{C}_{\mathcal{H}}$.

7.7. Theorem. *Assume that $\mathcal{H} = \{\mathbb{H}_1, \ldots, \mathbb{H}_k\}$ is a collection of finite graphs, where no $\mathbb{H}_i$ is a minor of $\mathbb{H}_j$, when $i \neq j$. If $\mathcal{C}_{\mathcal{H}}$ is trivial, then it is $\mathcal{L}_{\omega\omega}$-definable. If no $\mathbb{H}_i \in \mathcal{H}$ is trivial, for $1 \leq i \leq k$, then $\mathcal{C}_{\mathcal{H}}$ is not definable even in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.*

Proof. We have proved the first claim in Lemma 7.5. We prove now the second one. Let d be a positive integer. We show, using Lemma 7.6, that there are d-equivalent graphs $\mathbb{G}'$ and $\mathbb{G}$ such that $\mathbb{G}' \in \mathcal{C}_{\mathcal{H}}$ and $\mathbb{G} \notin \mathcal{C}_{\mathcal{H}}$. Then the claim follows from Corollary 5.2.

Choose one $\mathbb{H}_i \in \mathcal{H}$, which is not trivial. Let $\mathbb{G}$ and $\mathbb{G}'$ be given by Lemma 7.6 for $\mathbb{H}_i$ and d. Because $\mathbb{H}_i$ is a minor of $\mathbb{G}$, $\mathbb{G}$ is in $\overline{\mathcal{C}_{\mathcal{H}}}$. On the other hand, $\mathbb{G}'$ has no non-trivial minors. Because no graph in $\mathcal{H}$ is trivial, it follows that $\mathbb{G}'$ does not have any graph in

$\mathcal{H}$ as a minor. Hence $\mathbb{G}'$ is in $\mathcal{C}_{\mathcal{H}}$. Furthermore $\mathbb{G}$ and $\mathbb{G}'$ are d-equivalent, and hence the theorem follows from Corollary 5.2. □

Theorem 7.7 shows that many interesting classes $\mathcal{C}_{\mathcal{H}}$ are not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. The following corollary gives two such classes.

7.8. Corollary. *Let $\mathcal{C}$ be any of the following two classes of finite graphs:*

- *the class of acyclic graphs;*
- *the class of planar graphs.*

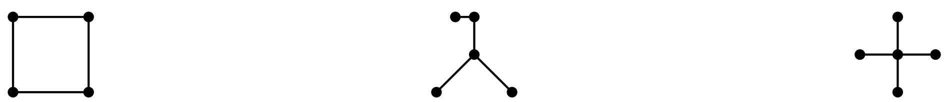
Then $\mathcal{C}$ is not definable in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

Proof. By Theorem 2.5 no graph in the obstruction sets for these classes is trivial. □

7.9. Remark. Gaifman proved in [7] inexpressibility in first-order logic of many graph properties, including planarity, using a certain kind of locality condition. Using Proposition 5.1, inexpressibility of many of these properties can easily be seen to extend to $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

If an obstruction set $\mathcal{H}$ contains both trivial and non-trivial graphs, then the proof of Theorem 7.7 cannot be applied. This is because the connected components in the graph $\mathbb{H}'_d$ constructed in the proof may have a trivial graph in $\mathcal{H}$ as a minor. In this case the logical definability of the class $\mathcal{C}_{\mathcal{H}}$ is not quite obvious. Indeed, in the following example there are two such classes, one of which is definable in $\mathcal{L}_{\omega\omega}$ and the other is not definable even in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

7.10. Example. Assume the graphs $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$ are as in Figure 7. Then $\mathbb{H}_1$ and $\mathbb{H}_3$ are non-trivial but $\mathbb{H}_2$ is trivial. Let $\mathcal{H} = \{\mathbb{H}_1, \mathbb{H}_2\}$. When the algorithm in the proof of Theorem 7.7 is repeated for the class $\mathcal{C}_{\mathcal{H}}$, the cycle $\mathbb{H}_1$ is chosen to be considered and the resulting graph $\mathbb{G}'$ consists of four connected components, where each of the components is a path of length $4d$. Thus the proof of Theorem 7.7 shows that $\mathcal{C}_{\mathcal{H}}$ is not definable in the logic $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$. Assume then that $\mathcal{H}' = \{\mathbb{H}_2, \mathbb{H}_3\}$. If we now try to repeat the algorithm



$\mathbb{H}_1$ $\mathbb{H}_2$ $\mathbb{H}_3$

Figure 7: Trivial and non-trivial graphs

in the proof of Theorem 7.7, we see that the graph $\mathbb{G}'$ has $\mathbb{H}_2$ as a minor. Thus this resulting graph is not in $\mathcal{C}_{\mathcal{H}'}$. But in this case, $\mathcal{C}_{\mathcal{H}'}$ is even first-order definable. Namely, if the vertex with degree four in the graph $\mathbb{H}_3$ is subdivided to several vertices with degree three, we see that this graph has $\mathbb{H}_2$ as a minor. Thus, if a graph has $\mathbb{H}_3$ as a minor, then it has $\mathbb{H}_2$ as a minor or has $\mathbb{H}_3$ as a subgraph. Hence, the first-order sentence, which says, that a graph has neither $\mathbb{H}_2$ nor $\mathbb{H}_3$ as a subgraph, defines the class $\mathcal{C}_{\mathcal{H}'}$.

8. Conclusions

In this paper we have proved that connectivity of finite graphs cannot be expressed in monadic existential second-order logic with any set of unary generalized quantifiers, even if sentences are allowed to contain built-in relations of moderate degree. Furthermore, we proved that for any class (if it is not in some sense trivial) defined by forbidden minors, there cannot be a sentence of $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ which defines this class.

In both proofs of these results bijective Ehrenfeucht-Fraïssé game was replaced by a combinatorial argument, namely counting d-types (Proposition 5.1). When this result is combined to a new game, which is essentially a combination of Ajtai-Fagin -game and bijective Ehrenfeucht-Fraïssé game with respect to unary quantifiers, we proved the desired result that connectivity is not in *Mon* $\Sigma_1^1(\mathbf{Q}_u)$. Combining these tools with probabilistic arguments developed by Fagin, Stockmeyer and Vardi ([6]), we were able to extend the result even in the presence of built-in relations of moderate degree. On the classes of finite graphs, which are defined by forbidden minors, we constructed a graph that obviously has the forbidden minors and on the other hand a graph which does not have the minors. However, the graphs are d-equivalent. This implies that, for example, the PTIME computable property of graphs of being planar, cannot be expressed in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$.

It is important to notice that Proposition 5.1 holds only for unary quantifiers and for bounded quantifier rank. It would be interesting to know, whether there is a modification of this result which would apply to quantifiers of greater arity, or can there be that kind of modifications at all.

Another interesting open question is, whether Theorem 7.7 can be extended to fixpoint logic with unary generalized quantifiers. This is, is the class $\mathcal{C}_{\mathcal{H}}$, where no graph in $\mathcal{H}$ is trivial, definable in fixpoint logic with unary quantifiers. If this is not the case, for example planarity would be a natural *PTIME* computable property, which is not definable in this logic.

Furthermore, it should be noted that none of the methods considered in this paper, can be applied in the presence of linear order. This is because bijective Ehrenfeucht-Fraïssé games cannot be applied in the presence of linear order. Hence, we can ask if, for example, connectivity of finite graphs is in $\mathcal{L}_{\omega\omega}(\mathbf{Q}_u)$ in the presence of linear order.

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