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NOTE

Monotone Subsequences in Any Dimension

Ryan Siders

Math Department, Princeton University, Princeton, New Jersey 08544
E-mail: rcsiders@math.princeton.edu

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We exhibit sequences of n points in d dimensions with no long monotone subsequences, by which we mean when projected in a general direction, our sequence has no monotone subsequences of length $\sqrt{n}+d$ or more. Previous work proved that this function of n would lie between $\sqrt{n}$ and $2\sqrt{n}$; this paper establishes that the coefficient of $\sqrt{n}$ is one. This resolves the question of how the Erdős–Szekeres result that a (one-dimensional) sequence has monotone subsequences of at most $\sqrt{n}$ generalizes to higher dimensions.

1. INTRODUCTION

Motivation

7, 8, 9, 4, 5, 6, 1, 2, 3 is Erdős–Szerkeres sequence of 9 numbers with short monotone subsequences. There are 30 subsequences of length 3, which can be easily seen in this representation:

```
        9
     8
  7
                 6
              5
           4
                          1
                       2
                    3
```

With this pattern one could list k^2 numbers with no subsequence longer than k. In the 60 years since this was proven, at least two papers have asked whether this could be generalized to higher dimensions. That is: can

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one list n points of a d dimensional vector space, no large subset of which tends in any direction? The idea that “no large subset tends in any direction” was considered a good generalization of the idea that Erdős–Szerkeres sequence had no long monotone subsequences. It means specifically, that when projected in any direction, the sequence of points has no long subsequences. In this paper we construct such sequences in arbitrary dimensions.

How would one pick 9 points in the plane which have no long subsequences? One might begin by declaring that, when projected along the x axis, the sequence should be the Erdős Szerkeres sequence. Unfortunately, this is the beginning of the end, because as one rotates the projection angle away from the x axis, one inevitably discovers a line of projection along which the sequence has long runs. So, the first step in our construction is to replace the Erdős–Szerkeres sequence the sequence:

```
      1
  2
              3
4
        5
                6
    7
            8
          9
```

We consider the diagram above to be more readable when presented in the following way:

```
    1
  2   3
4   5   6
  7   8
    9
```

This is nicely ambiguous as well; any of the sequences it might mean have minimally short subsequences.

To get a 2-dimensional sequence, we declaring that, when projected along the x axis, the sequence will be the sequence above. In the first section, below, we tell how to place the points in the plane so that as we swing the projection angle away from the x-axis, and ultimately back to it, the length of the longest monotone subsequence remains low. This is explained in the first section. In the opinion of the author, this is not very enlightening. But in Section 2, we create a sequence in 3 dimensions, starting with the assumption that when projected into, say, the x-y plane, the sequence is the 2-dimensional sequence we created in Section 1. Then we explain how to place points in 3 dimensions so that when projected to all other planes we get optimal subsequences. Then, just as we generalized from 2 to 3 dimensions, we generalize from 3 to 4, and on upto construct a sequence of points in arbitrary dimension with a small subsequence.

No such constructions have been created previously. In [1] it is shown that a random ordering will have a maximal subsequence of length $2\sqrt{n}$. In that paper we gave our extremal sequence of dimension 2 as an example of an ordering with subsequences of length at most $\sqrt{n}+1$, with an explicit description of where to plot points in the plane. Earlier, Martin Kruskal [2] conjectured what our function $\sqrt{n}+d$ would look like in higher dimensions; if k is the length of the longest monotone subsequence, he finds that $n \geqslant k^2 - kd - k + d + 1$ in many examples, and conjectures that this is true generally. Our result is a reply to this conjecture:

THEOREM. *Our sequence of n points in d dimensions has no monotone subsequences (when projected in any direction) of length* $\sqrt{n}+d$ *or more.*

Given [1], the surprise is that we can prove that the coefficient of $\sqrt{n}$ is one. Could there exist sequences with shorter monotone subsequences? Not by much. The coefficient of $\sqrt{n}$ is minimal, by the Erdős–Szekeres result. And, since any n points in $n-1$ dimensions can be projected so as to appear in any order, the coefficient one on d achieved by our construction is also minimal. I.e., no construction will best our theorem by building sequences with no monotone subsequence of length $f(n)+0.9d$, where $f(n)/n$ is a vanishingly small ratio, for take $d=n-1$ very large, and we conclude that there is no monotone subsequence of length $0.95d$. But, in reality, as there is just one more point than the number of dimensions, we can project so that the entire sequence runs in one direction. Previously, we knew that $f(n)$ was between $\sqrt{n}$ and $2\sqrt{n}$. The importance of this construction is that the coefficient on *sqrtn* is 1. In [1] we allowed projection in a direction in which two points may project to be equal; clearly this cannot increase the length of the longest run by more than d.

