# Approximations and limit theorems for likelihood ratio processes in the binary case

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**Abstract**

We study the asymptotic properties of the likelihood ratio processes for a sequence of binary filtered experiments. First we prove approximation results for the log-likelihood ratio processes and then apply them to obtain weak limit theorems. Here the limiting process is the stochastic exponential of a continuous martingale. Our results extend the corresponding results in the well-known monograph of Jacod and Shiryaev [16, Chapter X].

It turns out that the main results are valid for nonnegative supermartingales, too.

*Key words*: asymptotic mixed normality, asymptotic normality, binary filtered experiment, contiguity, Hellinger process, likelihood ratio process, nonnegative supermartingale, weak convergence

*AMS-classification:* **60F05, 60F17, 60G30, 60G44, 62E20**

## 1 Introduction

Let $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}^n_t)_{t\in\mathbb{R}_+}, (P^n, P'^n))$ be a sequence of binary filtered statistical experiments. Denote by $Z^n$ the (generalized) density process of $P'^n$ with respect to $P^n$. Then $Z^n$ is a $P^n$-supermartingale with $P^n$-a.s. finite values. In the paper we study approximations and limit theorems (finite-dimensional and functional) for $Z^n$ or the log-likelihood ratio processes $\log Z^n$. More precisely, we consider the situation where, first, *the contiguity* $(P_t^n) \lhd (P_t'^n)$ holds. This means that the cluster points of $\mathcal{L}(\log Z_t^n \mid P^n)$ are concentrated on $\mathbb{R}$. Second, *a Lindeberg-type condition* is imposed. As we shall see, it is closely related to the property $\sup_{s\le t} |\Delta Z_s^n| \xrightarrow{P^n} 0$. Thus, if additionally the laws $\mathcal{L}(\log Z^n \mid P^n)$ of the processes $\log Z^n$ are tight in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, [-\infty, \infty))$, then their cluster points necessarily have the support in $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$.

Such a situation has been extensively studied in various papers and books. Jacod and Shiryaev [16, Theorems X.1.12, X.1.16, and X.1.64] studied the conditions to achieve weak (finite-dimensional or functional) convergence of $Z^n$ to the stochastic exponential of a continuous Gaussian martingale, see also Greenwood and Shiryayev [7], Kordzakhia [18], Vostrikova [32, 33], Dzhaparidze and Valkeila [5]. The conditions implying the functional convergence of $Z^n$ under $P^n$ to the stochastic exponential of a certain continuous martingale were obtained by Jacod and Shiryaev [16, Theorems X.1.59 and X.1.65]. Mordecki [26] considered the functional convergence of $Z^n$ to the stochastic exponential of a conditionally Gaussian continuous martingale. Some other results of these types can be obtained as special cases of results proved in Mémin [25], Jacod [13], Coquet and Jacod [4], Kramkov [19] for more general situations.

In contrast to the papers mentioned, we first prove an *approximation* result for the processes $\log Z^n$. It says that in the considered situation we have a *uniform* (in $t$ over a fixed or every compact interval) approximation in probability of the processes $\log Z^n - \log Z_0^n$ by the processes

$$2m^n - 2\langle m^n, m^n \rangle, \tag{1.1}$$

where $m^n$ are $P^n$-locally square-integrable martingales easily constructed from $Z^n$; moreover, $\langle m^n, m^n \rangle$ are uniformly approximated by $2h^n$, where $h^n$ is the Hellinger process of order $1/2$ for $P^n$ and $P'^n$. The assumptions used to prove this approximation include the contiguity $(P_t^n) \lhd (P_t'^n)$, the functional Lindeberg condition on the jumps of $m^n$, and a mild additional condition on the Hellinger process of order 0 for $P^n$ and $P'^n$. These assumptions do not imply tightness properties of $Z^n$, $\log Z^n$, $m^n$, or $h^n$ in the corresponding Skorokhod spaces.

Having proved this approximation result, we can obtain functional or finite-dimensional weak convergence of $\log Z^n$ under $P^n$ in different settings in a unified manner, reducing the problem to corresponding results for weak convergence of locally square-integrable martingales $m^n$. Thus, our methodology of proving limit theorems for likelihood ratio processes is quite different from that used in the monograph by Jacod and Shiryaev [16], where limit theorems for semimartingales were applied (almost) directly to the log-likelihood processes $\log Z^n$. We find it more convenient to deal with the *stochastic* logarithm $y^n$ of the square root $\sqrt{Z^n}$ of the likelihood and with the components of the Doob–Meyer decomposition of $y^n$. Neglecting here technical difficulties arising from the fact that $Z^n$ may vanish, we have the representation

$$\sqrt{Z^n} = \sqrt{Z_0^n}\mathcal{E}(y^n), \tag{1.2}$$

where $\mathcal{E}(\cdot)$ stands for the stochastic exponential, and the Doob–Meyer decomposition of the locally square-integrable supermartingale $y^n$ has the form

$$y^n = m^n - h^n,$$

where $m^n$ and $h^n$ are the same as above.

Mémin [25] was the first who used the representation (1.2) to prove limit theorems for likelihood ratio processes in the general setup. He studied the functional convergence of $\log Z^n$ to a process with independent increments. Jacod and Shiryaev in Bibliographic Comments to their monograph [16] wrote that Mémin's method is "simpler, but it only gives the functional convergence". We agree with the first part of this statement but the second one seems to be too pessimistic. Of course, the functional convergence, say, of $y^n$ to $y$ implies the functional convergence of $\mathcal{E}(y^n)$ to $\mathcal{E}(y)$ under mild additional assumptions, whereas one cannot expect the same for the finite-dimensional convergence. However, it turns out that conditions providing the weak convergence of $y_t^n$ for a fixed $t$, may imply the weak convergence of $\mathcal{E}(y^n)_t$ as well; our approach illustrates this point.

A link between limiting behaviour of $y_t^n$ and $\log Z_t^n$ in nonfunctional setup is certainly not surprising because it is well known for independent observations (i.e. when $\mathbb{F}^n$ are discrete-time filtrations, and $\mathbb{F}^n$, $P^n$, and $P'^n$ have a product structure). Le Cam [20] used this link to describe all possible limiting distributions of likelihood ratios in this setting under the assumption that no single observation gives much information, see also Le Cam [21, Chapter 16] and Le Cam and Yang [22, Chapter 5]. In the case of Gaussian limits related references are also Oosterhoff and van Zwet [28] and Strasser [31, Chapter 12]. If the observations are discrete but not independent, a similar approach was used by Greenwood and Shiryayev [7], though they proved only the functional convergence, and by Fabian and Hannan [6] in nonfunctional setup.

Finally, we would like to note that all the assumptions in our main theorems (including the definitions of the processes entering therein) are expressed, essentially, in terms of the measures $P^n$. This is convenient in the case of parametric models when usually $P^n$ correspond to a fixed particular value $\vartheta_0$ of the parameter, while $P'^n$ correspond to parameter values of the form $\vartheta_0 + \delta_n \alpha_n$, where $\{\delta_n\}$ is a normalizing sequence of numbers or matrices. There is also another advantage of such an approach. It turns out to be unimportant that $Z^n$ are the density processes of some measures $P'^n$ with respect to $P^n$, and one can reformulate our results in the form of approximation or limit theorems for nonnegative supermartingales $Z^n$ on filtered spaces $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^n)$. This allows us to extend the area of statistical applications of our results to the case of partial likelihood processes, as in Jacod [15]. The obtained results can also be applied to problems in stochastic finance, where supermartingales appear naturally, for example in the case where we consider the discounted price process with respect to an equivalent local martingale measure. Thus, our limit theorems can be used to prove convergence of prices under different local martingale measures and/or in different market models. An example of such an application can be found in [8].

In statistical applications one is concerned with general parametric models rather than binary models. In the accompanying paper [9] we study the question when the approximation (1.1) (now depending on the parameter) is asymptotically linear in the first term and (hence) asymptotically quadratic in the second term. This allows us, in particular, to prove (functional or nonfunctional) local asymptotic normality, or local asymptotic mixed normality, or local asymptotic quadraticity.

The paper is organized as follows. In Section 2 we introduce basic objects and formulate the approximation results. We also discuss links between the assumptions in these theorems and conditions for contiguity, asymptotic negligibility of jumps, and conditions used in other papers. Section 3 contains six limit theorems. First two theorems deal with finite-dimensional and functional weak convergence of likelihood ratio processes to the stochastic exponential of a continuous Gaussian martingale; their assumptions are equivalent to those in Jacod and Shiryaev [16] with the difference that we derive necessary conditions even for the finite-dimensional convergence along a dense subset. In the next two theorems we consider the same types of convergence if the limiting process is the stochastic exponential of a conditionally Gaussian continuous martingale and a nesting condition is satisfied; the functional result extends that of Mordecki [26]. The last two theorems deal with the case where the limiting process is the stochastic exponential of a continuous martingale; only the functional convergence is considered. In the first theorem the stochastic logarithm of the limiting likelihood ratio process is assumed to be the unique solution of a martingale problem. In the second theorem the logarithm of the limiting likelihood ratio process is assumed to be the unique solution of a martingale problem; this theorem is close to the corresponding result in Jacod and Shiryaev [16] (we again strengthen the converse part about necessary conditions for the convergence). In Section 4 we discuss how the statements of two previous sections have to be interpreted in order to be valid for arbitrary nonnegative supermartingales. Moreover, we reformulate a counterpart (for this more general setting) of the contiguity criterion. We also formulate here a result about necessary conditions for the finite-dimensional convergence (along a dense subset) of nonnegative supermartingales to a continuous positive martingale, which is applied in the proofs of the theorems of Section 3. The proofs of all main results are collected in Section 5. In Section 6 we construct some counter-examples. In Appendix we compute the Hellinger and some similar processes for $P^n$ and $P'^n$ in terms of $P^n$.

For standard notation and results concerning general theory of processes, stochastic integration and semimartingales we refer to Jacod and Shiryaev [16]. We shall deal with different filtered spaces $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+})$; the filtration $\mathbb{F}$ is always right-continuous, but no completeness assumption is made, cf. [16]. If $P$ is a probability measure on $(\Omega, \mathcal{F})$ and $T$ is a stopping time relative to $\mathbb{F}$, then $P_T$ is the restriction of $P$ onto $\mathcal{F}_T$. An increasing process is always assumed to be right-continuous. If $X$ is a semimartingale, then $X^c$ is its continuous martingale part and $[X, X]$ is its quadratic variation; the angle brackets $\langle M, M\rangle$ stand for the quadratic characteristic of a locally square-integrable martingale $M$. The notation $H \cdot X_t = \int_0^t H_s\, dX_s$ and $W \star \mu_t = \int_0^t \int_{\mathbb{R}^d} W(s, x)\, \mu(ds, dx)$ is used for the (ordinary or stochastic) integral processes, where $X$ is a semimartingale and $\mu = \mu(\omega, ds, dx)$ is a random measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

We shall consider also stochastic processes defined only on predictable stochastic intervals of the form $\Gamma = \bigcup_k [\![0, T_k]\!]$, where $(T_k)$ is an increasing sequence of stopping times. We refer to Jacod [10, Ch. 5] for more details on such processes. If $X$ is defined on $\Gamma$, then the process $\mathrm{Var}(X)$ is defined also on $\Gamma$ in the following way: if $(\omega, t) \in \Gamma$ then $\mathrm{Var}(X)_t(\omega)$ is the total variation of $X_\cdot(\omega)$ on $[0, t]$.

Let $E$ be a Polish space and $\mathcal{E}$ its Borel $\sigma$-field. $\mathcal{P}(E)$ is the space of all probability measures on $(E, \mathcal{E})$ endowed with the weak topology. The distribution of a random element $X$ with values in $E$ under a probability measure $P$, i.e. the image $P \circ X^{-1} \in \mathcal{P}(E)$ of $P$ under $X$, is denoted by $\mathcal{L}(X \mid P)$. The space $\mathbb{D}(E)$ of all càdlàg functions $\alpha\colon \mathbb{R}_+ \to E$ equipped with the Skorokhod topology is also a Polish space. The Borel $\sigma$-field in

$\mathbb{D}(E)$ is denoted by $\mathcal{D}(E)$. The corresponding notation $\mathbb{C}(E)$ and $\mathcal{C}(E)$ is used for the space of all continuous functions from $\mathbb{R}_+$ to $E$ equipped with the local uniform topology and its Borel $\sigma$-field. If $E = \mathbb{R}$, we shall sometimes write $\mathbb{D}$ and $\mathbb{C}$ instead of $\mathbb{D}(\mathbb{R})$ and $\mathbb{C}(\mathbb{R})$ respectively.

The weak convergence of distributions in $\mathcal{P}(\mathbb{R}^d)$ is denoted by $\Rightarrow$, while the symbol $\xrightarrow{d}$ is used for the weak convergence of distributions in $\mathcal{P}(\mathbb{D}(E))$. To avoid ambiguities we shall add that the convergence is in $\mathbb{D}(E)$ if $E \neq \mathbb{R}$. On the other hand, the symbol $\xrightarrow{d_f(S)}$ is used for the finite-dimensional convergence along a set $S$, i.e. $\mathcal{L}(X^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(X \mid P)$, $n \to \infty$, means that $\mathcal{L}(X^n_{t_1}, \dots, X^n_{t_p} \mid P^n) \Rightarrow \mathcal{L}(X_{t_1}, \dots, X_{t_p} \mid P)$, $n \to \infty$, for any $p = 1, 2, \dots$ and $t_1, \dots, t_p \in S$.

We say that a sequence $(X^n \mid P^n)$ is $\mathbb{D}$-tight if $X^n$ are $P$-a.s. càdlàg processes with values in $\mathbb{R}$ and the laws $\mathcal{L}(X^n \mid P^n)$ are tight in $\mathbb{D}$; if, moreover, all cluster points of the sequence $\mathcal{L}(X^n \mid P^n)$ are laws of continuous processes, we say that the sequence $(X^n \mid P^n)$ is $\mathbb{C}$-tight.

If $\xi_n$ are random variables with values $[-\infty, +\infty]$ on probability spaces $(\Omega^n, \mathcal{F}^n, P^n)$, we write

$$\xi_n \xrightarrow{P^n} 0 \qquad \text{if} \qquad \lim_{n\to\infty} P^n(|\xi_n| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0,$$

and we say that the sequence

$$(\xi_n \mid P^n) \quad \text{is } \mathbb{R}\text{-tight} \qquad \text{if} \qquad \lim_{N\uparrow\infty} \limsup_{n\to\infty} P^n(|\xi_n| > N) = 0.$$

We shall also use this notation even if $\xi_n$ are not well defined everywhere on $\Omega^n$; it is sufficient to assume that $\xi_n$ are defined on subsets $B_n \in \mathcal{F}$ and $\lim_{n\to\infty} P^n(B_n) = 1$.

If $P^n$ and $P'^n$ are probability measures on measurable spaces $(\Omega^n, \mathcal{F}^n)$, $n = 1, 2, \dots$, then $(P^n) \lhd (P'^n)$ means that the sequence $(P^n)$ is contiguous to the sequence $(P'^n)$.

# 2 Main approximation result

## 2.1 Basic setup

Let $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}^n_t)_{t\in\mathbb{R}_+}, (P^n, P'^n))$ be a sequence of binary filtered experiments. If otherwise is not specified, for processes defined on $\Omega^n$ stochastic integrals and angle brackets are taken with respect to $P^n$. Denote by $Z^n$ the (generalized) density process of $P'^n$ with respect to $P^n$, i.e. a right-continuous (and admitting left-hand limits $P^n$- and $P'^n$-a.s) adapted process $(Z^n_t)_{t\in\mathbb{R}_+}$ with values in $[0,\infty]$ such that for any $\mathbb{F}^n$-stopping time $T$

$$P'^n(B) = \int_B Z^n_T\, dP^n + P'^n(B\cap\{Z^n_T=\infty\}), \quad B\in\mathcal{F}_T.$$

$Z^n$ is a $P^n$-supermartingale and

$$P^n(\sup_t Z^n_t \geq a) \leq a^{-1} \qquad \text{for any} \quad a>0, \tag{2.1}$$

in particular, the sequence $(\sup_t Z^n_t \mid P^n)$ is $\mathbb{R}$-tight.

Put $T^n_k := \inf\{t\colon Z^n_t < 1/k\}$ and $\Gamma^n := \bigcup_{k=1}^\infty [\![0,T^n_k]\!]$, then $\Gamma^n = [\![0]\!]\cup\{Z^n_- > 0\}$ $P^n$-a.s.

The processes $Z^n$ and $Y^n := \sqrt{Z^n}$ are $P^n$-supermartingales. Using the Doob–Meyer decomposition theorem and the standard arguments used to prove the existence of Hellinger processes, one can find predictable increasing processes $\iota^n$ and $h^n$ with values in $[0,\infty]$, $\iota^n_0 = h^n_0 = 0$, such that

$$Z^n + Z^n_-\cdot\iota^n \quad \text{is a } P^n\text{-local martingale}, \tag{2.2}$$

$$Y^n + Y^n_-\cdot h^n \quad \text{is a } P^n\text{-martingale}. \tag{2.3}$$

The processes $\iota^n$ and $h^n$ are $P^n$-a.s. unique on $\Gamma^n$. A process $h^n$ (resp. $\iota^n$) satisfies the above requirements if and only if $h^n$ (resp. $\iota^n$) is a Hellinger process of order 1/2 (resp. of order 0) for $P^n$ and $P'^n$, see Lemma A.1 and Corollary A.1 in Appendix. If $P'^n \overset{\text{loc}}{\ll} P^n$, then one can take $\iota^n = 0$. In what follows $\iota^n$ and $h^n$ are arbitrary processes satisfying the above requirements.

Define now a process $y^n$ on $\Gamma^n$ by $y^n := (1/Y^n_-)\cdot Y^n$. Put also $m^n := y^n + h^n$ on $\Gamma^n$. It follows from (2.3) that $m^n$ is a $P^n$-local martingale on $\Gamma^n$. In fact, $m^n$ is a $P^n$-locally square-integrable martingale on $\Gamma^n$, see Lemma 4.1.

