For exercises 4 and 5 recall that a reproducing kernel Hilbert space \( \mathcal{R} \) of a centered continuous Gaussian process \( X = (X_t)_{t \in T} \) with covariance \( K = (K(t, s))_{t, s \in T} \) is the Hilbert space characterized by

(i) \( K(t, \cdot) \in \mathcal{R} \) for all \( t \in T \),

(ii) \( \langle K(t, \cdot), K(s, \cdot) \rangle_{\mathcal{R}} = K(t, s) \).

1. Let \( p \geq 1 \). Define, for Wiener polynomials \( F \),

\[
\| F \|_{1,p}^p = \mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{H}^p].
\]

Show that \( \| \cdot \|_{1,p} \) is a norm, and consistent with an inner product if and only if \( p = 2 \).

2 & 3. Formulate appropriately and prove formulas 3.3.6(i)–3.3.6(iv) for \( F, G \in \mathcal{D}^{1,p} \).

4. Let \( X \) be a \( d \)-variate Gaussian random vector. What is its reproducing kernel Hilbert space?

5. Let \( X \) be a Gaussian process given by the Volterra equation

\[
X_t = \int_0^t k(t, s) \, dW_s,
\]

where \( k \in L^2([0, 1]^2) \cap C^{1,1}([0, 1]^2) \) and \( W \) is a standard Brownian motion. What is the “isonormal Hilbert space” \( H \) for this \( X \)? What is the reproducing kernel Hilbert space \( \mathcal{R} \) for this \( X \)? What is the isometry between \( H \) and \( \mathcal{R} \)?