SECTION 1: DIMENSIONS 1 AND 2

A list of n numbers with $\sqrt{n}$-long monotonic subsequences is found by writing the numbers in a diamond, and reading from left to right, taking the elements of a column in any order. The elements of each row appear in arbitrary order, but if $a<b$, a does not appear in a row lower than b does.

```
                                                    1
                        1                      2         3
   1               2         3            5         4         6
2       3     4         5         6   10        7         9         8
   4               7         8            13        12        11
                        9                      15        14
                                                    16

 2143           427159836          ...15, 2, 7, 1, 12, 16, 4...
```

In these sequences, no monotonic subsequence is longer than $\sqrt{n}$. If $a < b$ then b appears either to the right of a or below a in the diagram. So each step along an increasing subsequence is a step down or to the right in the diagram. But no path of such steps is longer than $\sqrt{n}$. This solves the one dimensional problem: we exhibit a sequence with no monotone subsequences of length $\sqrt{n}+1$ or more.

In two dimensions, place the first point of our sequence at a great distance from the rest. This can be interpreted two ways. We can place the latter points first, and then put the first one a great distance away. Or we can place the first point arbitrarily, and then restrict ourselves to placing later points inside some disk far from the first point. Because the first point is far from the later points, the projection vectors along which this first point projects equal to other points lie in a small arc in the circle of projection vectors (Fig. 1). If we swing a time-dependent projection vector through this arc, the first point (4 in the example below) move from being first to being last in the projected sequence of points. We have moved the first number to be last, disturbing the diamond pattern. As the time-dependent projection vector continues to swing, we want the diamond pattern to be repaired. What is required is that the last number (6 in the example below) move to be first.

```
     1                                  1
  2     3                            2     3
     5     6  4                  6      5     4
  7     8                            7     8
     9                                  9

4 moves to be last...  6 moves and the diamond is restored.
```

We need to place 6 relative to 4 and to the remaining points so that this occurs. Place 6 far from the remaining points. so that its arc (the arc of projection vectors along which it appears among the remaining points) is separate from 4's arc, but is adjacent to it on the circle of projection vectors.

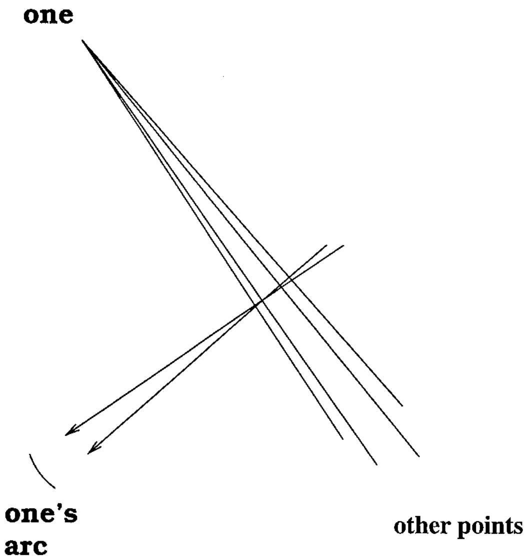

FIG. 1. As the point "one" is far from the other points, the projections in which it appears equal to another lie close together on the circle of projection angles.

As a time-dependent projection vector sweeps across this arc, the second number, 6, moves in the diagram from its initial position to the position 4 used to occupy. Continue in this way to space the points so that each has an arc during which it crosses over the remaining points.