The jump measures $\mu^{y^n}$ and $\mu^{m^n}$ of the processes $y^n$ and $m^n$ as well as their $P^n$-compensators $\nu^{y^n}$ and $\nu^{m^n}$ respectively are well defined on the stochastic interval $\Gamma^n$. These measures charge only the predictable set $\Gamma^n\cap\Big(\mathbb{R}_+\times\big([-1,0)\cup(0,\infty)\big)\Big)$. In what follows it is convenient for us to consider the extensions of $\nu^{y^n}$ and $\nu^{m^n}$ (denoted by the same symbols) to $\mathbb{R}_+\times\mathbb{R}$ in an arbitrary way assuming that these extensions are predictable random measures that charge only the set $\mathbb{R}_+\times\big([-1,0)\cup(0,\infty)\big)$.

## 2.2 The approximation theorem

In this subsection we fix a number $t>0$.

Introduce the following assumptions:

$$(P_0^n) \lhd (P_0'^n); \tag{2.4}$$

$$\text{the sequence} \quad (h_t^n \mid P^n) \quad \text{is} \quad \mathbb{R}\text{-tight}; \tag{2.5}$$

$$x^2 \mathbf{1}_{\{|x|>\varepsilon\}} \star \nu_t^{m^n} \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad \varepsilon > 0; \tag{2.6}$$

$$\iota_t^n \xrightarrow{P^n} 0, \quad n \to \infty. \tag{2.7}$$

We start with our main approximation result.

**Theorem 2.1** *Assume that conditions* (2.4)–(2.7) *hold. Then*

$$(P_t^n) \lhd (P_t'^n), \tag{2.8}$$

$$\operatorname{Var}\Big(h^n - \frac{1}{2}\langle m^n, m^n\rangle\Big)_t \xrightarrow{P^n} 0, \quad n \to \infty, \tag{2.9}$$

*and*

$$\sup_{s\le t}\Big|\log Z_s^n - \log Z_0^n - \big((2m_s^n - 2\langle m^n, m^n\rangle_s\big)\Big| \xrightarrow{P^n} 0, \quad n \to \infty. \tag{2.10}$$

**Remark 2.1** The processes $m^n$ and $\langle m^n, m^n\rangle$ in (2.9) and (2.10) are defined only on a part of the interval $[0,t]$ if $(\omega, t) \notin \Gamma^n$. But the contiguity (2.8) implies

$$\lim_{n\to\infty} P^n\{\omega\colon (\omega,t) \in \Gamma^n\} = 1. \tag{2.11}$$

Indeed, (2.8) is equivalent to the property

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(\inf_{s\le t} Z_s^n < \varepsilon) = 0, \tag{2.12}$$

see [16, Lemma V.1.19].

**Remark 2.2** Under the assumptions of Theorem 2.1 we cannot assert that $(P_t'^n) \lhd (P_t^n)$ even if $(P_0'^n) \lhd (P_0^n)$. We give a corresponding example in Section 6.

**Remark 2.3** If $Z^n$ is continuous and a local martingale with respect to $P^n$ for every $n$, then $Z^n = Z_0^n \mathcal{E}(N^n) = Z_0^n \exp\big(N^n - \frac{1}{2}\langle N^n, N^n\rangle\big)$ on $\Gamma^n$, where $N^n$ is a continuous $P^n$-local martingale on $\Gamma^n$. Hence $m^n = \frac{1}{2}N^n$ and $h^n = \frac{1}{8}\langle N^n, N^n\rangle$ on $\Gamma^n$ and the approximations (2.9) and (2.10) are trivial under (2.11).

Now we want to examine in more details the rôle of the conditions in Theorem 2.1 and to link them to contiguity and other properties.

The next proposition is a reformulation of the well-known criterion for contiguity, see Liptser and Shiryayev [23], Jacod [11], Jacod and Shiryaev [16, Theorem V.2.3].

**Proposition 2.1** *The properties* (i) *and* (ii) *are equivalent*:

(i) $(P^n_t) \lhd (P'^n_t)$.

(ii) *Conditions* (2.4) *and* (2.5) *hold and*

$$\lim_{\beta\downarrow 0}\limsup_{n\to\infty} P^n\bigl(\mathbf{1}_{\{x<-1+\beta\}} \star \nu^{y^n}_t > \varepsilon\bigr) = 0 \quad \textit{for all} \quad \varepsilon > 0. \tag{2.13}$$

*In particular*, *conditions* (2.4), (2.5), *and*

$$x^2\mathbf{1}_{\{|x|>\varepsilon\}} \star \nu^{y^n}_t \xrightarrow{P^n} 0, \quad n\to\infty, \quad \textit{for all} \quad \varepsilon > 0. \tag{2.6$^*$}$$

*imply* (2.8).

Note that, as soon as the contiguity (2.8) is established, all the conditions (2.5)–(2.7), (2.6$^*$), and (2.13) do not depend on the choice of $h^n$, $\iota^n$, $\nu^{y^n}$, and $\nu^{m^n}$ in view of Remark 2.1, see (2.11).

Proposition 2.1 immediately follows from equivalence (i)$\Leftrightarrow$(ii) in Jacod and Shiryaev [16, Theorem V.2.3] (with $T^n = t$, $P^n$ instead of $P'^n$ and vice versa), if one takes into account that $h^n$ is a Hellinger process of order 1/2 for $P^n$ and $P'^n$, see Lemma A.1, and $i(\mathbf{1}_{[0,\beta]}; P'^n, P^n) = \mathbf{1}_{\{x\le -1+\beta^{1/2}\}} \star \nu^{y^n}$, $\beta \in (0,1)$, see Corollary A.1. Indeed, then (ii.1) in that theorem coincides with (2.4), (ii.2) coincides with (2.5), and (ii.3) coincides with (2.13). However, we shall give a direct (rather short) proof below, see Theorem 4.1 and its proof in Section 5.

**Proposition 2.2** *Assume that* (2.4) *and* (2.5) *hold. Then*:

(a) (2.6$^*$) *implies* (2.6);

(b) (2.6) *and* (2.7) *are sufficient for having* (2.6$^*$).

*In the both cases*

$$\sup_{s\le t}\Delta h^n_s \xrightarrow{P^n} 0, \quad n\to\infty. \tag{2.14}$$

In the setting of independent discrete observations Le Cam considers (2.14) as the assumption of negligibility of information in a single observation, see e.g. [21, Assumption (A), p. 458]. Proposition 2.2 says, in particular, that it is automatically satisfied under our basic assumptions.

Due to Proposition 2.2, we can replace (2.6) by (2.6$^*$) in Theorem 2.1. However, although (2.4), (2.5) and (2.6$^*$) imply the first statement (2.8) of Theorem 2.1 by Proposition 2.1, (2.9) and (2.10) may not be true if (2.7) is not assumed, see Theorem 2.2 and Example 6.4. It may also happen that $P^n_t \ll P'^n_t$ for all $n$ (so that all the considered processes are $P^n$-a.s. uniquely defined on $[0,t]$), (2.4), (2.5) and (2.6) hold, but the sequence $(P^n_t)$ is not even contiguous with respect to $(P'^n_t)$, see Example 6.2. As for condition (2.7), it is satisfied if $(P'^n_t) \lhd \rhd (P^n_t)$. Indeed, it follows from the contiguity criterion that the condition $\iota^n_t \xrightarrow{P'^n} 0$, $n \to \infty$, is necessary for the contiguity $(P'^n_t) \lhd (P^n_t)$. A simple direct proof is also possible, see the proof of Theorem 4.2.

The following remark might be also useful. Since $\Delta h^n \le 1$ on $\Gamma^n$ $P^n$- and $P'^n$-a.s., without loss of generality we may assume that $\Delta h^n \le 1$ everywhere. Define then

$$\mathcal{E}(-h^n)_t = \begin{cases} e^{-h^n_t} \prod_{s \le t} (1 - \Delta h^n_s) e^{\Delta h^n_s} & \text{if } h^n_t < \infty, \\ 0 & \text{if } h^n_t = \infty. \end{cases}$$

Since $\mathcal{E}(-h^n) \le e^{-h^n}$, the condition

$$\text{the sequence} \quad (\mathcal{E}(-h^n)^{-1}_t \mid P^n) \quad \text{is} \quad \mathbb{R}\text{-tight} \tag{2.15}$$

is stronger than (2.5). But it is still necessary for the contiguity $(P^n_t) \lhd (P'^n_t)$, use the multiplicative decomposition connected with Hellinger processes [16, Proposition V.4.16]. Using arguments similar to those in the proof of Theorem 4.1, one can show that (2.4), (2.15) and (2.6) imply (2.8).

The next proposition indicates a link to some conditions used in other works.

**Proposition 2.3** *Under* (2.4) *and* (2.5) *we have equivalence between*:

(i) (2.6) *and* (2.7);

(ii) *condition* $[L\text{-}\{t\}]$ *in* [16, p. 595], *see also Section* 4;

(iii) $k^n_t(p) \xrightarrow{P^n} 0$, $n \to \infty$, *for* $p > 2$, *where* $k^n(p)$ *is the* $p$*-divergency process for* $P^n$ *and* $P'^n$ *introduced in* [5], *see also Appendix*;

(iv) *for* $\alpha \in (0,1)$, $\alpha \neq 1/2$,

$$h^n(\alpha)_t + h^n(1-\alpha)_t - 8\alpha(1-\alpha)h^n_t \xrightarrow{P^n} 0, \quad n \to \infty,$$

*where* $h^n(\alpha)$ *is the Hellinger process of order* $\alpha$ *for* $P^n$ *and* $P'^n$.

*If these conditions are satisfied then*

$$\mathrm{Var}\left(h^n(\alpha) - 4\alpha(1-\alpha)h^n\right)_t \xrightarrow{P^n} 0, \quad n \to \infty. \tag{2.16}$$

Now let us discuss a connection between our Lindeberg conditions (2.6) and (2.6*) and the property that $Z^n$ have small jumps as $n \to \infty$. Remark, first, that (with $L^n := \log Z^n$)

$$\Delta L^n = \log\Bigl(1 + \frac{\Delta Z^n}{Z^n_-}\Bigr) \qquad (\text{where } (-\infty) - (-\infty) = \frac{0}{0} = 0)$$

and

$$1 + \frac{\Delta Z^n}{Z^n_-} = (1 + \Delta y^n)^2 \quad \text{on } \Gamma^n. \tag{2.17}$$

Taking into account (2.1) and (2.12), we see that under (2.8)

$$\sup_{s \le t} |\Delta Z^n_s| \xrightarrow{P^n} 0 \iff \sup_{s \le t} |\Delta L^n_s| \xrightarrow{P^n} 0 \iff \sup_{s \le t} |\Delta y^n_s| \xrightarrow{P^n} 0.$$

The second part of the following proposition seems to be surprising.

**Proposition 2.4** (a) *Assume that*

$$\mathbf{1}_{\{|x|>\varepsilon\}} \star \nu^{y^n}_t \xrightarrow{P^n} 0, \quad n \to \infty, \quad \textit{for all} \quad \varepsilon > 0. \tag{2.18}$$

*Then* $\sup_{s \le t} |\Delta Z^n_s| \xrightarrow{P^n} 0$.

(b) *Assume that* $(P'^n_t) \lhd \rhd (P^n_t)$ *and* $\sup_{s \le t} |\Delta Z^n_s| \xrightarrow{P^n} 0$. *Then we have* (2.6*).

In Section 6 we shall construct an example showing that the contiguity $(P'^n_t) \lhd (P^n_t)$ is essential in the statement (b).

Finally, we want to formulate the result (which seems to be of secondary importance) similar to Theorem 2.1, but without condition (2.7).

**Theorem 2.2** *Assume that conditions* (2.4), (2.5), *and* (2.6*) *hold. Then*

$$(P^n_t) \lhd (P'^n_t),$$

$$\mathrm{Var}\Big(h^n - \frac{1}{2}\langle m^n, m^n\rangle - \frac{1}{2}\iota^n\Big)_t \xrightarrow{P^n} 0, \quad n \to \infty,$$

*and*

$$\sup_{s\le t}\Big|\log Z^n_s - \log Z^n_0 - \big(2m^n_s - 2\langle m^n, m^n\rangle_s - \iota^n_s\big)\Big| \xrightarrow{P^n} 0, \quad n \to \infty.$$

It may happen that the assumptions of Theorem 2.2 are satisfied, while those of Theorem 2.1 are not: see Example 6.4. In Theorem 2.2 we cannot replace (2.6*) by (2.6) even under (2.15).

# 3 Limit theorems

We consider the same setting as in Section 2. Our aim is to formulate a number of results on the weak convergence of the sequence $Z^n$ to different limits. Though some of these results are known or extend previous ones, in Section 5 we shall give new proofs for all of them based on Theorem 2.1.

## 3.1 Convergence to the stochastic exponential of a Gaussian martingale

In this subsection we assume that $t \rightsquigarrow C_t$ is a nondecreasing continuous function with $C_0 = 0$, $M$ is a continuous Gaussian martingale with $M_0 = 0$ and $\langle M, M\rangle_t = C_t$ on some stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Put $Z := e^{M-C/2}$ and $Z' := e^{M+C/2}$.

**Theorem 3.1** *Let $S$ be a subset of $\mathbb{R}_+$. Assume that $Z^n_0 \xrightarrow{P^n} 1$, (2.6) and (2.7) hold for all $t \in S$ and*

$$h^n_t \xrightarrow{P^n} \frac{1}{8}C_t \quad \textit{for all} \quad t \in S. \tag{3.1}$$

*Then*

$$(P'^n_t) \lhd \rhd (P^n_t) \quad \textit{for all} \quad t \quad \textit{such that} \quad [t,\infty) \cap S \neq \emptyset, \tag{3.2}$$

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(Z \mid P), \quad n \to \infty, \tag{3.3}$$

*and*

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d_f(S)} \mathcal{L}(Z' \mid P), \quad n \to \infty. \tag{3.4}$$

In view of Proposition 2.3 and the lemmas in Appendix, Theorem 3.1 is an equivalent version of Theorems X.1.16 and X.1.64 b) in Jacod and Shiryaev [16].

**Theorem 3.2** *Let $S$ be a dense subset in $\mathbb{R}_+$ containing $0$. If the assumptions of Theorem 3.1 are satisfied for $S$ then*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d} \mathcal{L}(Z \mid P), \quad n \to \infty. \tag{3.5}$$

*Conversely, if*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(Z \mid P), \quad n \to \infty, \tag{3.6}$$

*then $Z_0^n \xrightarrow{P^n} 1$, we have (2.6), (2.7), $h_t^n \xrightarrow{P^n} \frac{1}{8}C_t$, and $(P_t'^n) \lhd \rhd (P_t^n)$ for all $t \in \mathbb{R}_+$, and the relations (3.5) and*

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d} \mathcal{L}(Z' \mid P), \quad n \to \infty, \tag{3.7}$$

*hold.*

**Remark 3.1** The processes $Z^n$ may take the value $+\infty$ with a positive $P'^n$-probability, so the convergence in (3.7) means the weak convergence in $\mathbb{D}([0,\infty])$.

Theorem 3.2 strengthens Theorems X.1.12 and X.1.64 a) in Jacod and Shiryaev [16]: in the converse part we assume only the finite-dimensional convergence along $S$.

## 3.2 Convergence to the stochastic exponential of a conditionally Gaussian martingale

In this subsection we assume that $(\Omega^n, \mathcal{F}^n)$ and $P^n$ do not depend on $n$, and the *nesting* condition is satisfied. More precisely, we introduce the

*Hypothesis* N: there are a sequence $(\Omega, \mathcal{F}, \mathbb{F}^n, (P, P'^n))$ of binary filtered experiments and a sequence of numbers $(s_n)$, decreasing to 0, such that $\mathcal{F}^n_{s_n} \subseteq \mathcal{F}^{n+1}_{s_{n+1}}$ for all $n$ and $\mathcal{G} := \bigvee_n \mathcal{F}^n_\infty = \bigvee_n \mathcal{F}^n_{s_n}$.

Next, let $C = (C_t)_{t \in \mathbb{R}_+}$ be a continuous increasing process with $C_0 = 0$ and such that $C_t$ are $\mathcal{G}$-measurable for all $t$. Recall that $\mathbb{C} = \mathbb{C}(\mathbb{R})$ is the space of real-valued continuous functions $\alpha = (\alpha(t))$ on $\mathbb{R}_+$ and $\mathcal{C} = \mathcal{C}(\mathbb{R})$ is the Borel $\sigma$-field. There is a Markov kernel $Q(\omega, d\alpha)$ from $(\Omega, \mathcal{G})$ into $(\mathbb{C}, \mathcal{C})$ such that for each $\omega \in \Omega$ the canonical process $X$ on $\mathbb{C}$, i.e. $X_t(\alpha) = \alpha(t)$, is a continuous Gaussian martingale under $Q(\omega, d\alpha)$ with $X_0 = 0$ $Q(\omega, d\alpha)$-a.s. and $\langle X, X \rangle_t = C_t(\omega)$. Similarly, let $Q'(\omega, d\alpha)$ be a Markov kernel from $(\Omega, \mathcal{G})$ into $(\mathbb{C}, \mathcal{C})$ such that for each $\omega \in \Omega$ $X_0 = 0$ $Q'(\omega, d\alpha)$-a.s. and $X_t - C_t(\omega)$ is a continuous Gaussian martingale under $Q'(\omega, d\alpha)$with the quadratic characteristic $C_t(\omega)$. Put $\mathbf{P}(d\omega, d\alpha) = P(d\omega)Q(\omega, d\alpha)$, $\mathbf{P}'(d\omega, d\alpha) = P(d\omega)Q'(\omega, d\alpha)$.