We can imagine a nested family of disks. Each disk is free to spin about its center, through which it is pinned near the edge of the next larger disk. Each smaller disk is just a fraction of the size of the next larger disk. At the center of each disk is written one number. 4 is at the center of the large disk, so 4 is far from the remaining numbers; 6 is at the center of the next larger disk; we rotate 6's disk so that the arc of lines between 6 and the remaining numbers is separate from the arc of lines between 4 and the remaining numbers. At the center of the next disk is written, say, 2. We turn this disk so that 2's arc follows 4's and 6's and is separate from them (Fig. 2). We determine thus a sequence of arcs, say, 462378195. We include 5 in this list so as to have listed all numbers 1..9, although no numbers after five remain to be placed, so there is no arc of lines between 5 and all of them. As we move to each higher dimension, one more of the numbers we list to describe our construction will become vacuous, until in $n+1$ dimensions the whole sequence is meaningless.

The sequence 462378195 in which the arcs occur around the circle describes enough about our 2-dimensional construction for us to verify that it will

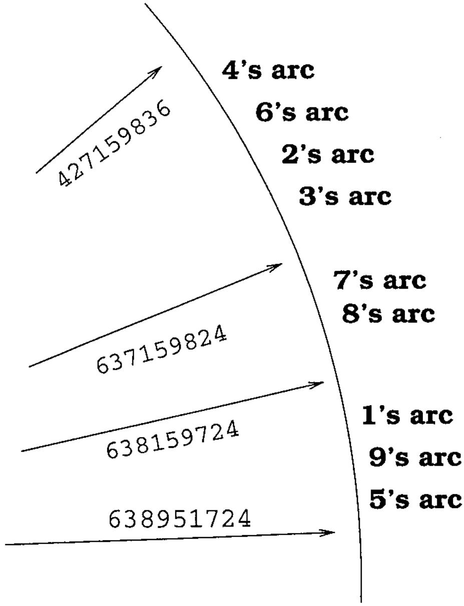

FIG. 2. When the projection angle lies on one side of 4's arc, 4 appears as the leftmost point. Projecting on the other side of 4's arc, 4 appears last. As we sweep *n*'s arc, *n* moves across the remaining points. After sweeping across the arc corresponding to each point, the sequence in which points appear is inverted.

have no long monotone subsequences: After our time-dependent projection vector swings past pairs of arcs, the diamond pattern is first broken and then restored. The broken diamond pattern has runs no longer than a diamond pattern with an additional, $(n+1)$th, element thrown in. So the longest monotone subsequence will trace a diagonal across our diamond pattern, and involve the extra number, so the maximal length is $\sqrt{n}+1$.

We list the arcs in pairs so that the first of each pair moves point a to point b's place in the diamond. We want the second of this pair to move b to a's place in the diamond. In the construction given above, also want to move each point across all the remaining points. So a and b must be the first and last points in the sequence which we haven't yet moved. This is guaranteed if we read the diamond

```
        1
    2       3
4       5       6
    7       8
        9
```

from left to right, ordering the opposite columns (here $\{2, 7\}$ and $\{3, 8\}$) in the opposite order. So we start with the sequence 427159836 but not 427159386 along one line in our two-dimensional construction.

SECTION 2: DIMENSIONS 3 AND HIGHER

In three dimensions we place the first point far from the others, relative to their distances apart. The vectors along which the first point projects equal to another point lie, on the sphere of projection vectors, in a narrow cylinder about some circle (Fig. 3). Along one longitude of the sphere, these cylinders cut out arcs. We order these arcs according to our optimal two-dimensional construction (Fig. 4, the vertical column on the right hand).

Along this longitude we encounter the cylinders of 4, then 6, then 2, 3, 7, 8, 1, 9, 5. How should this order on cylinders change as we move from this longitude around to various other longitues? We index the longitudinal lines with the points of the equator. At each point of the equator we write the order in which the cylinders occur when viewed down that longitudinal line. Now, how should these orderings change as we move around the equator? Along the initial longitudinal line, the cylinder of 4 appears at the top, and so is written first on the list. Let us have 4 move from first to last. Initially, 6 was second; let 6 move to be 2nd to last. Now, in order, take the nth cylinder of the ordering along the initial longitudinal line and move it to be nth from the last. First one cylinder moves past the others. Then

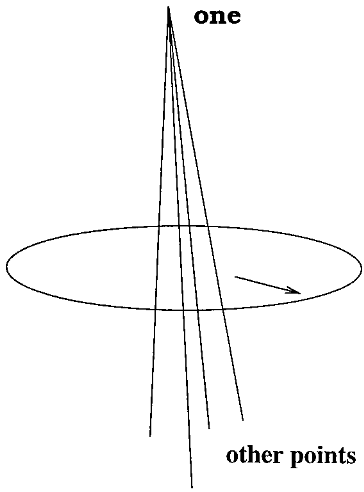

FIG. 3. As the point "one" is far from other points, the projections in which it appears equal to another gather near a circle on the sphere of projection angles.

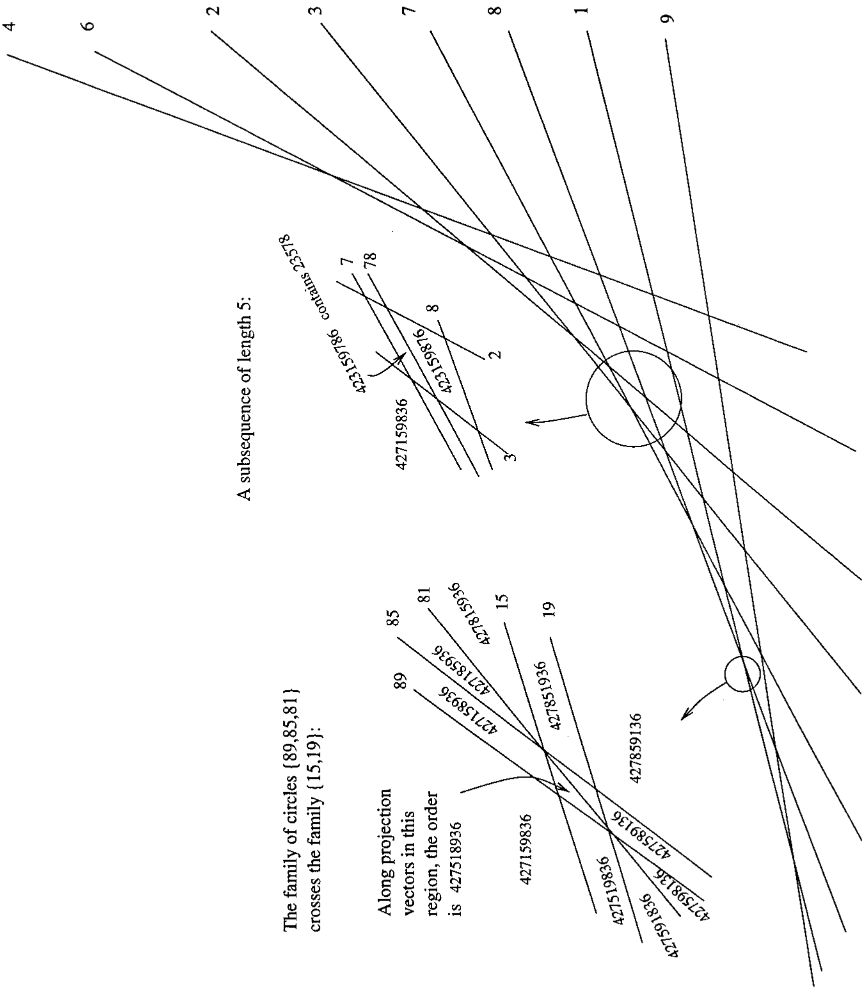

FIG. 4. On the sphere of projection vectors in three dimensions lie circles where two of n ($=9$ here) points appear to be equal. The pair 89 labels the circle of projections under which $8=9$. On the right, the number 8 indicates where the three circles 89, 85, 81 lie close together. Where 8 crosses 1 in the grid, we observe the circles 89, 0.85, 81 crossing the circles 15, 19. Where 85 crosses 15, $8=5=1$, so this point meets 81 as well.