Consider the filtered model $(\Omega\times\mathbb{C},\mathcal{G}\otimes\mathcal{C},\mathbf{F},(\mathbf{P},\mathbf{P}'))$, where $\mathbf{F}=(\boldsymbol{\mathcal{F}}_t)$ is the smallest filtration of $\Omega\times\mathbb{C}$ to which $X$ (naturally extended to $\Omega\times\mathbb{C}$) is adapted and such that $\mathcal{G}\subseteq\boldsymbol{\mathcal{F}}_0$. It is clear that $\mathbf{P}'\overset{\text{loc}}{\sim}\mathbf{P}$ and the density process $Z=(Z_t(\omega,\alpha))$ is given by

$$Z_t(\omega,\alpha)=\exp(X_t(\omega)-\frac{1}{2}C_t(\alpha)).$$

It is also clear that the distribution $\mathcal{L}(Z\mid\mathbf{P}')$ coincides with $\mathcal{L}(Z'\mid\mathbf{P})$, where

$$Z'_t(\omega,\alpha)=\exp(X_t(\omega)+\frac{1}{2}C_t(\alpha)).$$

If $Y^n$ and $Y$ are cádlág stochastic processes on $(\Omega,\mathcal{F},P)$ and $(\Omega\times\mathbb{C},\mathcal{G}\otimes\mathcal{C},\mathbf{P})$ respectively with values in $\mathbb{R}^d$ then we shall write

$$\mathcal{L}(Y^n\mid P)\xrightarrow{d_f(S)}\mathcal{L}(Y\mid\mathbf{P}),\quad n\to\infty,\quad(\mathcal{G}\text{-stably})\tag{3.8}$$

if

$$\lim_{n\to\infty}\mathcal{L}(\xi,Y^n_{t_1},\dots,Y^n_{t_p}\mid P)\Rightarrow\mathcal{L}(\xi,Y_{t_1},\dots,Y_{t_p}\mid\mathbf{P})$$

for any $p=1,2,\dots,$ $t_1,\dots,t_p\in S$, and any bounded $\mathcal{G}$-measurable random variable $\xi$. Similarly,

$$\mathcal{L}(Y^n\mid P)\xrightarrow{d}\mathcal{L}(Y\mid\mathbf{P}),\quad n\to\infty,\quad(\mathcal{G}\text{-stably})\tag{3.9}$$

if $\mathcal{L}(\xi,Y^n\mid P)$ converges in law to $\mathcal{L}(\xi,Y\mid\mathbf{P})$ in $\mathbb{R}\times\mathbb{D}(\mathbb{R}^d)$ for any bounded $\mathcal{G}$-measurable $\xi$.

**Remark 3.2** Let $\mathcal{P}(\mathcal{G})$ be the set of all probability measures $\widetilde{P}$ which are absolutely continuous with respect to $P$ with a bounded $\mathcal{G}$-measurable density; for $\widetilde{P}\in\mathcal{P}(\mathcal{G})$ put $\widetilde{\mathbf{P}}(d\omega,d\alpha)=\widetilde{P}(d\omega)Q(\omega,d\alpha)$. It is easy to see that (3.8) is equivalent to

$$\mathcal{L}(Y^n\mid\widetilde{P})\xrightarrow{d_f(S)}\mathcal{L}(Y\mid\widetilde{\mathbf{P}}),\quad n\to\infty,\quad\text{for every }\widetilde{P}\in\mathcal{P}(\mathcal{G}),$$

and (3.9) is equivalent to

$$\mathcal{L}(Y^n\mid\widetilde{P})\xrightarrow{d}\mathcal{L}(Y\mid\widetilde{\mathbf{P}}),\quad n\to\infty,\quad\text{for every }\widetilde{P}\in\mathcal{P}(\mathcal{G}).$$

Since, evidently, a change of measure with a bounded density preserves tightness, (3.8) for a dense subset $S\subseteq\mathbb{R}_+$ and $\mathcal{L}(Y^n\mid P)\xrightarrow{d}\mathcal{L}(Y\mid\mathbf{P})$, $n\to\infty$, are sufficient for (3.9).

We start with the finite-dimensional convergence.

**Theorem 3.3** *Assume Hypothesis* N. *Let* $S = \{t_1, \ldots, t_p\}$, $0 < t_1 < \cdots < t_p$, *be a subset of* $\mathbb{R}_+$. *Assume that* $Z_0^n \xrightarrow{P} 1$, (2.6) *and* (2.7) *hold for* $t = t_p$ *and*

$$h_t^n \xrightarrow{P} \frac{1}{8} C_t, \quad n \to \infty, \quad \textit{for all} \quad t \in S.$$

*Moreover, assume that* $h_{s_n}^n \xrightarrow{P} 0$, $n \to \infty$. *Then*

$$(P_t'^n) \lhd \rhd (P_t^n) \quad \textit{for all} \quad t \le t_p,$$

$$\mathcal{L}(Z^n \mid P) \xrightarrow{d_f(S)} \mathcal{L}(Z \mid \mathbf{P}), \quad n \to \infty, \quad (\mathcal{G}\textit{-stably}) \tag{3.10}$$

*and*

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d_f(S)} \mathcal{L}(Z' \mid \mathbf{P}), \quad n \to \infty. \tag{3.11}$$

**Remark 3.3** A trivial counter-example (see Section 6) shows that we cannot assert that the convergence in (3.11) is $\mathcal{G}$-stable (with a natural interpretation of what does it mean). However, we shall show after the proof of Theorem 3.3 that, if $(\xi^n)$ is a sequence of random vectors converging in $P$-probability to $\xi$ and $\xi^n$ are $\mathcal{F}_{t_p}^n$-measurable for all $n$, then $\mathcal{L}(\xi^n, Z_{t_1}^n, \ldots, Z_{t_p}^n \mid P'^n) \Rightarrow \mathcal{L}(\xi, Z'_{t_1}, \ldots, Z'_{t_p} \mid \mathbf{P})$.

The next theorem extends Theorem 1 in Mordecki [26].

**Theorem 3.4** *Assume Hypothesis* N. *Let* $S$ *be a dense subset in* $\mathbb{R}_+$ *containing* 0. *Assume that* $Z_0^n \xrightarrow{P} 1$, (2.6) *and* (2.7) *hold for all* $t \in S$ *and*

$$h_t^n \xrightarrow{P} \frac{1}{8} C_t, \quad n \to \infty, \quad \textit{for all} \quad t \in S. \tag{3.12}$$

*Then*

$$\mathcal{L}(Z^n \mid P) \xrightarrow{d} \mathcal{L}(Z \mid \mathbf{P}), \quad n \to \infty \quad (\mathcal{G}\textit{-stably}). \tag{3.13}$$

*Conversely, if*

$$\mathcal{L}(Z^n \mid P) \xrightarrow{d_f(S)} \mathcal{L}(Z \mid \mathbf{P}), \quad n \to \infty, \quad (\mathcal{G}\textit{-stably}) \tag{3.14}$$

*then* $Z_0^n \xrightarrow{P} 1$, *we have* (2.6), (2.7), $h_t^n \xrightarrow{P} \frac{1}{8} C_t$, *and* $(P_t'^n) \lhd \rhd (P_t^n)$ *for all* $t \in \mathbb{R}_+$, *and the relations* (3.13) *and*

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d} \mathcal{L}(Z' \mid \mathbf{P}), \quad n \to \infty, \quad \textit{in } \mathbb{D}([0,\infty]) \tag{3.15}$$

*hold.*

## 3.3 Convergence to a continuous martingale

In this subsection we consider a general sequence $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, (P^n, P'^n))$ of binary filtered models as in Section 2 and Subsection 3.1 and give two results on the functional convergence of $Z^n$ to a continuous martingale.

We start with a specification of the limiting process in the first theorem.

*Hypotheses* M: a) $(\Omega, \mathcal{F}, \mathbb{F}) = (\mathbb{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}), \mathfrak{D}(\mathbb{R}))$ is the Skorokhod space with the Borel $\sigma$-field $\mathcal{D}(\mathbb{R})$ and the filtration $\mathfrak{D}(\mathbb{R})$ generated by the canonical process denoted by $M$.

b) $C$ is an adapted continuous increasing process with $C_0 = 0$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$.

c) There is a continuous increasing function $t \rightsquigarrow F_t$ with $F_0 = 0$, such that $F - C(\alpha)$ is nondecreasing for all $\alpha \in \Omega$.

d) $\alpha \rightsquigarrow C_t(\alpha)$ is Skorokhod-continuous for all $t \in \mathbb{R}_+$.

e) There is a unique probability measure $P$ on $(\Omega, \mathcal{F})$ under which $M$ is a continuous local martingale with $M_0 = 0$ and $\langle M, M \rangle = C$.

**Lemma 3.1** *Assume Hypotheses* M. *There is a unique measure $P'$ on $(\Omega, \mathcal{F})$ under which $M' := M - 2C$ is a continuous local martingale with $M'_0 = 0$ and $\langle M', M' \rangle = C$. Moreover, $P' \overset{\text{loc}}{\sim} P$, and the density process of $P'$ with respect to $P$ is $Z := e^{2M-2C}$. In particular, $Z$ is a $P$-martingale.*

Let the processes $m^n$ be constructed from $Z^n$ as in Subsection 2.1 and $T^n := \inf\{s\colon Z^n_s < 1/n\}$. Then the process $\widetilde{m}^n := (m^n)^{T^n}$ is $P^n$-a.s. with paths in $\mathbb{D}(\mathbb{R})$, so the process $C \circ \widetilde{m}^n$ is $P^n$-a.s. well defined.

**Theorem 3.5** *Assume Hypotheses* M. *Let $S$ be a dense subset in $\mathbb{R}_+$ containing* 0. *Assume that $Z^n_0 \xrightarrow{P^n} 1$,* (2.6), (2.7) *and*

$$h^n_t - \frac{1}{2} C_t \circ \widetilde{m}^n \xrightarrow{P^n} 0, \quad n \to \infty, \tag{3.16}$$

*hold for all $t \in S$. Then*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d} \mathcal{L}(e^{2M-2C} \mid P), \quad n \to \infty. \tag{3.17}$$

*Conversely, if*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(e^{2M-2C} \mid P), \quad n \to \infty, \tag{3.18}$$

*then $Z^n_0 \xrightarrow{P^n} 1$, we have* (2.6), (2.7), (3.16), *and $(P'^n_t) \lhd \rhd (P^n_t)$ for all $t \in \mathbb{R}_+$, and the relations* (3.17) *and*

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d} \mathcal{L}(e^{2M-2C} \mid P'), \quad n \to \infty, \quad \textit{in } \mathbb{D}([0,\infty]) \tag{3.19}$$

*hold, where $P'$ is defined in Lemma* 3.1.

The second theorem is essentially the result proved by Jacod and Shiryaev [16, Theorems X.1.59 and X.1.65]; we strengthen only the converse statement.

*Hypotheses* X: a) $(\Omega, \mathcal{F}, \mathbb{F}) = (\mathbb{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}), \mathscr{D}(\mathbb{R}))$ is the Skorokhod space with the Borel $\sigma$-field $\mathcal{D}(\mathbb{R})$ and the filtration $\mathscr{D}(\mathbb{R})$ generated by the canonical process denoted by $X$.

b) $C$ is an adapted continuous increasing process with $C_0 = 0$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$.

c) There is a continuous increasing function $t \rightsquigarrow F_t$ with $F_0 = 0$, such that $F - C(\alpha)$ is nondecreasing for all $\alpha \in \Omega$.

d) $\alpha \rightsquigarrow C_t(\alpha)$ is Skorokhod-continuous for all $t \in \mathbb{R}_+$.

e) There is a unique probability measure $P$ on $(\Omega, \mathcal{F})$ under which $M := X + C/2$ is a continuous local martingale with $M_0 = 0$ and $\langle M, M \rangle = C$.

**Lemma 3.2 (Jacod and Shiryaev [16, Lemma X.1.58])** *Assume Hypotheses* X. *There is a unique measure $P'$ on $(\Omega, \mathcal{F})$ under which $M' := X - C/2$ is a continuous local martingale with $M'_0 = 0$ and $\langle M', M' \rangle = C$. Moreover, $P' \overset{\text{loc}}{\ll} P$, and the density process of $P'$ with respect to $P$ is $e^X$. In particular, $e^X$ is a $P$-martingale.*

**Theorem 3.6** *Assume Hypotheses* X. *Let $S$ be a dense subset in $\mathbb{R}_+$ containing* 0. *Assume that $Z_0^n \xrightarrow{P^n} 1$,* (2.6), (2.7) *and*

$$h_t^n - \frac{1}{8} C_t \circ \log(Z^n \vee \frac{1}{n}) \xrightarrow{P^n} 0, \quad n \to \infty, \tag{3.20}$$

*hold for all $t \in S$. Then*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d} \mathcal{L}(e^X \mid P), \quad n \to \infty. \tag{3.21}$$

*Conversely, if*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(e^X \mid P), \quad n \to \infty,$$

*then $Z_0^n \xrightarrow{P^n} 1$, we have* (2.6), (2.7), (3.20), *and $(P_t'^n) \lhd\rhd (P_t^n)$ for all $t \in \mathbb{R}_+$, and the relations* (3.21) *and*

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d} \mathcal{L}(e^X \mid P'), \quad n \to \infty, \quad \textit{in } \mathbb{D}([0, \infty]) \tag{3.22}$$

*hold, where $P'$ is defined in Lemma* 3.2.

Theorem 3.2 is a special case of Theorems 3.5 and 3.6.

# 4 Nonnegative supermartingales

## 4.1 Fundamental lemma

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space and $Z$ a right-continuous nonnegative supermartingale on it. Put $T_k := \inf\{t\colon Z_t < 1/k\}$, $\Gamma := \bigcup_{k=1}^{\infty}[\![0, T_k]\!]$, $Y := Z^{1/2}$. As in subsection 2.1, there are predictable increasing processes $\iota$ and $h$ with values in $[0,\infty]$, $\iota_0 = h_0 = 0$, such that

$$Z + Z_- \cdot \iota \quad \text{is a } P\text{-local martingale,}$$

$$Y + Y_- \cdot h \quad \text{is a } P\text{-martingale.}$$

The processes $\iota$ and $h$ are $P$-a.s. unique on $\Gamma$. Define the processes $y := \frac{1}{Y_-} \cdot Y$ and $m := y + h$ on $\Gamma$. It follows from the definition of $h$ that $m$ is a $P$-local martingale on $\Gamma$.

The next lemma is due to Mémin, see [25, Lemma 2.1]. It plays a central rôle in the proof our main results.

**Lemma 4.1** *The process $m$ is a $P$-locally square-integrable martingale on $\Gamma$ and*

$$\langle m, m\rangle = 2h - [h, h] - \iota \quad P\text{-a.s.} \quad \text{on } \Gamma. \tag{4.1}$$

**Proof:** Let $N := Z + Z_- \cdot \iota$ and define on $\Gamma$ the process $n := \frac{1}{Z_-} \cdot N$, then $n$ is a $P$-local martingale on $\Gamma$. On $\Gamma$ we have

$$Z = Z_0\mathcal{E}(n - \iota) \quad \text{and} \quad Z = Z_0\{\mathcal{E}(m - h)\}^2,$$

which gives

$$n - \iota = 2m - 2h + [m, m] - 2[m, h] + [h, h]$$

on $\Gamma$ by Yor's formula. Note that $[m, h] = (\Delta h) \cdot m$ is a $P$-local martingale on $\Gamma$. Then it follows that $[m, m]$ is a $P$-special semimartingale on $\Gamma$ with the Doob–Meyer decomposition $[m, m] = (n - 2m + 2[m, h]) + (2h - [h, h] - \iota)$. The claim follows.

## 4.2 Extensions to nonnegative supermartingales

Throughout this subsection it is assumed that for every $n = 1, 2, \dots$ we have a stochastic basis $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}^n_t)_{t\in\mathbb{R}_+}, P^n)$ and a nonnegative right-continuous supermartingale $Z^n$ on it with $E^nZ^n_0 \leq 1$. We do not assume that there are probability measures $P'^n$ on $(\Omega^n, \mathcal{F}^n)$ such that $Z^n$ is the generalized density process of $P'^n$ with respect to $P^n$. If the latter takes place for all $n$, we shall say that we are in the setting of subsection 2.1 in contrast to the current more general setting.

Let us introduce the sets $\Gamma^n$, the processes $Y^n$, $\iota^n$, $h^n$, $y^n$, $m^n$, and the random measures $\mu^{y^n}$, $\mu^{m^n}$, $\nu^{y^n}$, and $\nu^{m^n}$ exactly as in subsection 2.1. Note that the inequality (2.1) holds by the optional sampling theorem.

It turns out that our results in Sections 2 and 3 are valid in the current more general setting. Of course, one has to reformulate the assertions that use the measures $P'^n$ explicitly or implicitly.

We start with conditions (2.4) and (2.8). In the setting of subsection 2.1 they are equivalent to

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(Z^n_0 < \varepsilon) = 0 \tag{4.2}$$

and

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(Z^n_t < \varepsilon) = 0 \tag{4.3}$$

respectively, see [16, Lemma V.1.6].

The relations (2.9) and (2.10) are meaningful if (2.11) holds. The next lemma shows, in particular, that in the current setting (4.3) implies (2.12) and hence (2.11) .

**Lemma 4.2** *For every* $n = 1, 2, \ldots$ *let* $T^n$ *be a stopping time relative to* $\mathbb{F}^n$. *There is equivalence between* (i) *and* (ii) *below*:

(i) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(Z^n_{T^n} < \varepsilon) = 0$.

(ii) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(\inf_{s\le T^n} Z^n_s < \varepsilon) = 0$.

**Proof:** We show that (i)$\Rightarrow$(ii), the converse is trivial. For $\varepsilon > 0$ put $S^n := \inf\{t\colon Z^n_t < \varepsilon/2\} \wedge T^n$. By the optional sampling theorem $E^n(Z^n_{T^n} \mid \mathcal{F}^n_{S^n}) \le Z^n_{S^n}$. Hence

$$\begin{aligned}\frac{\varepsilon}{2}P^n(Z^n_{S^n} \le \varepsilon/2) &\ge E^n Z^n_{S^n}\mathbf{1}_{\{Z^n_{S^n}\le\varepsilon/2\}} \ge E^n Z^n_{T^n}\mathbf{1}_{\{Z^n_{S^n}\le\varepsilon/2\}}\\ &\ge E^n Z^n_{T^n}\mathbf{1}_{\{Z^n_{S^n}\le\varepsilon/2,\ Z^n_{T^n}\ge\varepsilon\}} \ge \varepsilon P^n(Z^n_{S^n} \le \varepsilon/2,\ Z^n_{T^n} \ge \varepsilon).\end{aligned}$$

Therefore,

$$\begin{aligned}P^n(Z^n_{T^n} < \varepsilon) &\ge P^n(Z^n_{T^n} < \varepsilon,\ Z^n_{S^n} \le \varepsilon/2)\\ &= P^n(Z^n_{S^n} \le \varepsilon/2) - P^n(Z^n_{S^n} \le \varepsilon/2,\ Z^n_{T^n} \ge \varepsilon)\\ &\ge \frac{1}{2}P^n(Z^n_{S^n} \le \varepsilon/2).\end{aligned}$$

Since $\{\inf_{s\le T^n} Z^n_s < \varepsilon/2\} \subseteq \{Z^n_{S^n} \le \varepsilon/2\}$, the claim follows.

In Proposition 2.4 the assumption $(P'^n_t)\lhd(P^n_t)$ is made. It can be replaced by

$$\lim_{N\uparrow\infty}\limsup_{n\to\infty} E^n Z^n_t \mathbf{1}_{\{Z^n_t>N\}} = 0, \qquad \lim_{n\to\infty} E^n Z^n_t = 1, \tag{4.4}$$

which is equivalent to $(P'^n_t) \lhd (P^n_t)$ in the setting of subsection 2.1, see [16, Lemma V.1.10].

There are some special conditions in Proposition 2.3. Condition [$L$-$\{t\}$] in Jacod and Shiryaev [16, Theorem X.1.12, p. 595] says that $I^n(1+\varepsilon)_t \xrightarrow{P^n} 0$, $n \to \infty$, for all $\varepsilon > 0$, where $I^n(a)$, $a > 1$, is an arbitrary version of $i(\varphi_a; P^n, P'^n) + i(\varphi_a; P'^n, P^n)$, $\varphi_a(x) := (1-x)\mathbf{1}_{\{x\le 1/a\}}$, see Jacod and Shiryaev [16, Chapter IV, § 1d] or Appendix for the definition of the processes $i(\psi; P, P')$. Let

$$\widehat{I}^n(a) := \theta_a((1+x)^2) \star \nu^{y^n} + \iota^n,$$

where $\theta_a(x) := |x-1|\mathbf{1}_{\{1/a<x<a\}^c}$. It follows from Lemma A.3 and (2.17) that in the setting of subsection 2.1 $I^n(a) = \widehat{I}^n(a)$ on $\Gamma^n$ $P^n$-a.s., and we reformulate (ii) in Proposition 2.3 as

$$\widehat{I}^n(1+\varepsilon)_t \xrightarrow{P^n} 0, \quad n\to\infty, \quad \text{for all} \quad \varepsilon > 0. \tag{4.5}$$

Let

$$\widehat{k}^n(p) = \left|(1+x)^{2/p} - 1\right|^p \star \nu^{y^n} + \iota^n, \quad p \ge 2.$$

It follows from Corollary A.2 and (2.17) that in the setting of subsection 2.1 $k^n(p) = \widehat{k}^n(p)$ on $\Gamma^n$ $P^n$-a.s., and we reformulate (iii) in Proposition 2.3 as

$$\widehat{k}^n(p)_t \xrightarrow{P^n} 0, \quad n\to\infty, \quad \text{for some} \quad p > 2. \tag{4.6}$$

Introduce now $\widehat{h}^n(1-\alpha)$, $\alpha \in (0,1)$, as a predictable increasing process with values in $[0,\infty]$ such that $h^n(1-\alpha)_0 = 0$ and $(Z^n)^\alpha + (Z^n_-)^\alpha \cdot \widehat{h}^n(1-\alpha)$ is a $P^n$-martingale. Its existence follows from the same arguments as for $\alpha = 1/2$. By Lemma A.1 any versions of $h^n(1-\alpha)$ and $\widehat{h}^n(1-\alpha)$ are $P^n$-indistinguishable in the setting of subsection 2.1, and we replace $h^n(1-\alpha)$ by $\widehat{h}^n(1-\alpha)$ in (iv) and (2.16) in Proposition 2.3.

Each theorem in Section 3 contains also a convergence statement with respect to alternatives $P'^n$. Though these statements admit reformulations in terms of $P^n$ and $Z^n$, we do not consider them in the current setting in view of minor interest.

As a result, we state that

> After replacing the relations $(P_0^n)\lhd(P_0'^n)$, $(P_t^n)\lhd(P_t'^n)$ and $(P_t'^n)\lhd(P_t^n)$ by (4.2), (4.3) and (4.4) respectively and excluding relations (3.4), (3.7), (3.11), (3.15), (3.19), and (3.22) from the consideration, all the statements of Propositions 2.1–2.4 and Theorems 2.1, 2.2, 3.1–3.6, are valid for nonnegative supermartingales $Z^n$ in the current setting; in Proposition 2.3 (ii) is replaced by (4.5), (iii) is replaced by (4.6), and $h^n(1-\alpha)$ are replaced by $\widehat{h}^n(1-\alpha)$.

The proofs in Section 5 are adapted to the current setting, but there are only few points mentioned there, where they can be simplified in the setting of subsection 2.1.

Since the contiguity criterion (Proposition 2.1) is a result of primary importance, we give a separate and slightly more general formulation for it in the current setting.

**Theorem 4.1** *For every* $n = 1, 2, \dots$ *let* $T^n$ *be a stopping time relative to* $\mathbb{F}^n$*. The properties* (i) *and* (ii) *are equivalent*:

(i) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(Z^n_{T^n} < \varepsilon) = 0$.

(ii) (1) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(Z^n_0 < \varepsilon) = 0$;

(2) the sequence $(h^n_{T^n} \mid P^n)$ is $\mathbb{R}$-tight;

(3) $\lim_{\beta\downarrow 0}\limsup_{n\to\infty} P^n\big(\mathbf{1}_{\{x<-1+\beta\}} \star \nu^{y^n}_{T^n} > \varepsilon\big) = 0$ for all $\varepsilon > 0$.

**Remark 4.1** If $Z^n$ are the generalized density processes of $P'^n$ with respect to $P^n$ and $\mathcal{F}^n = \bigvee_t \mathcal{F}^n_t$, Theorem 4.1 reduces to equivalence (i)⇔(ii) in Jacod and Shiryaev [16, Theorem V.2.3], $P^n$ and $P'^n$ being interchanged. That theorem contains also another equivalent condition (iii) for the contiguity, which admits such a form that remains valid for nonnegative supermartingales as well. Namely, both (i) and (ii) in Theorem 4.1 are equivalent to

(iii) (1) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} P^n(Z^n_0 < \varepsilon) = 0$;

(2) $\lim_{\alpha\downarrow 0}\limsup_{n\to\infty} P^n(\widehat{h}^n(1-\alpha)_{T^n} > \eta) = 0$ for all $\eta > 0$.

(Recall that the processes $\widehat{h}^n(1-\alpha)$ coincide with the Hellinger processes of order $1-\alpha$ for $P^n$ and $P'^n$ in the setting of subsection 2.1). We do not prove this result here but only mention that to prove (ii)⇔(iii) one can use essentially the same arguments as in Jacod and Shiryaev [16, Lemmas V.2.16 and V.2.19].

**Remark 4.2** Consider the "stationary case": $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^n) = (\Omega, \mathcal{F}, \mathbb{F}, P)$, $Z^n = Z$, and $T^n = T$. Put $S := \inf\{t\colon Z_t = 0\}$. Then, with the notation of subsection 4.1, Theorem 4.1 says that $P(Z_T = 0) = 0$ (or $P(T \geq S) = 0$) if and only if $P(Z_0 = 0) = 0$ and $P(h_T < \infty) = 1$ and $P(h(1)_T > 0) = 0$, where $h(1) := \mathbf{1}_{\{x=-1\}} \star \nu^y$ coincides with the $P$-compensator of $\mathbf{1}_{\{Z_{S-}>0\}}\mathbf{1}_{[\![S,\infty]\!]}$ or $\mathbf{1}_{[\![S,\infty]\!]}$ on $\Gamma$ and is a version of the Hellinger process $h(1; P, P') = h(0; P', P)$ if $Z$ is the generalized density process of a probability measure $P'$ with respect to $P$, see Corollary A.1. In the latter case this statement coincides with the criterion for absolute continuity $P_T \ll P'_T$ in the form due to Jacod [11], see Jacod and Shiryaev [16, Theorem IV.2.1] (where $P$ and $P'$ are interchanged).

Moreover, our proof, written in the stationary case, allows us to obtain much more. The first part of the proof gives

$$\{S > 0,\ Z_{S-} > 0\} \subseteq \{h_S < \infty\} \quad P\text{-a.s.} \tag{4.7}$$

(this is similar to the proof of Lemma IV.2.16 in [16]). Taking into account that a locally square-integrable martingale and its quadratic variation converge a.s. on the set where its quadratic characteristic converges, see e.g. Liptser and Shiryayev [24, § 2.6], the second part of the proof shows that

$$P(S > 0,\ Z_{S-} = 0,\ h_{S-} < \infty) = 0. \tag{4.8}$$

In the case where $Z$ is the generalized density process, the combination of (4.7) and (4.8) reduces to Theorem 3 in Schachermayer and Schachinger [30], where the proof is rather laborious. Results about singularity of $P_T$ and $P'_T$ from Jacod and Shiryaev [16, Chapter IV, § 2a] also follow from (4.7) and (4.8).

Finally we formulate a general result which is used in the proofs of converse parts in Theorems 3.2 and 3.4–3.6. The most nontrivial part (finite-dimensional convergence implies functional convergence) is due to Aldous [1] who proved such an assertion for uniformly integrable sequences of martingales converging to a continuous martingale. In our setting the processes $Z^n$ are only supermartingales, but only minor changes in Aldous' proof are needed.

**Theorem 4.2** *Let $Z$ be a continuous process on a probability space $(\Omega, \mathcal{F}, P)$ such that $P(Z_t > 0) = 1$ and $EZ_t = 1$ for all $t \in \mathbb{R}_+$. Assume that*

$$\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(Z \mid P), \quad n \to \infty, \tag{4.9}$$

*where $S$ is a dense subset of $\mathbb{R}_+$ containing $0$. Then*:

(i) $\mathcal{L}(Z^n \mid P^n) \xrightarrow{d} \mathcal{L}(Z \mid P)$, $n \to \infty$.

(ii) *$Z$ is a $P$-martingale with respect to the filtration $\mathbb{F}^Z$ generated by $Z$.*

(iii) (2.5), (2.6), (2.7), (2.9) *and* (2.10) *hold.*

(iv) $\lim_{n\to\infty} P^n(T^n < t) = 0$ *for any* $t \in \mathbb{R}_+$, *where* $T^n := \inf\{s\colon Z^n_s < 1/n\}$.

(v) $\mathcal{L}(\widetilde{m}^n, \langle \widetilde{m}^n, \widetilde{m}^n\rangle \mid P^n) \xrightarrow{d} \mathcal{L}(M/2, \langle M, M\rangle/4 \mid P)$ *in* $\mathbb{D}(\mathbb{R}^2)$, $n \to \infty$, *where* $\widetilde{m}^n := (m^n)^{T^n}$, $M := \frac{1}{Z_-} \cdot Z$ *is a $P$-local martingale so that* $Z = Z_0\mathcal{E}(M)$.

# 5 Proofs of main results

## 5.1 Proof of Theorem 4.1

Put $S^n := \inf\{t\colon Z^n_t < 1/n\} \wedge T^n$. Then $[\![0, S^n]\!] \subseteq \Gamma^n$, in particular, the processes $h^n$, $y^n$, and $m^n$ are $P^n$-a.s. uniquely defined on $[\![0, S^n]\!]$. For $\beta \in (0, 1)$ define increasing processes $A^n = A^n(\beta)$ and $B^n = B^n(\beta)$ by $A^n := \mathbf{1}_{\{x<-1+\beta\}}\mathbf{1}_{[\![0,S^n]\!]}\star\nu^{y^n}$ and $B^n := \mathbf{1}_{\{x<-1+\beta\}}\mathbf{1}_{[\![0,S^n]\!]}\star\mu^{y^n}$. Then $B^n$ is a counting process and $A^n$ is its $P^n$-compensator. By Lenglart's inequality, see [24, Theorem 1.9.3],

$$P^n(B^n_{S^n} \geq 1) \leq \varepsilon + P^n(A^n_{S^n} \geq \varepsilon)$$

and

$$P^n(A^n_{S^n} \geq \varepsilon) \leq \frac{1}{\varepsilon}E^n(B^n_{S^n} \wedge 2) + P^n(B^n_{S^n} \geq 1) \leq \Big(1 + \frac{2}{\varepsilon}\Big)P^n(B^n_{S^n} \geq 1)$$

for any $\varepsilon > 0$, hence

$$\begin{aligned} &\lim_{\beta\downarrow 0}\limsup_{n\to\infty} P^n(A^n_{S^n} > \varepsilon) = 0 \quad \text{for any } \varepsilon > 0 \\ &\qquad\iff \quad \lim_{\beta\downarrow 0}\limsup_{n\to\infty} P^n(B^n_{S^n} \geq 1) = 0. \qquad (5.1) \end{aligned}$$

Assume (i). Then (ii.1) follows from Lemma 4.2. Next, $E^n(Y^n_T)^2 = E^nZ^n_T \leq E^nZ^n_0 \leq 1$ for any finite-valued $\mathbb{F}^n$-stopping time $T$, hence the process $Y^n$ is of class (D) and $Y^n + Y^n_- \cdot h^n$ is a $P^n$-uniformly integrable martingale. Therefore, $E^nY^n_- \cdot h^n_\infty = E^nY^n_0 - E^nY^n_\infty \leq 1$ for all $n$, and the inequality

$$\begin{aligned} P^n(h^n_{T^n} > N) &\leq P^n(Y^n_- \cdot h^n_\infty \geq N^{1/2}) + P^n(\inf_{s\leq T^n} Y^n_{s-} \leq N^{-1/2}) \\ &\leq N^{-1/2} + P^n(\inf_{s\leq T^n} Z^n_s \leq N^{-1}) \end{aligned}$$

implies (ii.2) due to Lemma 4.2.

Furthermore, $\{B^n_{S^n} \geq 1\} = \{\inf_{s\leq S^n} \Delta y^n_s < -1+\beta\} = \{\inf_{s\leq S^n} Z^n_s/Z^n_{s-} < \beta^2\} \subseteq \{\sup_{s\leq S^n} Z^n_s > a\} \cup \{\inf_{s\leq S^n} Z^n_s < a\beta^2\}$ for any $a > 0$. In view of (2.1) and Lemma 4.2 the relation on the right in (5.1) takes place, and we obtain (ii.3) from the left-hand relation in (5.1) and from the inequality $P^n(S^n < T^n) \leq P^n(\inf_{s\leq T^n} Z^n_s < 1/n) \to 0$, $n \to \infty$.

Assume now that (ii) holds. Using the inequalities $\sum_{s\leq S^n}(\Delta h^n_s)^2 \leq (h^n_{S^n})^2$ and $\langle m^n, m^n\rangle_{S^n} \leq 2h^n_{S^n}$ (Lemma 4.1), we obtain from (ii.2) that the sequences $\big(\sum_{s\leq S^n}(\Delta h^n_s)^2\big|P^n\big)$ and $\big(\langle m^n, m^n\rangle_{S^n}\big|P^n\big)$ are $\mathbb{R}$-tight. Moreover, by Lenglart's inequality the sequences $\big(\sup_{s\leq S^n}|m^n_s|\big|P^n\big)$ and $\big([m^n, m^n]_{S^n}\big|P^n\big)$ are $\mathbb{R}$-tight, hence $\big(\sup_{s\leq S^n}|y^n_s|\big|P^n\big)$ and $\big(\sum_{s\leq S^n}(\Delta y^n_s)^2\big|P^n\big)$ are $\mathbb{R}$-tight.

On the set $\{S^n < \infty\}$ we have

$$Z^n_{S^n} = Z^n_0 \mathcal{E}(y^n)^2_{S^n} = Z^n_0 \exp\big(2y^n_{S^n} - \langle m^{n,c}, m^{n,c}\rangle_{S^n}\big) \prod_{s\leq S^n}(1+\Delta y^n_s)^2 e^{-2\Delta y^n_s}.$$

For $\beta \in (0,1)$ let $\eta^n(\beta) := 1$ on $\{S^n = \infty\}$ and

$$\eta^n(\beta) := Z^n_0 \exp\big(2y^n_{S^n} - \langle m^{n,c}, m^{n,c}\rangle_{S^n}\big) \prod_{\substack{s\leq S^n\\ \Delta y^n_s \geq -1+\beta}} (1+\Delta y^n_s)^2 e^{-2\Delta y^n_s}$$

on $\{S^n < \infty\}$. Since $0 \leq x - \log(1+x) \leq C_\beta x^2$, $x \geq -1+\beta$, the product in the definition of $\eta^n(\beta)$ is bounded from below by $\exp\big(-2C_\beta \sum_{s\leq S^n}(\Delta y^n_s)^2\big)$. Hence, due to (ii.1) and the above tightness statements

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} P^n(\eta^n(\beta) < \varepsilon) = 0 \quad \text{for all } \beta \in (0,1). \tag{5.2}$$

Evidently, if $n > \varepsilon^{-1}$,

$$\{Z^n_{T^n} < \varepsilon\} \subseteq \{Z^n_{S^n} < \varepsilon,\ S^n < \infty\} \subseteq \{Z^n_{S^n} \neq \eta^n(\beta),\ S^n < \infty\} \cup \{\eta^n(\beta) < \varepsilon\}$$

and

$$\{Z^n_{S^n} \neq \eta^n(\beta),\ S^n < \infty\} \subseteq \{\inf_{s\leq S^n} \Delta y^n_s < -1+\beta\} = \{B^n_{S^n} \geq 1\}.$$

(i) follows now from (ii.3), (5.1) and (5.2).

## 5.2 Proofs of Propositions 2.2–2.4

**Proof of Proposition 2.2:** Let us first show that in the both cases (2.14) is valid for a "strict" version $h'^n := \mathbf{1}_{\Gamma^n} \cdot h^n$ of $h^n$.

We start with case (a). Since

$$\Delta h'^n_s = -\int_{\mathbb{R}} x\mathbf{1}_{\Gamma^n}\, \nu^{y^n}(\{s\}\times dx),$$

we obtain

$$\Delta h'^n_s \le \delta + \int_{\mathbb{R}} |x|\mathbf{1}_{\{x<-\delta\}}\, \nu^{y^n}(\{s\}\times dx) \le \delta + \delta^{-1}\int_{\mathbb{R}} x^2\mathbf{1}_{\{|x|>\delta\}}\, \nu^{y^n}(\{s\}\times dx).$$

Applying this inequality with $\delta=\varepsilon/2$, we get

$$P^n\bigl(\sup_{s\le t}\Delta h'^n_s > \varepsilon\bigr) \le P^n\bigl(x^2\mathbf{1}_{\{|x|>\varepsilon/2\}} \star \nu^{y^n}_t > \varepsilon^2/4\bigr) \to 0, \quad n\to\infty,$$

due to (2.6*).