another moves past all remaining ones, and so on... the order in which they cylinders move is 462378195, the same as the order in which they appeared down the initial longitudinal line. This is surely possible, if the last cylinders are very close to the equator, and cylinder 4 varies dramatically away from the origin. Each cylinder containes a great circle about the equator. We specify the cylinder's location by its point along the initial longitudinal line, and the angle it subtends with the initial longitudinal line. We require, without loss of generality, that all our cylinders intersect the initial longitudinal line on one side of the equator. We will have 4 slanting most dramatically, so that it passes through the other cylinders and on through the equator, within only a few degrees of arc about the equator. 6 will slant less dramatically, so that it begins to intersec the other cylinders only after 4 has passed through the equator. Only after 6 has passed through all cylinders, and on through the equator, will 2 begin to intersect any other cylinders. Each cylinder in turn passes the remaining ones. The whole process can be scaled to occur within a few degrees of arc on one side of our initial longitudinal line. But, for visibility, we have drawn Fig. 4 so that circle 4 does not pass through all other circles before circle 6 begins to drop through some others. Two great circles cross only once on each side of the sphere. So once they have crossed in the proscribed manner, they remain disjoint until on the other side of the sphere they cross again, in exactly the same pattern, 4 crossing all, then 6 crossing the remaining cylinders, and so on. Figure 4 shows the sphere of projection angles in three dimensions; the projection of nine points with a monotone subsequence of length $\sqrt{n}+2=5$ is indicated.

The constructions are inductively carried out in higher dimensions; dimension d generalizes the construction of dimension $d-1$ just as the construction in three dimensions generalizes that in two. In d dimensions we choose that one codimension one subspace of the space of all projection vectors matches the $d-1$ dimensional example we have already constructed. There remains a circle of cospheres. The picture is just as in dimension 3 (Fig. 5). The label $\gamma =$ "462378195" marking a point on the circle indicates a $d-1$-dimensional pattern, and the order "462378195" dictates in what order the numbers in γ are to move from front to back as we move around the circle in Fig. 5.

In each subsequent dimension we knock one more number out of the diamond pattern than we did in the previous dimension because we sometimes split up a single pair of numbers which together effect a trading of two numbers in the diamond pattern. In d dimensions where $d-1$ $(d-2)$-dimensional hyperspheres intersect in the great $d-1$-dimensional sphere of projection vectors, $d-1$ elements will be dislodged from the diamond pattern; one for each circle, and perhaps simultaneously dislodged as we simultaneously cross $d-1$ circles. For this reason, we may simultaneously

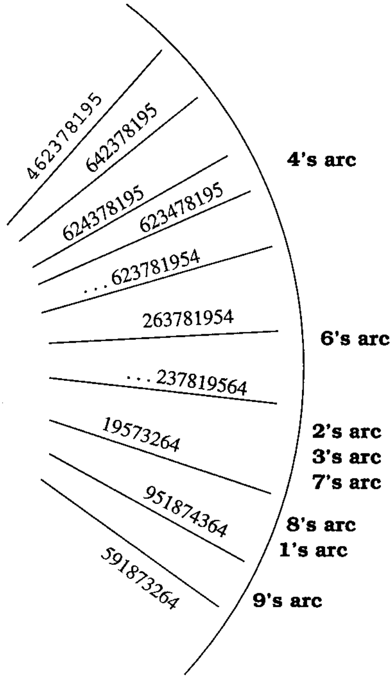

FIG. 5. To induct to dimension n consider the circle of $n-1$-spheres. On one, the arcs appear in the order 462378195. Moving down, we move each number across the remaining ones. This sphere, indexed by the sequence in bold, becomes the first n-sphere of our construction in dimension $n+1$.

knock one more number out of the diamond pattern that it was possible to in the previous dimension. In each subsequent dimension, the length of the longest monotone subsequence may increase by at most one over the previous dimension, so we know that in these constructions the monotone subsequences are shorter than $\sqrt{n}+d$.

We finish with a comment about constructibility. We argued before that it was possible to put circles on spheres with our desired property. But is it possible to put points inside spheres so that their bisectors will cut the sphere in these circles? Yes. Each point is placed "far away" from all subsequent points. This can be achieved if the distances between each point and those to be placed later decreases by an order of magnitude each time we place a point. This difference of scale gives us great freedom to move the latter points around significantly. Each circle may be placed on the sphere in any general position whatsoever. So, in particular, we have the freedom to put the circles on the sphere in the manner desired in our construction.

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