In case (b) we start with the following estimate coming from (4.1):

$$\Delta h'^n \le \Delta h'^n(2-\Delta h'^n) = (\Delta\langle m^n,m^n\rangle + \Delta\iota^n)\mathbf{1}_{\Gamma^n}. \tag{5.3}$$

Next,

$$\Delta\langle m^n,m^n\rangle_s \le \int_{\mathbb{R}} x^2\, \nu^{m^n}(\{s\}\times dx) \le \delta^2 + \int_{\mathbb{R}} x^2\mathbf{1}_{\{|x|>\delta\}}\, \nu^{m^n}(\{s\}\times dx) \quad \text{on } \Gamma^n,$$

hence

$$\sup_{s\le t,\, s\in\Gamma^n} \Delta\langle m^n,m^n\rangle_s \le \delta^2 + x^2\mathbf{1}_{\{|x|>\delta\}}\star\nu^{m^n}_t. \tag{5.4}$$

Combining (5.3) and (5.4) with $\delta^2=\varepsilon/2$, we get

$$P^n\bigl(\sup_{s\le t}\Delta h'^n_s > \varepsilon\bigr) \le P^n\bigl(x^2\mathbf{1}_{\{x^2>\varepsilon/2\}}\star\nu^{m^n}_t > \varepsilon/4\bigr) + P^n\bigl(\iota^n_t > \varepsilon/4\bigr) \to 0,$$

$n\to\infty$, due to (2.6) and (2.7).

Since $[h'^n,h'^n]_t \le \bigl(\sup_{s\le t}\Delta h'^n_s\bigr)h'^n_t$, it follows now from (2.5) that in the both cases

$$[h'^n,h'^n]_t \xrightarrow{P^n} 0, \quad n\to\infty. \tag{5.5}$$

Using (5.5), the inequality

$$(a+b)^2\mathbf{1}_{\{|a+b|>\varepsilon\}} \le 4a^2\mathbf{1}_{\{|a|>\varepsilon/2\}} + 4b^2\mathbf{1}_{\{|b|>\varepsilon/2\}}$$

and the identity

$$\Delta y^n = \Delta m^n - \Delta h^n \quad \text{on } \Gamma^n,$$

we then obtain that in case (a) (resp. in case (b)) (2.6) (resp. (2.6*)) holds for the “strict” version $\nu'^{m^n} = \mathbf{1}_{\Gamma^n}\cdot\nu^{m^n}$ of $\nu^{m^n}$ (resp. for the “strict” version $\nu'^{y^n} = \mathbf{1}_{\Gamma^n}\cdot\nu^{y^n}$ of $\nu^{y^n}$).

Now, in the both cases we may assert that the condition (ii.3) in Theorem 4.1 is fulfilled at least for a certain version $\nu^{y^n}$. By Theorem 4.1 and Lemma 4.2 we have (2.11), therefore, (2.6), (2.6*) and (2.14) are fulfilled for any version of $\nu^{m^n}$, $\nu^{y^n}$ and $h^n$ respectively.

**Proof of Proposition 2.3:** According to subsection 4.2 and Lemma A.2

$$\widehat{I}^n(a) := \theta_a((1+x)^2) \star \nu^{y^n} + \iota^n, \quad \widehat{k}^n(p) = \chi_p((1+x)^2) \star \nu^{y^n} + \iota^n,$$

$$h^n = \langle m^{n,c}, m^{n,c} \rangle /2 + \varphi_{1/2}((1+x)^2) \star \nu^{y^n} + \iota^n/2,$$

$$\widehat{h}^n(\alpha) - 4\alpha(1-\alpha)h^n = \psi_\alpha((1+x)^2) \star \nu^{y^n} + (1-\alpha)(1-2\alpha)\iota^n,$$

$$\widehat{h}^n(\alpha) + \widehat{h}^n(1-\alpha) - 8\alpha(1-\alpha)h^n = \rho_\alpha((1+x)^2) \star \nu^{y^n} + (1-2\alpha)^2\iota^n$$

$P^n$-a.s. on $\Gamma^n$, where $\theta_a(x) := |x-1|\mathbf{1}_{\{1/a<x<a\}^c}$, $\chi_p(x) := |x^{1/p}-1|^p$, $\varphi_\alpha(x) := \alpha + (1-\alpha)x - x^{1-\alpha}$, $\psi_\alpha(x) := \varphi_\alpha(x) - 4\alpha(1-\alpha)\varphi_{1/2}(x)$, $\rho_\alpha(x) := \psi_\alpha(x) + \psi_{1-\alpha}(x)$, $x \in \mathbb{R}$; the same is true for $I^n(a)$, $k^n(p)$ and $h^n(\alpha)$ in the setting of subsection 2.1. For any $\varepsilon \in (0,1)$, $\alpha \in (0,1)$, $\alpha \neq 1/2$, and $p > 2$ there are positive constants $\delta = \delta(\varepsilon)$, $B = B(\varepsilon,\delta)$, $C = C(\varepsilon,p)$, $K = K(\alpha,\varepsilon)$, $\varkappa = \varkappa(\alpha,\varepsilon)$, and $L = L(\alpha,\varkappa)$ such that

$$\theta_{1+\varepsilon}((1+x)^2) \le Bx^2\mathbf{1}_{\{|x|>\delta\}}, \quad x^2\mathbf{1}_{\{|x|>\varepsilon\}} \le \theta_{1+2\varepsilon}((1+x)^2), \quad x \ge -1,$$

$$\chi_p(x) \le 2\left(\frac{2\varepsilon}{1-\varepsilon}\right)^{p-2}\varphi_{1/2}(x) + \theta_{\frac{1+\varepsilon}{1-\varepsilon}}(x), \quad \theta_{1+\varepsilon}(x) \le C\chi_p(x), \quad x \in \mathbb{R}_+,$$

see the proof of Theorem 4.1 in Dzhaparidze and Valkeila [5],

$$\theta_{1+\varepsilon}(x) \le K\rho_\alpha(x), \quad |\psi_\alpha(x)| \le \varepsilon\varphi_{1/2}(x) + L\theta_{1+\varkappa}(x), \quad x \in \mathbb{R}_+,$$

see the proof of Theorems X.1.64 and X.1.65 in Jacod and Shiryaev [16]. The statement follows easily from the above facts, Proposition 2.1, and Theorem 4.1.

**Proof of Proposition 2.4:** Let us first show that

$$\mathbf{1}_{\Gamma^n}\mathbf{1}_{\{|x|>\varepsilon\}} \star \nu_t^{y^n} \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad \varepsilon > 0, \tag{5.6}$$

is equivalent to

$$\sup_{s\le t,\, s\in\Gamma^n} |\Delta y_s^n| \xrightarrow{P^n} 0, \quad n \to \infty. \tag{5.7}$$

We cannot formally apply standard results such as [16, Lemma VI.4.22] or [24, Lemma 5.5.1] because $y^n$ are defined only on the predictable intervals $\Gamma^n$ and cannot be extended, in general, to càdlàg processes on $\mathbb{R}_+$.

Let $R^n := \inf\{s\colon s \in \Gamma^n, |\Delta y^n_s| > \varepsilon\}$. Then ($\mu^{y^n}$ is the jump measure of $y^n$ on $\Gamma^n$)

$$B^n := \mathbf{1}_{\Gamma^n \cap [\![0,R^n]\!]} \mathbf{1}_{\{|x|>\varepsilon\}} \star \mu^{y^n}$$

is a bounded (by 1) increasing process with the $P^n$-compensator

$$A^n := \mathbf{1}_{\Gamma^n \cap [\![0,R^n]\!]} \mathbf{1}_{\{|x|>\varepsilon\}} \star \nu^{y^n}.$$

Applying the above mentioned results, we get

$$B^n_t \xrightarrow{P^n} 0 \qquad \Longleftrightarrow \qquad A^n_t \xrightarrow{P^n} 0 \qquad (n \to \infty). \tag{5.8}$$

Now $\{\sup_{s\le t,\, s\in\Gamma^n} |\Delta y^n_s| > \varepsilon\} = \{B^n_t = 1\}$, and the implication (5.6)⇒(5.7) follows from (5.8). Conversely,

$$P^n\Big(\mathbf{1}_{\Gamma^n} \mathbf{1}_{\{|x|>\varepsilon\}} \star \nu^{y^n}_t > \delta\Big) \le P^n(R^n < t) + P^n(A^n_t > \delta),$$

and (5.7) implies $P^n(R^n < t) \to 0$ and $B^n_t \xrightarrow{P^n} 0$. $n \to \infty$. The implication (5.7)⇒(5.6) follows from (5.8).

(a): Due to the above arguments we have (5.7), which implies

$$\sup_{s\le t,\, s\in\Gamma^n} |\Delta Z^n_s| \xrightarrow{P^n} 0, \quad n \to \infty,$$

due to (2.17) and (2.1). It remains to note that $(\Delta Z^n)\mathbf{1}_{(\Gamma^n)^c} = 0$ $P^n$-a.s.

(b): Assume (4.3), (4.4) and

$$\sup_{s\le t} |\Delta Z^n_s| \xrightarrow{P^n} 0, \quad n \to \infty. \tag{5.9}$$

By Lemma 4.2 we have (2.12), which allows us to deduce (5.7) from (5.9) and (2.17). Therefore we have (5.6) and even (2.18). Since $\nu^{y^n}$ charges only the set $\mathbb{R}_+ \times \big([-1,0) \cup (0,\infty)\big)$, (2.18) implies, in particular,

$$x^2 \mathbf{1}_{\{x<-\varepsilon\}} \star \nu^{y^n}_t \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{5.10}$$

Now we want to show that

$$\lim_{N\uparrow\infty} \limsup_{n\to\infty} P^n\big(x^2 \mathbf{1}_{\{x>N\}} \star \nu^{y^n}_t > \varepsilon\big) = 0 \quad \text{for all} \quad \varepsilon > 0. \tag{5.11}$$

In the setting of subsection 2.1 the contiguity criterion [16, Theorem V.2.3] for $(P'^n_t) \lhd (P^n_t)$ and the representation (A.5) for the process $i\big(\mathbf{1}_{[0,\beta]}; P^n, P'^n\big)$ give (5.11) with $P'^n$ instead of $P^n$, and (5.11) follows from $(P^n_t) \lhd (P'^n_t)$. In the general case we need a separate proof.

If $S^n$ are $\mathbb{F}^n$-stopping times and $S^n \le t$ then

$$0 \le E^n Z^n_{S^n} \mathbf{1}_C - E^n Z^n_t \mathbf{1}_C \le E^n Z^n_{S^n} - E^n Z^n_t$$

for $C \in \mathcal{F}^n_{S^n}$ by the optional sampling theorem. Hence,

$$\begin{aligned} E^n Z^n_{S^n} \mathbf{1}_{\{Z^n_{S^n} > N\}} &\le E^n Z^n_t \mathbf{1}_{\{Z^n_{S^n} > N\}} + (1 - E^n Z^n_t) \\ &\le E^n Z^n_t \mathbf{1}_{\{Z^n_t > M\}} + M/N + 1 - E^n Z^n_t. \qquad (5.12) \end{aligned}$$

Given $\delta \in (0,1)$, take $N > \delta^{-1}$ and define $S^n = S^n(N,\delta)$ by $S^n := \inf\{s\colon Z^n_s < \delta \text{ or } Z^n_s > N\} \wedge t$. Define also $A^n := x^2 \mathbf{1}_{\{x>N\}} \star \nu^{y^n}$. The stopped process $(A^n)^{S^n}$ is the $P^n$-compensator of $x^2 \mathbf{1}_{\{x>N\}} \mathbf{1}_{\llbracket 0,S^n \rrbracket} \star \mu^{y^n}$ and hence is majorized by the $P^n$-compensator of

$$\begin{aligned} B^n &:= (x+1)^2 \mathbf{1}_{\{x>N\}} \mathbf{1}_{\llbracket 0,S^n \rrbracket} \star \mu^{y^n} = \sum_{s \le S^n \wedge \cdot} \frac{Z^n_s}{Z^n_{s-}} \mathbf{1}_{\{Z^n_s/Z^n_{s-} > (N+1)^2\}} \\ &= \frac{Z^n_{S^n}}{Z^n_{S^n-}} \mathbf{1}_{\{Z^n_{S^n}/Z^n_{S^n-} > (N+1)^2\}} \mathbf{1}_{\llbracket S^n,\infty \rrbracket}, \end{aligned}$$

where the last equality comes from the definition of $S^n$ and the inequality $N\delta > 1$. Now we obtain

$$\begin{aligned} P^n(A^n_t > \varepsilon) &\le P^n(S^n < t) + P^n(A^n_{S^n} > \varepsilon) \\ &\le P^n\Big(\sup_{s \le t} Z^n_s > N\Big) + P^n\Big(\inf_{s \le t} Z^n_s < \delta\Big) + \frac{1}{\varepsilon} E^n B^n_{S^n} \\ &\le P^n\Big(\sup_{s \le t} Z^n_s > N\Big) + P^n\Big(\inf_{t \le T^n} Z^n_t < \delta\Big) + \frac{1}{\varepsilon\delta} E^n Z^n_{S^n} \mathbf{1}_{\{Z^n_{S^n} > N\}}. \end{aligned}$$

Replacing the last term by its estimate from (5.12) and taking $\limsup_{n\to\infty}$, $\limsup_{N\uparrow\infty}$, $\limsup_{M\uparrow\infty}$, and finally $\limsup_{\delta\downarrow 0}$, we obtain (5.11) from (4.4), (2.1) and (2.12).

Concluding the proof, we have

$$\begin{aligned} &P^n\big(x^2 \mathbf{1}_{\{x>\varepsilon\}} \star \nu^{y^n}_t > \delta\big) \\ &\quad \le P^n\big(x^2 \mathbf{1}_{\{\varepsilon < x \le N\}} \star \nu^{y^n}_t > \delta/2\big) + P^n\big(x^2 \mathbf{1}_{\{x>N\}} \star \nu^{y^n}_t > \delta/2\big) \\ &\quad \le P^n\big(\mathbf{1}_{\{x>\varepsilon\}} \star \nu^{y^n}_t > \delta/(2N^2)\big) + P^n\big(x^2 \mathbf{1}_{\{x>N\}} \star \nu^{y^n}_t > \delta/2\big), \end{aligned}$$

and we obtain from (2.18) and (5.11) that

$$\lim_{n\to\infty} P^n\big(x^2 \mathbf{1}_{\{x>\varepsilon\}} \star \nu^{y^n}_t > \delta\big) = 0 \quad \text{for all} \quad \delta > 0.$$

This and (5.10) give (2.6*).

## 5.3 Proofs of Theorems 2.1 and 2.2

**Proof of Theorem 2.1:** The first statement (2.8), or (4.3) in the general case, was established during the proof of Proposition 2.2 (b) with the help of Theorem 4.1. It follows from (2.5) and (2.14) that

$$[h^n, h^n]_t \xrightarrow{P^n} 0, \quad n \to \infty, \tag{5.13}$$

and we obtain the second statement (2.9) from Lemma 4.1, (5.13) and (2.7).

On the set $\Gamma^n$

$$Z^n = Z_0^n \mathcal{E}(y^n)^2 = Z_0^n \exp(2y^n - \langle y^{n,c}, y^{n,c}\rangle) \prod_{s\le\cdot} (1 + \Delta y_s^n)^2 e^{-2\Delta y_s^n},$$

hence on the set $\{Z^n > 0\} \subseteq \Gamma^n$

$$\begin{aligned} \log Z^n &= \log Z_0^n + 2m^n - 2h^n - \langle y^{n,c}, y^{n,c}\rangle + 2\sum_{s\le\cdot} \big\{\log(1 + \Delta y_s^n) - \Delta y_s^n\big\} \\ &= \log Z_0^n + 2m^n - 2h^n - [y^n, y^n] + 2\sum_{s\le\cdot} \psi(\Delta y_s^n), \end{aligned}$$

where $\psi(x) = \log(1 + x) - x + x^2/2$, $x > -1$.

Put $T^n := \inf\{s\colon Z_s^n < 1/n\} \wedge t$. Then

$$\begin{aligned} &P^n\Big\{\sup_{s\le t} |\log Z_s^n - \log Z_0^n - 2m_s^n + 2\langle m^n, m^n\rangle_s| > \varepsilon\Big\} \\ &\quad\le P^n(Z_{T^n}^n \le 1/n) + P^n\Big(\sup_{s\le T^n} |2h_s^n - \langle m^n, m^n\rangle_s| > \varepsilon/3\Big) \\ &\qquad + P^n\Big(\sup_{s\le T^n} \big|[y^n, y^n]_s - \langle m^n, m^n\rangle_s\big| > \varepsilon/3\Big) + P^n\Big(\sum_{s\le T^n} |\psi(\Delta y_s^n)| > \varepsilon/6\Big). \end{aligned}$$

Since $\lim_{n\to\infty} P^n(Z_{T^n}^n \le 1/n) = 0$ by (2.12) and we have proved (2.9), it is enough to show that

$$\sup_{s\le T^n} \big|[y^n, y^n]_s - \langle m^n, m^n\rangle_s\big| \xrightarrow{P^n} 0, \quad n \to \infty, \tag{5.14}$$

and

$$\sum_{s\le T^n} |\psi(\Delta y_s^n)| \xrightarrow{P^n} 0, \quad n \to \infty. \tag{5.15}$$

The sequence $\big(\langle m^n, m^n\rangle_{T^n} \big| P^n\big)$ is $\mathbb{R}$-tight in view of Lemma 4.1 and (2.5). This property combined with the Lindeberg condition (2.6) implies

$$\sup_{s\le T^n} \big|[m^n, m^n]_s - \langle m^n, m^n\rangle_s\big| \xrightarrow{P^n} 0, \quad n \to \infty, \tag{5.16}$$

see e.g. the proof of Lemma V.5.5 in Liptser and Shiryayev [24]. In particular, the sequence $\big([m^n,m^n]_{T^n}\big|P^n\big)$ is $\mathbb{R}$-tight. Since

$$\big|[y^n,y^n]-[m^n,m^n]\big| = \big|[h^n,h^n]-2[m^n,h^n]\big| \le [h^n,h^n]+2[m^n,m^n]^{1/2}[h^n,h^n]^{1/2}$$

on the interval $[\![0,T^n]\!]$, (5.14) follows from (5.16) and (5.13).

Put $\rho(\delta)=\sup_{0<|x|\le\delta}\frac{\psi(x)}{x^3}$, then $\lim_{\delta\downarrow 0}\rho(\delta)=\frac{1}{3}$. We have

$$P^n\Big(\sum_{s\le T^n}|\psi(\Delta y^n_s)|>\varepsilon\Big)\le P^n\Big(\sup_{s\le T^n}|\Delta y^n_s|>\delta\Big)+P^n\Big([y^n,y^n]_{T^n}>\frac{\varepsilon}{\delta\rho(\delta)}\Big). \tag{5.17}$$

It follows easily from (2.6*), which holds by Proposition 2.2, that

$$\lim_{n\to\infty}P^n\Big(\sup_{s\le T^n}|\Delta y^n_s|>\delta\Big)=0\quad\text{for all }\delta>0, \tag{5.18}$$

see e.g. the proof of Proposition 2.4. On the other hand, the sequence $\big([y^n,y^n]_{T^n}\big|P^n\big)$ is $\mathbb{R}$-tight in view of (5.14), and (5.15) follows from (5.17) and (5.18).

**Proof of Theorem 2.2** repeats the proof of Theorem 2.1 with obvious changes.

## 5.4 Proof of Theorem 4.2

We start with simple consequences of the assumptions. Since $P(Z_t>0)=1$, we have (4.3) for $t\in S$. Hence

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}P^n(\inf_{s\le t}Z^n_s<\varepsilon)=0\quad\text{for every}\quad t\in\mathbb{R}_+ \tag{5.19}$$

by Lemma 4.2, in particular, we have (iv).

By Fatou's lemma $1=EZ_t\le\liminf_{n\to\infty}E^nZ^n_t\le 1$, $t\in S$. Hence the sequence $(Z^n_t\mid P^n)_{n\ge1}$ is uniformly integrable and we have (4.4) for $t\in S$. Fix $L>0$ and let $\mathfrak{T}^n(L)$ be the set of all $\mathbb{F}^n$-stopping times $T$ such that $T\le L$. Applying inequality (5.12) from the proof of Proposition 2.4 for $S^n=T$ and $t\in S$, $t\ge L$, we get

$$\lim_{N\uparrow\infty}\limsup_{n\to\infty}\sup_{T\in\mathfrak{T}^n(L)}E^nZ^n_T\mathbf{1}_{\{Z^n_T>N\}}=0. \tag{5.20}$$

This implies

$$\lim_{N\uparrow\infty}\sup_{t\le L,\,t\in S}EZ_t\mathbf{1}_{\{Z_t>N\}}=0$$

and hence the uniform integrability of $(Z_t \mid P)_{t\in[0,L]}$ due to continuity of $Z$. In particular, $Z$ is continuous in $L^1(P)$.

Our next step is to show that $\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(\mathbb{R}_+)} \mathcal{L}(Z \mid P)$. Let $0 \le s < t < u$. Then $Z^n_t - Z^n_s = \{E^n(Z^n_u - Z^n_s \mid \mathcal{F}^n_t)\} + \{Z^n_t - E^n(Z^n_u \mid \mathcal{F}^n_t)\}$. Since the second term on the right is nonnegative, we have

$$E^n|Z^n_t - Z^n_s| \le E^n|Z^n_u - Z^n_s| + E^n Z^n_t - E^n Z^n_u.$$

If $s, u \in S$, then using finite-dimensional convergence of $Z^n$ along $S$ we get

$$\limsup_{n\to\infty} E^n|Z^n_t - Z^n_s| \le E|Z_u - Z_s|. \tag{5.21}$$

Let now $0 = t_0 < t_1 < \cdots < t_p$. Choose $s_i, u_i \in S$ such that $0 < s_i < t_i < u_i$, $i = 1, \dots, p$. Take a uniformly continuous bounded real-valued function $f$ on $\mathbb{R}^{p+1}$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ if $\|x - y\| \le \delta$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{p+1}$. Then, with $K = \sup_x |f(x)|$,

$$\begin{aligned}
&E^n|f(Z^n_{t_0}, Z^n_{t_1}, \dots, Z^n_{t_p}) - f(Z^n_{t_0}, Z^n_{s_1}, \dots, Z^n_{s_p})| \\
&\quad \le \varepsilon + 2KP^n\big(\big\|(Z^n_{t_0}, Z^n_{t_1}, \dots, Z^n_{t_p}) - (Z^n_{t_0}, Z^n_{s_1}, \dots, Z^n_{s_p})\big\| > \delta\big) \\
&\quad \le \varepsilon + 2K\sum_{i=1}^p P^n\big(|Z^n_{t_i} - Z^n_{s_i}| > \delta p^{-1/2}\big) \le \varepsilon + 2K\delta^{-1}p^{1/2}\sum_{i=1}^p E^n|Z^n_{t_i} - Z^n_{s_i}|.
\end{aligned}$$

Using (5.21) and finite-dimensional convergence of $Z^n$ along $S$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} &|E^n f(Z^n_{t_0}, Z^n_{t_1}, \dots, Z^n_{t_p}) - Ef(Z_{t_0}, Z_{t_1}, \dots, Z_{t_p})| \\
&\le \limsup_{n\to\infty} E^n|f(Z^n_{t_0}, Z^n_{t_1}, \dots, Z^n_{t_p}) - f(Z^n_{t_0}, Z^n_{s_1}, \dots, Z^n_{s_p})| \\
&\qquad + \limsup_{n\to\infty} |E^n f(Z^n_{t_0}, Z^n_{s_1}, \dots, Z^n_{s_p}) - Ef(Z_{t_0}, Z_{s_1}, \dots, Z_{s_p})| \\
&\qquad + E|f(Z_{t_0}, Z_{s_1}, \dots, Z_{s_p}) - Ef(Z_{t_0}, Z_{t_1}, \dots, Z_{t_p})| \\
&\le \varepsilon + 2K\delta^{-1}p^{1/2}\sum_{i=1}^p E|Z_{u_i} - Z_{s_i}| \\
&\qquad + E|f(Z_{t_0}, Z_{s_1}, \dots, Z_{s_p}) - Ef(Z_{t_0}, Z_{t_1}, \dots, Z_{t_p})|.
\end{aligned}$$

Due to continuity $Z$ in $L^1$ we can choose the points $s_i, u_i \in S$ close enough to $t_i$ so that the expression on the right in the last inequality is less than $2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mathcal{L}(Z^n_{t_0}, Z^n_{t_1}, \dots, Z^n_{t_p} \mid P^n) \Rightarrow \mathcal{L}(Z_{t_0}, Z_{t_1}, \dots, Z_{t_p} \mid P)$.

The next step is to show that the finite-dimensional convergence $\mathcal{L}(Z^n \mid P^n) \xrightarrow{d_f(\mathbb{R}_+)} \mathcal{L}(Z \mid P)$ implies the weak convergence $\mathcal{L}(Z^n \mid P^n) \xrightarrow{d} \mathcal{L}(Z \mid P)$

in the Skorokhod space. This is very close to Proposition 1.2 in Aldous [1] but, unlike that proposition, $Z^n$ are only supermartingales and not martingales in general. Nevertheless, it is possible to adapt Aldous' proof.

Recall that, given a cádlág process $X$ on a probability space $(\Omega, \mathcal{F}, P)$, the quantities $\Gamma_X(L, \varepsilon, \delta)$ and $\phi_X(L, \varepsilon)$, $L > 0$, $0 < \varepsilon < 1/2$, $0 < \delta \leq 1$, are defined in Aldous [1, p. 589 and 590] as follows. $\Gamma_X(L, \varepsilon, \delta)$ is the supremum of $\Gamma \geq 0$ such that: for each stopping time $T$ for $X$ satisfying $P(T \leq L) \geq \varepsilon$ we have

$$P(X_{T+\delta'} - X_T \leq \varepsilon \mid T \leq L) \geq \Gamma, \quad \text{all } 0 < \delta' \leq \delta, \quad \text{and}$$

$$P(X_{T+\delta'} - X_T \geq -\varepsilon \mid T \leq L) \geq \Gamma, \quad \text{all } 0 < \delta' \leq \delta;$$

$\phi_X(L, \varepsilon)$ is the infimum of $\phi \geq 1$ such that, for all stopping times $0 \leq T_1 \leq T_2 \leq L$,

$$E|X_{T_2} - X_{T_1}|\mathbf{1}_{\{|X_{T_2} - X_{T_1}| \geq \phi\}} \leq \varepsilon.$$

The same proof as in Lemma 2.3 in Aldous [1] shows that in our setting

$$\Gamma_{Z^n}(L, \varepsilon, 1) \geq \frac{\varepsilon^2 - 2(1 - E^n Z^n_{L+1})}{4\phi_{Z^n}(L+1, \varepsilon^2/2)}. \tag{5.22}$$

It follows from (5.20) that

$$\limsup_{n\to\infty} \phi_{Z^n}(L, \varepsilon) < \infty. \tag{5.23}$$

By (5.22) and (5.23)

$$\liminf_{n\to\infty} \Gamma_{Z^n}(L, \varepsilon, 1) > 0 \quad \text{for every } L > 0 \quad \text{and} \quad 0 < \varepsilon < 1/2,$$

and we are in a position to apply Proposition 2.2 in Aldous [1] to obtain the convergence $\mathcal{L}(Z^n \mid P^n) \xrightarrow{d} \mathcal{L}(Z \mid P)$.

It is easy to check that under the uniform integrability the weak limit of supermartingales is a supermartingale with respect to the filtration that the limiting process generates; since $EZ_t \equiv 1$, $Z$ is a martingale.

It follows from (5.19) and Theorem 4.1 that the sequence $(h^n_t \mid P^n)$ is $\mathbb{R}$-tight for any $t \in \mathbb{R}_+$. Since $Z^n + Z^n_- \cdot \iota^n$ is a nonnegative $P^n$-local martingale and hence a supermartingale, we have $E^n Z^n_- \cdot \iota^n_t \leq E^n Z^n_0 - E^n Z^n_t \to 0$ for any $t \in \mathbb{R}_+$, and it follows easily from (5.19) that $\iota^n_t \xrightarrow{P^n} 0$, $n \to \infty$. Proposition 2.4 (b) implies (2.6*), and (2.6) follows from Proposition 2.2. Thus, we are in a position to apply Theorem 2.1 to get (2.9) and (2.10).

It follows from (2.6) that $\sup_{s\le t}|\Delta\widetilde{m}^n_s| \xrightarrow{P^n} 0$, $n\to\infty$, cf. the proof of Proposition 2.4. As in the proof of Theorem 2.1, the sequence $\big(\langle\widetilde{m}^n,\widetilde{m}^n\rangle_t\big|P^n\big)$ is $\mathbb{R}$-tight and

$$\sup_{s\le t}\big|[\widetilde{m}^n,\widetilde{m}^n]_s-\langle\widetilde{m}^n,\widetilde{m}^n\rangle_s\big| \xrightarrow{P^n} 0,\quad n\to\infty,\quad t\in\mathbb{R}_+. \tag{5.24}$$

Put $U^n := \widetilde{m}^n-[\widetilde{m}^n,\widetilde{m}^n]$. Since $\log Z=\log Z_0+M-\frac{1}{2}\langle M,M\rangle$ by Itô's formula, it follows from (i), (iv), (2.10), and (5.24) that

$$\mathcal{L}(U^n\mid P^n)\xrightarrow{d}\mathcal{L}(M/2-\langle M,M\rangle/4\mid P),\quad n\to\infty. \tag{5.25}$$

Take the truncation function $h(x):=x\mathbf{1}_{\{|x|\le 2\}}$. Since $\Delta U^n\mathbf{1}_{\{|\Delta U^n|>2\}}=\Delta\widetilde{m}^n(1-\Delta\widetilde{m}^n)\mathbf{1}_{\{\Delta\widetilde{m}^n\notin[-1,2]\}}$, the first characteristics $B^n(h)$ of $U^n$ is given by

$$B^n(h)=-\langle\widetilde{m}^n,\widetilde{m}^n\rangle-x(1-x)\mathbf{1}_{\{x\notin[-1,2]\}}\mathbf{1}_{[\![0,T^n]\!]}\star\nu^{m^n},$$

and it is clear that the sequence

$$\big(\mathrm{Var}\,(B^n(h))_t\mid P^n\big)\quad\text{is }\mathbb{R}\text{-tight},\quad t\in\mathbb{R}_+. \tag{5.26}$$

By Theorem VI.6.26 in Jacod and Shiryaev [16] we have

$$\begin{aligned}&\mathcal{L}(U^n,[U^n,U^n]\mid P^n)\\ &\quad\xrightarrow{d}\mathcal{L}(M/2-\langle M,M\rangle/4,\langle M,M\rangle/4\mid P)\quad\text{in }\mathbb{D}(\mathbb{R}^2),\quad n\to\infty.\end{aligned} \tag{5.27}$$

Note that $\mathrm{Var}\,\big\{[U^n,U^n]-[\widetilde{m}^n,\widetilde{m}^n]\big\}_t\le\sum_{s\le t}(\Delta\widetilde{m}^n_s)^4+2\sum_{s\le t}|\Delta\widetilde{m}^n_s|^3\xrightarrow{P^n}0$, $n\to\infty$. Thus, we obtain from (5.24), (5.27), and the continuity of the limiting processes that

$$\mathcal{L}(\widetilde{m}^n,\langle\widetilde{m}^n,\widetilde{m}^n\rangle\mid P^n)\xrightarrow{d}\mathcal{L}(M/2,\langle M,M\rangle/4\mid P)\quad\text{in }\mathbb{D}(\mathbb{R}^2),\quad n\to\infty.$$

## 5.5 Proofs of Theorems 3.1–3.6

**Proof of Theorem 3.1:** The hypotheses imply that the assumptions of Theorem 2.1 are satisfied for all $t$ such that $[t,\infty)\cap S\ne\emptyset$. Then (2.8), or (4.3) in the setting of subsection 4.2, holds for all $t$ such that $[t,\infty)\cap S\ne\emptyset$, and we have (2.9) and (2.10) for all $t\in S$. Furthermore,

$$\lim_{n\to\infty}P^n(T^n<t)=0\quad\text{for all }t\in S, \tag{5.28}$$

where $T^n:=\inf\{s\colon Z^n_s<1/n\}$.

Let us stop $m^n$ at $T^n$: $\widetilde{m}^n := (m^n)^{T^n}$. Then $\widetilde{m}^n$ is a $P^n$-locally square-integrable martingale, we have

$$\langle \widetilde{m}^n, \widetilde{m}^n \rangle_t \xrightarrow{P^n} C_t/4, \quad t \in S, \tag{5.29}$$

use (3.1), (2.9) and (5.28), and the Lindeberg condition for $\widetilde{m}^n$ is satisfied due to (2.6). Thus, all the assumptions of the central limit theorem for locally square-integrable martingales are satisfied, see e.g. Liptser and Shiryayev [24, Theorem 5.5.4], and

$$\mathcal{L}(\widetilde{m}^n \mid P^n) \xrightarrow{d_f(S)} \mathcal{L}(M/2 \mid P), \quad n \to \infty. \tag{5.30}$$

Now (3.3) follows from (2.10), (5.28), (5.29) and (5.30). Since $EZ_t = 1$ and $E^n Z_t^n \leq 1$, (4.4) follows for $t \in S$; the same assertion for $t$ satisfying $[t, \infty) \cap S \neq \emptyset$ follows from inequality (5.12) (in the setting of subsection 2.1 one can apply Le Cam's first lemma, see e.g. Jacod and Shiryaev [16, Corollary V.1.12]).

To prove (3.4) it is sufficient to take $S = \{t_1, \dots, t_p\}$, $0 < t_1 < \cdots < t_p$. Define a measure $P'$ on $(\Omega, \mathcal{F})$ by $dP' = Z_{t_p}\, dP$. Then we obtain from (3.3) and Le Cam's third lemma, see e.g. Jacod and Shiryaev [16, Lemma V.1.13], that

$$\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d_f(S)} \mathcal{L}(Z \mid P'), \quad n \to \infty.$$

By Girsanov's theorem, under $P'$, $(M_t - C_t)_{t \leq t_p}$ is a Gaussian martingale with the quadratic characteristic $C_t$, hence $\mathcal{L}((Z_t)_{t \in S} \mid P') = \mathcal{L}((Z'_t)_{t \in S} \mid P)$, and the claim follows.

**Proof of Theorem 3.2:** To prove (3.5) we follow the same lines as in the proof of (3.3) in Theorem 3.1 with the only difference that now we use the functional limit theorem, see e.g. Jacod and Shiryaev [16, Theorem VIII.3.22] or Liptser and Shiryayev [24, Theorem 7.1.4], thus getting

$$\mathcal{L}(\widetilde{m}^n \mid P^n) \xrightarrow{d} \mathcal{L}(M/2 \mid P), \quad n \to \infty,$$

instead of (5.30). The statement follows also from Theorems 3.1 and 4.2.

If (3.6) holds then we are in a position to apply Theorem 4.2 to obtain all the assertions except (3.7); in particular,

$$h_t^n \xrightarrow{P^n} \frac{1}{8} C_t, \quad n \to \infty,$$

in view of part (v) of Theorem 4.2 and (2.9).

The last statement (3.7) follows from (3.4) and from the tightness of $\mathcal{L}(Z^n \mid P'^n)$ in $\mathbb{D}([0, \infty])$, see Jacod and Shiryaev [16, Theorem X.3.1].

**Proof of Theorem 3.3:** The argument follows almost exactly that of Theorem 3.1. Here we use Theorem 5.5.5 in Liptser and Shiryayev [24] (instead of Theorem 5.5.4) to obtain

$$\mathcal{L}(\widetilde{m}^n \mid P) \xrightarrow{d_f(S)} \mathcal{L}(M/2 \mid \mathbf{P}), \quad n \to \infty \quad (\mathcal{G}\text{-stably}).$$

This relation and known properties of the stable convergence, see e.g. Aldous and Eagleson [2], Jacod and Shiryaev [16, Proposition VIII.5.33], imply (3.10).

Let us also prove the assertion stated in Remark 3.3: if $(\xi^n)$ is a sequence of random vectors converging in $P$-probability to $\xi$ and $\xi^n$ are $\mathcal{F}^n_{t_p}$-measurable for all $n$, then $\mathcal{L}(\xi^n, Z^n_{t_1}, \dots, Z^n_{t_p} \mid P'^n) \Rightarrow \mathcal{L}(\xi, Z'_{t_1}, \dots, Z'_{t_p} \mid \mathbf{P})$. Indeed, let $f$ and $g$ be arbitrary bounded continuous functions defined on the corresponding Euclidean spaces. Then

$$\begin{aligned} E'^n\{f(\xi^n)g(Z^n_{t_1}, \dots, Z^n_{t_p})\} &= E\{Z^n_{t_p} f(\xi^n) g(Z^n_{t_1}, \dots, Z^n_{t_p})\} \\ &\quad + E'^n\{f(\xi^n)g(Z^n_{t_1}, \dots, Z^n_{t_p})\mathbf{1}_{\{Z^n_{t_p}=\infty\}}\}. \end{aligned} \tag{5.31}$$

Since $(P'^n_{t_p}) \lhd (P^n_{t_p})$, the sequence $(Z^n_{t_p} \mid P)$ is uniformly integrable and $\lim_n P'^n(Z^n_{t_p} = \infty) = 0$. Therefore, in (5.31) the second term on the right vanishes and we can pass to the limit under the expectation sign in the first term due to (3.10). Thus, we obtain

$$\begin{aligned} \lim_n E'^n\{f(\xi^n)g(Z^n_{t_1}, \dots, Z^n_{t_p})\} &= \mathbf{E}\{Z_{t_p} f(\xi) g(Z_{t_1}, \dots, Z_{t_p})\} \\ &= \mathbf{E}'\{f(\xi)g(Z_{t_1}, \dots, Z_{t_p})\} \\ &= \int f(\xi(\omega))\Big\{\int g(Z_{t_1}(\omega,\alpha), \dots, Z_{t_p}(\omega,\alpha))Q'(\omega, d\alpha)\Big\}P(d\omega) \\ &= \int f(\xi(\omega))\Big\{\int g(Z'_{t_1}(\omega,\alpha), \dots, Z'_{t_p}(\omega,\alpha))Q(\omega, d\alpha)\Big\}P(d\omega) \\ &= \mathbf{E}\{f(\xi)g(Z'_{t_1}, \dots, Z'_{t_p})\}. \end{aligned}$$

**Proof of Theorem 3.4:** The first part of the proof repeats that of Theorem 3.2 including the reference to Theorem 7.1.4 in Liptser and Shiryayev [24], which gives

$$\mathcal{L}(\widetilde{m}^n \mid P) \xrightarrow{d} \mathcal{L}(X/2 \mid \mathbf{P}), \quad n \to \infty \quad (\mathcal{G}\text{-stably}). \tag{5.32}$$

Since $C$ is a continuous process, the convergence (3.12) implies

$$\sup_{s \le t} \Big| h^n_s - \frac{1}{8} C_s \Big| \xrightarrow{P} 0, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+. \tag{5.33}$$

The relations (5.32) and (5.33), together with properties of the stable convergence and our previous elaborations, give (3.13).

There is again an alternative proof of this part of the theorem based on Theorem 4.2. Indeed, by Theorem 3.3 we have (3.10), hence

$$\mathcal{L}(Z^n \mid P) \xrightarrow{d} \mathcal{L}(Z \mid \mathbf{P}), \quad n \to \infty,$$

by Theorem 4.2, and (3.13) follows from Remark 3.2.

To prove the converse part of the theorem including (3.15) we use the same arguments as in the proof of Theorem 3.2; here are some additional necessary explanations. If (3.14) holds, we apply Theorem 4.2 and obtain all its statements; in particular, (i) combined with Remark 3.2 yields (3.13). It remains to show that the convergence in (v) is also $\mathcal{G}$-stable, i.e.

$$\mathcal{L}(\widetilde{m}^n, \langle \widetilde{m}^n, \widetilde{m}^n \rangle \mid P) \xrightarrow{d} \mathcal{L}(X/2, C/4 \mid \mathbf{P}), \quad n \to \infty \quad (\mathcal{G}\text{-stably}). \tag{5.34}$$

Indeed, since $C$ is continuous and the variables $C_t$ are $\mathcal{G}$-measurable, it follows from (5.34) that

$$\langle \widetilde{m}^n, \widetilde{m}^n \rangle_t \xrightarrow{P} C_t/4, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+.$$

and (2.9) implies

$$h_t^n \xrightarrow{P} C_t/8, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+.$$

Reproducing the proof of Theorem 4.2, we see that the convergence in (5.25) is $\mathcal{G}$-stable, i.e.

$$\mathcal{L}(U^n \mid P) \xrightarrow{d} \mathcal{L}(X/2 - C/4 \mid \mathbf{P}), \quad n \to \infty \quad (\mathcal{G}\text{-stably}). \tag{5.35}$$

Therefore, $\mathcal{L}(U^n \mid \widetilde{P}) \xrightarrow{d} \mathcal{L}(X/2 - C/4 \mid \widetilde{\mathbf{P}})$ for any $\widetilde{P} \in \mathcal{P}(\mathcal{G})$, cf. Remark 3.2. Furthermore, in view of (5.35) and (5.26) (with $P$ instead of $P^n$), the sequence $(U^n \mid P)$ is predictably uniformly tight (P-UT), see Jakubowski et al. [17, Lemma 3.1] (where the abbreviation UT was used) or Jacod and Shiryaev [16, Theorem VI.6.21]. Evidently, P-UT is preserved under a change of measure with a bounded density, hence it holds under $\widetilde{P}$ as well. Thus, the assumptions of Theorem VI.6.26 in Jacod and Shiryaev [16] are satisfied for $U^n$ under $\widetilde{P}$, which yields $\mathcal{L}(U^n, [U^n, U^n] \mid \widetilde{P}) \xrightarrow{d} \mathcal{L}(X/2 - C/4, C/4 \mid \widetilde{\mathbf{P}})$ in $\mathbb{D}(\mathbb{R}^2)$. According to Remark 3.2,

$$\mathcal{L}(U^n, [U^n, U^n] \mid P) \xrightarrow{d} \mathcal{L}(X/2 - C/4, C/4 \mid \mathbf{P}), \quad n \to \infty \quad (\mathcal{G}\text{-stably}), \tag{5.36}$$

which implies (5.34) following the lines in the proof of Theorem 4.2.

**Remark 5.1** To apply Theorem VI.6.26 in Jacod and Shiryaev [16] for $U^n$ under $\widetilde{P}$ it is not sufficient to check (5.26) with $P^n \equiv \widetilde{P}$: the triplet of predictable characteristics of a semimartingale depends on a measure and $B^n(h)$ may not be the first characteristic of $U^n$ under $\widetilde{P}$. Such a wrong argument was used by Mordecki in the proof of Theorem 1 in [27]. Our way of arguing shows that his result is nevertheless true.

**Proof of Lemma 3.1:** Repeating the arguments in the proof of Lemma X.1.58 in Jacod and Shiryaev [16], we show that $Z = \mathcal{E}(2M)$ is a $P$-martingale. Then there is a probability measure $P'$ on $(\Omega, \mathcal{F}, \mathbb{F})$ such that $P' \overset{\text{loc}}{\sim} P$ and $Z$ is the density process of $P'$ with respect to $P$, see e.g. Cherny [3]. By Girsanov's theorem $M'$ is a continuous local martingale under $P'$ with the quadratic characteristic $C$. It remains to show the unicity of $P'$.

Let $\widetilde{P}'$ be a probability measure such that $M'$ is a continuous local martingale with $M'_0 = 1$ and $\langle M', M' \rangle = C$ under $\widetilde{P}'$. As above one obtains that $Z' := \mathcal{E}(-2M')$ is a $\widetilde{P}'$-martingale, that there is a probability measure $\widetilde{P}$ such that $\widetilde{P} \overset{\text{loc}}{\sim} \widetilde{P}'$ and $Z'$ is the density process of $\widetilde{P}$ with respect to $\widetilde{P}'$, and that $M = M' + 2C$ is a continuous local martingale under $\widetilde{P}$ with the quadratic characteristic $C$. Thus, $\widetilde{P} = P$, which implies $\widetilde{P}' = P'$.

**Proof of Theorem 3.5:** We start with the first part of the theorem. It follows from (3.16) and M c) that $\lim_n P^n(h^n_t > \frac{1}{2}F_t + 1) = 0$, hence (2.5) is satisfied for all $t \in \mathbb{R}_+$. Thus, we are in a position to apply Theorem 2.1 to get (4.3), (2.9) and (2.10) for all $t \in \mathbb{R}_+$. In particular,

$$\lim_{n\to\infty} P^n(T^n < t) = 0 \quad \text{for all } t \in \mathbb{R}_+. \tag{5.37}$$

Because of (3.16), (2.9) and (5.37), we have

$$\langle \widetilde{m}^n, \widetilde{m}^n \rangle_t - C_t \circ \widetilde{m}^n \xrightarrow{P^n} 0, \quad n \to \infty. \tag{5.38}$$

This allows us to apply Theorem IX.3.27 in Jacod and Shiryaev [16] to obtain

$$\mathcal{L}(\widetilde{m}^n \mid P^n) \xrightarrow{d} \mathcal{L}(M \mid P). \tag{5.39}$$

It remains to note that the hypotheses M b) and M d) imply the continuity of $\alpha \rightsquigarrow C(\alpha)$ in the Skorokhod topology, and we obtain (3.17) from (2.10), (5.37)–(5.39), and from the continuity of the limiting processes.

Let us prove the converse part. Assume that (3.18) holds. Taking into account that $Z = e^{2M-2C}$ is a positive $P$-martingale by Lemma 3.1, we can apply Theorem 4.2 obtaining immediately all the statements except (3.16) and (3.19). Next, (v) says that

$$\mathcal{L}(\widetilde{m}^n, \langle \widetilde{m}^n, \widetilde{m}^n \rangle \mid P^n) \xrightarrow{d} \mathcal{L}(M, \langle M, M \rangle \mid P) \quad \text{in } \mathbb{D}(\mathbb{R}^2), \quad n \to \infty.$$

Moreover, $\langle M, M\rangle = C$, and using M d) we obtain

$$\langle \widetilde{m}^n, \widetilde{m}^n\rangle_t - C_t \circ \widetilde{m}^n \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+,$$

which implies (3.16) due to (2.9).

To prove (3.19) it is enough to note that the sequence $(Z^n \mid P'^n)$ is tight in $\mathbb{D}([0,\infty])$ by Theorem X.3.1 in Jacod and Shiryaev [16], and the finite-dimensional convergence $\mathcal{L}(Z^n \mid P'^n) \xrightarrow{d_f(\mathbb{R}_+)} \mathcal{L}(e^{2M-2C} \mid P')$ follows from Le Cam's third lemma and from Lemma 3.1.

**Proof of Theorem 3.6:** As in the beginning of the proof of Theorem 3.5 we apply Theorem 2.1 to get (4.3), (2.9), (2.10), and

$$\lim_{n\to\infty} P^n(T^n < t) = 0 \quad \text{for all } t \in \mathbb{R}_+, \tag{5.40}$$

where $T^n := \inf\{s\colon Z^n_s < 1/n\}$.

Put $\widetilde{m}^n := (m^n)^{T^n}$, $X^n := 2\widetilde{m}^n - 2\langle \widetilde{m}^n, \widetilde{m}^n\rangle$, $\widetilde{C}^n := \langle \widetilde{m}^n, \widetilde{m}^n\rangle$. It is easy to obtain from the Lindeberg condition (2.6) that

$$\sup_{s\le t} |\Delta \widetilde{m}^n_s| \xrightarrow{P^n} 0 \quad \text{and} \quad \sup_{s\le t} |\Delta \widetilde{C}^n_s| \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+. \tag{5.41}$$

Because of (3.20), (2.9) and (5.40), we have

$$\widetilde{C}^n_t - \frac{1}{4} C_t \circ \log(Z^n \vee \frac{1}{n}) \xrightarrow{P^n} 0; \tag{5.42}$$

moreover, due to X c),

$$\sup_{s\le t} \left| \widetilde{C}^n_s - \frac{1}{4} C_s \circ \log(Z^n \vee \frac{1}{n}) \right| \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+, \tag{5.43}$$

see e.g. Jacod and Shiryaev [16, Lemma IX.3.4].

Using the strong majoration hypothesis X c) again, the sequence $(\widetilde{C}^n \mid P^n)$ is $\mathbb{C}$-tight, use e.g. [16, Lemma IX.3.3]. Since $\widetilde{m}^n$ is a $P^n$-locally square integrable martingale and $\widetilde{C}^n$ is its quadratic characteristic, the sequence $(\widetilde{m}^n \mid P^n)$ is $\mathbb{D}$-tight, use Theorem VI.4.13 in [16], and hence is $\mathbb{C}$-tight because of (5.41).

Thus, the sequence $(X^n \mid P^n)$ is $\mathbb{C}$-tight. Since $\sup_{s\le t} \left|\log(Z^n_s \vee \frac{1}{n}) - X^n_s\right| \xrightarrow{P^n} 0$ due to (2.10) and (5.40), we deduce from the continuity property X d) that one can replace $Z^n \vee \frac{1}{n}$ by $X^n$ in (5.42). The argument used to prove (5.43) allows us to obtain

$$\sup_{s\le t} \left| \widetilde{C}^n_s - \frac{1}{4} C_s \circ X^n \right| \xrightarrow{P^n} 0, \quad n \to \infty, \quad \text{for all} \quad t \in \mathbb{R}_+.$$

Now it is easy to check, using (5.41), that all the assumptions of Theorem IX.3.27 in Jacod and Shiryaev [16] are satisfied for the sequence $(X^n \mid P^n)$, and we conclude that $\mathcal{L}(X^n \mid P^n) \xrightarrow{d} \mathcal{L}(X \mid P)$, which implies (3.21) due to (2.10) and (5.40).

To prove the converse part of the theorem we use Theorem 4.2. In particular, we have, cf. (v) or (5.27),

$$\mathcal{L}(X^n, \langle \widetilde{m}^n, \widetilde{m}^n \rangle \mid P^n) \xrightarrow{d} \mathcal{L}(X, C/4 \mid P) \quad \text{in } \mathbb{D}(\mathbb{R}^2), \quad n \to \infty.$$

In view of X d) we get

$$\langle \widetilde{m}^n, \widetilde{m}^n \rangle_t - \frac{1}{4} C_t \circ X^n \xrightarrow{P^n} 0. \tag{5.44}$$

Since the assumptions of Theorem 2.1 are satisfied for all $t \in \mathbb{R}_+$, we deduce (3.20) from (5.44) using the arguments from the first part of the proof.

The proof of (3.22) is similar to the proof of the corresponding statement in Theorem 3.5.

# 6 Counter-examples

**Example 6.1 (cf. Theorem 2.1)** Let $B$ be a standard Brownian motion in $\mathbb{R}^3$ with $B_0 = (1,0,0)$ on some probability space $(\Omega, \mathcal{F}, P)$ with respect to the natural filtration $\mathbb{F}$. It is well known that $Z_t = 1/\|B_t\|$, where $\|\cdot\|$ is standard Euclidean norm in $\mathbb{R}^3$, is a positive continuous local martingale and the function $t \rightsquigarrow EZ_t$ is strictly decreasing, see e.g. Protter [29], so $Z$ is a supermartingale and not a martingale. Put $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^n) := (\Omega, \mathcal{F}, \mathbb{F}, P)$ and define $P'^n$ by $dP'^n = Z_{T_n}\, dP$, where $T_n = \inf\{t \colon Z_t > n\}$. Since $Z^{T_n}$ is a bounded martingale with expectation 1, $P'^n$ is a probability measure and $P'^n \sim P$. The density process of $P'^n$ with respect to $P$ is $Z^n := Z^{T_n}$. It is clear that $P'^n_0 = P_0$ for all $n$, for all $t$ (2.6) and (2.7) hold and we have (2.12), which implies (2.8) and hence (2.5) by Proposition 2.1. So all the assumptions of Theorem 2.1 are satisfied. But we do not have the contiguity $(P'^n_t) \lhd (P^n_t)$ for any $t > 0$. Indeed, $\lim_n P(T_n < t) = 0$, hence $Z^n_t \to Z_t$ a.s. and $EZ_t < 1$, which contradicts the contiguity because of Le Cam's first lemma [16, Corollary V.1.12].

**Example 6.2 (cf. Propositions 2.1 and 2.2)** Let $\Omega = \{0,1\}$, $\mathcal{F} = 2^\Omega$, $\mathcal{F}_t = \{\emptyset, \Omega\}$ if $t < 1$ and $\mathcal{F}_t = \mathcal{F}$ if $t \geq 1$, $P^n(\{0\}) = 1$, $P'^n(\{0\}) = n^{-1}$. Then $P^n_t = P'^n_t$ for $t < 1$ and $P^n_t \ll P'^n_t$ for $t \geq 1$, $h^n_t = -y^n_t = (1 - n^{-1/2})\mathbf{1}_{\{t \geq 1\}}$ and $m^n_t \equiv 0$ $P^n$-a.s. Therefore, (2.4)–(2.6) hold for all $t \in \mathbb{R}_+$ but, evidently, if

$t \geq 1$, the variational distance $\|P_t^n - P_t'^n\| = 2(1-n^{-1}) \to 2$, $n \to \infty$, hence the sequence $(P_t^n)$ is not contiguous with respect to $(P_t'^n)$. In particular, (2.13), (2.6*), and (2.14) do not hold.

**Example 6.3 (cf. Proposition 2.4 (b))** Let $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n) = (\mathbb{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}), \mathfrak{D}(\mathbb{R}))$ be the Skorokhod space with the canonical process $X$. Assume that $X$ is a Poisson process with intensity $n^{-1}$ (resp. $1+n^{-1}$) under $P^n$ (resp. $P'^n$). Then $P'^n \overset{\text{loc}}{\sim} P^n$ and the density process $Z^n$ of $P'^n$ with respect to $P^n$ is $Z_t^n = e^{-t}(1+n)^{X_t}$. Evidently, $\sup_{s \leq t} |\Delta Z_s^n| \xrightarrow{P^n} 0$ for all $t \in \mathbb{R}_+$ and we have (2.12) and hence the contiguity $(P_t^n) \lhd (P_t'^n)$. But (2.6*) is not satisfied for $t > 0$. Indeed, $y^n = (\sqrt{1+n}-1)X_t - t/2$, hence $\nu^{y^n}(dt, dx) = \frac{1}{n}\delta_{\sqrt{1+n}-1}(dx)\,dt$ (where $\delta_a$ is the Dirac measure at $a$) and $x^2 \mathbf{1}_{\{|x|>\varepsilon\}} \star \nu_t^{y^n} = n^{-1}(\sqrt{1+n}-1)^2 t \to t$.

**Example 6.4 (cf. Theorems 2.1 and 2.2)** This example is well known in the statistical literature, see e.g. Le Cam [21, p. 470]. Let $(\Omega^n, \mathcal{F}^n) = (\Omega, \mathcal{F})$ be the product space $\prod_{k=1}^{\infty}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $P^n = \mu \times \cdots \times \mu \times \cdots$, $P'^n = \mu_n \times \cdots \times \mu_n \times \cdots$, where $\mu$ is the uniform distribution on $[0,1]$ and $\mu_n$ is the uniform distribution on $[0, (1-n^{-1})^{-2}]$. Let $\mathcal{F}_t^n$ be the $\sigma$-field generated by the first $[nt]$ coordinates (here $[\cdot]$ is the integer part of a number). Then ($P^n$-a.s.) $Z_t^n = (1-n^{-1})^{2[nt]}$, $y_t^n = -n^{-1}[nt]$, $h_t^n = -y_t^n$, $m_t^n = 0$, $\iota_t^n = (2n^{-1}-n^{-2})[nt]$. It is clear that conditions (2.4), (2.5), and (2.6*) are satisfied, but (2.7) does not hold. The limiting behaviour of $h^n$ and $Z^n$ is described by Theorem 2.2, and (2.9) and (2.10) are not true.

**Example 6.5 (cf. Theorems 3.3 and 3.4)** Let $(\Omega, \mathcal{F}) = (\mathbb{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ be the Skorokhod space with the canonical process $X$. Assume that $P$ is the Dirac measure at $\alpha_0(t) = 0$ and $P'^n$ is the Dirac measure at $\alpha_n(t) = \mathbf{1}_{\{t \geq n^2\}}$. Define the filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \geq 0}$ as $\mathcal{F}_t^n = \mathcal{D}_{nt}$, where $\mathfrak{D}(\mathbb{R}) = (\mathcal{D}_t)_{t \geq 0}$ is the canonical filtration on $(\Omega, \mathcal{F})$ generated by $X$. Then the nesting condition N is satisfied with $s_n = n^{-1/2}$, $\mathcal{G} = \mathcal{D}(\mathbb{R})$. Then $P_t^n$ and $P_t'^n$ coincide if $t \in [0, n[$, $Z$ is identically equal to one on this interval $P^n$- and $P'^n$-a.s., and all the assumptions of Theorems 3.3 and 3.4 are satisfied with $C \equiv 0$, $Z \equiv 1$, and $Z' \equiv 1$. But $E^n\xi$ may or may not converge for a bounded $\mathcal{G}$-measurable $\xi$.

# A Appendix

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (P, P'))$ be a binary filtered statistical experiment, and let $Q$ be a probability measure on $(\Omega, \mathcal{F})$ such that $P \overset{\text{loc}}{\ll} Q$ and $P' \overset{\text{loc}}{\ll} Q$. The density processes of $P$ and $P'$ with respect to $Q$ are denoted $\mathfrak{z}$ and $\mathfrak{z}'$ respectively. Then the generalized density process $Z$ of $P'$ with respect to $P$ satisfies $Z = \mathfrak{z}'/\mathfrak{z}$ $P$- and $P'$-a.s. Put

$$S_k := \inf\{t\colon \mathfrak{z}_t < 1/k\}, \quad S'_k := \inf\{t\colon \mathfrak{z}'_t < 1/k\}, \quad T_k := \inf\{t\colon Z_t < 1/k\},$$

$$\Sigma := \bigcup_{k=1}^{\infty} [\![0, S_k]\!], \quad \Sigma' := \bigcup_{k=1}^{\infty} [\![0, S'_k]\!], \quad \Gamma := \bigcup_{k=1}^{\infty} [\![0, T_k]\!].$$

Then $\Sigma = [\![0]\!] \cup \{\mathfrak{z}_- > 0\}$ and $\Sigma' = [\![0]\!] \cup \{\mathfrak{z}'_- > 0\}$ $Q$-a.s., $\Gamma = [\![0]\!] \cup \{Z_- > 0\}$ $P$-a.s. Since $P(\inf_t \mathfrak{z}_t > 0) = 1$, we have $\Gamma = \Sigma'$ $P$-a.s.

Let $\alpha \in (0,1)$, $Y(\alpha) := \mathfrak{z}^{\alpha}\mathfrak{z}'^{1-\alpha}$. Recall that a predictable increasing process $h(\alpha)$, $h(\alpha)_0 = 0$, with values in $[0,\infty]$ is a Hellinger process of order $\alpha$ for $P$ and $P'$ if and only if

$$Y(\alpha) + Y(\alpha)_- \cdot h(\alpha) \quad \text{is a } Q\text{-martingale,}$$

see Jacod and Shiryaev [16, Chapter IV, § 1b].

**Lemma A.1** *A predictable increasing process* $\widehat{h}(\alpha)$, $\widehat{h}(\alpha)_0 = 0$, *with values in* $[0,\infty]$ *is a Hellinger process of order* $\alpha$ *for* $P$ *and* $P'$ *if and only if*

$$Z^{1-\alpha} + Z_-^{1-\alpha} \cdot \widehat{h}(\alpha) \quad \textit{is a } P\textit{-martingale.}$$

For the proof see Jacod [14, Lemma 5.8].

To compute the Hellinger processes in terms of $P$ and $Z$ let us introduce the processes $\iota$, $y$, $N$ and $n$ as in subsection 4.1; put also $z := n - \iota$ on $\Gamma$. Let $\nu^y$ be the $P$-compensator of the jump measure of $y$ on $\Gamma$.

**Lemma A.2** *A predictable increasing process* $\widehat{h}(\alpha)$, $\widehat{h}(\alpha)_0 = 0$, *with values in* $[0,\infty]$ *is a Hellinger process of order* $\alpha$ *for* $P$ *and* $P'$ *if and only if*

$$\begin{aligned}\widehat{h}(\alpha) &= (1-\alpha)\iota + 2\alpha(1-\alpha)\langle m^c, m^c\rangle \\ &\quad + \{\alpha + (1-\alpha)(1+x)^2 - (1+x)^{2(1-\alpha)}\} \star \nu^y \quad \textit{on } \Gamma \quad P\textit{-a.s.} \qquad (A.1)\end{aligned}$$

**Proof:** A similar formula expressed in terms of $\langle z^c, z^c\rangle$ and $\nu^z$ instead of $\langle m^c, m^c\rangle$ and $\nu^y$ is proved in Jacod [12, Proposition 3.19]; it reduces to (A.1) due to $z^c = 2m^c$ (see the proof of Lemma 4.1) and $\Delta z = (1 + \Delta y)^2 - 1$ on $\Gamma$.

One can also use the characterization of $\widehat{h}(\alpha)$ from Lemma A.1 and standard computations based on Itô's formula as in the proof of Theorem IV.1.33 in Jacod and Shiryaev [16].

Now we want to compute the processes $i(\psi; P, P')$ and $i(\psi; P', P)$, introduced by Jacod and Shiryaev [16, Chapter IV, § 1d], in terms of $P$ and $Z$. Let $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\frac{\psi(x)}{|x-1|^2\wedge|x-1|}$ is bounded (in particular, $\psi(1) = 0$). We also set $\psi(\infty) = 0$ and use the conventions $0/0 = 0$ and $a/0 = \infty$ if $a \in (0, \infty]$. Define

$$j(\psi; P, P') := \sum_{s\le\cdot}\Big(1+\frac{\Delta\mathfrak{z}'_s}{\mathfrak{z}'_{s-}}\Big)\psi\left(\frac{1+\frac{\Delta\mathfrak{z}_s}{\mathfrak{z}_{s-}}}{1+\frac{\Delta\mathfrak{z}'_s}{\mathfrak{z}'_{s-}}}\right) = \Big(1+\frac{y}{\mathfrak{z}'_-}\Big)\psi\left(\frac{1+\frac{x}{\mathfrak{z}_-}}{1+\frac{y}{\mathfrak{z}'_-}}\right)\star\mu^{\mathfrak{z},\mathfrak{z}'},$$

where $\mu^{\mathfrak{z},\mathfrak{z}'}(dx, dy)$ is the jump measure of $(\mathfrak{z}, \mathfrak{z}')$. It is shown in the proof of Proposition IV.1.42 in [16] that the process $j(\psi; P, P')$ is a $Q$-locally integrable increasing process on $\Sigma\cap\Sigma'$. A predictable increasing process $i(\psi)$, $i(\psi)_0 = 0$, with values in $[0, \infty]$ is said to be a version of the process $i(\psi; P, P')$ if $i(\psi)$ is the $Q$-compensator of $j(\psi; P, P')$ on $\Sigma\cap\Sigma'$. This definition does not depend on the choice of $Q$ and

$$i(\psi) = \Big(1+\frac{y}{\mathfrak{z}'_-}\Big)\psi\left(\frac{1+\frac{x}{\mathfrak{z}_-}}{1+\frac{y}{\mathfrak{z}'_-}}\right)\star\nu^{\mathfrak{z},\mathfrak{z}'}$$

is a version of $i(\psi; P, P')$, where $\nu^{\mathfrak{z},\mathfrak{z}'}(dx, dy)$ is the $Q$-compensator of $\mu^{\mathfrak{z},\mathfrak{z}'}(dx, dy)$. Note also that $i(\psi; P, P')$ is $P$- and $P'$-a.s. unique on $\Sigma\cap\Sigma'$, hence it is $P$-a.s. unique on $\Gamma$.

Let $\mu^Z$ be the jump measure of $Z$ and $\nu^Z$ its $P$-compensator.

**Lemma A.3** (a) *A predictable increasing process $i(\psi)$, $i(\psi)_0 = 0$, with values in $[0, \infty]$ is a version of the process $i(\psi; P, P')$ if and only if*

$$i(\psi) = \Big(1+\frac{x}{Z_-}\Big)\psi\left(\frac{1}{1+\frac{x}{Z_-}}\right)\star\nu^Z + \psi(0)\iota \quad \textit{on } \Gamma \quad P\textit{-a.s.} \tag{A.2}$$

(b) *A predictable increasing process $i(\psi)$, $i(\psi)_0 = 0$, with values in $[0, \infty]$ is a version of the process $i(\psi; P', P)$ if and only if*

$$i(\psi) = \psi\Big(1+\frac{x}{Z_-}\Big)\star\nu^Z \quad \textit{on } \Gamma \quad P\textit{-a.s.} \tag{A.3}$$

**Proof:** Take $Q = (P + P')/2$. We start with statement (b).

The process $j(\psi) := j(\psi; P', P)$ does not jump when $\mathfrak{z}$ jumps to 0. Hence the process

$$A := \sum_{s\leq \cdot} \frac{1}{\mathfrak{z}_{s-}} \psi \left( \frac{1 + \frac{\Delta \mathfrak{z}'_s}{\mathfrak{z}'_{s-}}}{1 + \frac{\Delta \mathfrak{z}_s}{\mathfrak{z}_{s-}}} \right)$$

is a finite increasing process on $\Sigma \cap \Sigma'$, where $j(\psi) = \mathfrak{z} \cdot A$ holds. By Itô's formula,

$$j(\psi)^{S_k \wedge S'_m} - (\mathfrak{z}A)^{S_k \wedge S'_m} \text{ is a } Q\text{-local martingale.}$$

On the other hand, let $i(\psi)$ be any version of $i(\psi; P', P)$. Define the predictable process $B := (1/\mathfrak{z}_-) \cdot i(\psi)$ on $\Sigma \cap \Sigma'$. Itô's formula gives $\mathfrak{z}B = i(\psi) + B \cdot \mathfrak{z}$ on $\Sigma \cap \Sigma'$, hence

$$i(\psi)^{S_k \wedge S'_m} - (\mathfrak{z}B)^{S_k \wedge S'_m} \text{ is a } Q\text{-local martingale.}$$

Therefore,

$$\{\mathfrak{z}(A - B)\}^{S_k \wedge S'_m} \text{ is a } Q\text{-local martingale.}$$

By Corollary III.3.10 in Jacod and Shiryaev [16] $(A - B)^{S'_m}$ is a $P$-local martingale. From the definitions of $A$ and $B$ we see that $i(\psi)^{S'_m}$ is the $P$-compensator of the process

$$\sum_{s\leq \cdot \wedge S'_m} \psi \left( \frac{1 + \frac{\Delta \mathfrak{z}'_s}{\mathfrak{z}'_{s-}}}{1 + \frac{\Delta \mathfrak{z}_s}{\mathfrak{z}_{s-}}} \right) = \sum_{s\leq \cdot \wedge S'_m} \psi\Big(1 + \frac{\Delta Z_s}{Z_{s-}}\Big) = \psi\Big(1 + \frac{x}{Z_-}\Big) \mathbf{1}_{[\![0, S'_m]\!]} \star \mu^Z,$$

where $\mu^Z$ is the jump measure of $Z$. Since $\bigcup_{m=1}^{\infty} [\![0, S'_m]\!] = \Sigma' = \Gamma$ $P$-a.s., we arrive at (A.3).

If, conversely, $i(\psi)$ satisfies (A.3) then, according to what has just been proved, it coincides with any version of $i(\psi; P', P)$ $P$-a.s. on $\Gamma$, hence $Q$-a.s. on $\Gamma \cap \Sigma = \Sigma \cap \Sigma'$.

To prove (a) we first note that $j(\psi; P, P') = j(\psi^*; P', P) + j(\psi \mathbf{1}_{\{0\}}; P, P')$, where

$$\psi^*(x) := \begin{cases} 0, & \text{if } x = 0, \\ x\psi(1/x), & \text{if } x > 0. \end{cases}$$

Therefore, in view of (b), it is enough to consider the case $\psi(x) = \mathbf{1}_{\{x=0\}}$. For this $\psi$

$$j(\psi) := j(\psi; P, P') = \frac{\mathfrak{z}'_S}{\mathfrak{z}'_{S-}} \mathbf{1}_{\{\mathfrak{z}_{S-} > 0\}} \mathbf{1}_{[\![S,\infty[\![},$$

where $S := \lim_n S_n$.

For any version $i(\psi)$ of $i(\psi; P, P')$

$$\mathfrak{z}'_- \cdot j(\psi)^{S_k \wedge S'_m} - \mathfrak{z}'_- \cdot i(\psi)^{S_k \wedge S'_m} \text{ is a } Q\text{-local martingale for all } k \text{ and } m.$$

We have
$$\mathfrak{z}'_- \cdot j(\psi)^{S_k\wedge S'_m} = (\mathfrak{z}'\mathbf{1}_{\{\mathfrak{z}=0\}})^{S_k\wedge S'_m} = (\mathfrak{z}' - \mathfrak{z}Z)^{S_k\wedge S'_m}$$
and
$$\begin{aligned}\mathfrak{z}'_- \cdot i(\psi)^{S_k\wedge S'_m} &= \mathfrak{z}_-^{S_k\wedge S'_m} \cdot \big(Z_- \cdot i(\psi)^{S_k\wedge S'_m}\big)\\ &= \mathfrak{z}^{S_k\wedge S'_m}\big(Z_- \cdot i(\psi)^{S_k\wedge S'_m}\big) - \big(Z_- \cdot i(\psi)^{S_k\wedge S'_m}\big) \cdot \mathfrak{z}^{S_k\wedge S'_m}\end{aligned}$$

Hence
$$\big\{\mathfrak{z}Z + \mathfrak{z}(Z_- \cdot i(\psi))\big\}^{S_k\wedge S'_m} \text{ is a } Q\text{-local martingale.}$$
By Corollary III.3.10 in Jacod and Shiryaev [16] $(Z + Z_- \cdot i(\psi))^{S'_m}$ is a $P$-local martingale, i.e. $Z + Z_- \cdot i(\psi)$ is a $P$-local martingale on $\Gamma$. From the unicity of the Doob–Meyer decomposition the processes $i(\psi)$ and $\iota$ are $P$-indistinguishable on $\Gamma$, and (A.2) for $\psi(x) = \mathbf{1}_{\{x=0\}}$ follows.

Finally, as $\iota$ coincides with any version of $i(\psi; P, P')$ $P$-a.s. on $\Gamma$, the same holds $Q$-a.s. on $\Gamma \cap \Sigma = \Sigma \cap \Sigma'$.

Recall that the Hellinger process $h(0; P, P')$ (resp. $h(1; P, P') = h(0; P', P)$) of order 0 (resp. of order 1) for $P$ and $P'$ is defined as $i(\psi; P, P')$ (resp. $i(\psi; P', P)$) with $\psi(x) = \mathbf{1}_{\{x=0\}}$. Let $\nu^y$ be a predictable random measure, charging only the set $\mathbb{R}_+ \times \big([-1, 0) \cup (0, \infty)\big)$, which coincides with the $P$-compensator of the jump measure of $y$ on $\Gamma$. In the next corollary, which immediately follows from Lemma A.3 and the equality $1+\Delta Z/Z_- = (1+\Delta y)^2$ on $\Gamma$, the equalities are understood as follows: any version of the process on the left is a version of the process on the right and vice versa.

**Corollary A.1** *The following equalities hold:*
$$h(0; P, P') = \iota,$$
$$h(0; P', P) = \mathbf{1}_{\{x=-1\}} \star \nu^y,$$
$$i(\mathbf{1}_{[0,\beta]}; P', P) = \mathbf{1}_{\{x\le -1+\beta^{1/2}\}} \star \nu^y, \quad \beta \in (0, 1), \tag{A.4}$$
$$i(\mathbf{1}_{[0,\beta]}; P, P') = (1 + x)^2\mathbf{1}_{\{x\ge \beta^{-1/2}-1\}} \star \nu^y + \iota, \quad \beta \in (0, 1). \tag{A.5}$$

Finally let us recall that the $p$-divergency process $k(p) = k(p; P, P') = k(p; P', P)$, $p \ge 2$, was defined by Dzhaparidze and Valkeila [5] as
$$k(p) = |(1 + x/\mathfrak{z}_-)^{1/p} - (1 + y/\mathfrak{z}'_-)^{1/p}|^p \star \nu^{\mathfrak{z},\mathfrak{z}'}.$$
In other words, $k(p)$ is the "strict" version of $i(\chi_p; P, P') + i(\psi; P', P)$ or $i(\chi_p; P', P) + i(\psi; P, P')$ with $\chi_p(x) := |x^{1/p} - 1|^p$ and $\psi(x) := \mathbf{1}_{\{x=0\}}$. The next result follows immediately from Lemma A.3.

**Corollary A.2**

$$k(p;P,P') = \left| \left(1 + \frac{x}{Z_-}\right)^{1/p} - 1 \right|^p \mathbf{1}_\Gamma \star \nu^Z + \mathbf{1}_\Gamma \cdot \iota \quad P\text{-}a.s.$$

# Acknowledgements

This work was supported in part by INTAS Project 99-00016, by the University of Helsinki and by a grant from Suomalainen Tiedeakatemia.

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